Boundary Conformal Field Theory from Strings to Percolation to Quantum Quenches

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Dirac Medal Lecture
ICTP Trieste, July 2012

Quantum Theory. Prof. Dirac.

Basic notions: dynamical system which possesses states represented by vectors $|A\rangle$ (ket vector). Superposition of states means that particles are not necessarily definitely in one place.

(i) The kets are complex i.e. $c|A\rangle$ is of the same nature as $|A\rangle$ for any complex no. c.

(ii) A state corresponds to the direction of $|A\rangle$. i.e. $|A\rangle$, $c|A\rangle$ correspond to the same state, for $c \neq 0$.

Superposition of $|A\rangle$ & $|B\rangle$ corresponds to $c_1|A\rangle + c_2|B\rangle$ direction of which is fixed by c_1/c_2. $\therefore$ there is a 2-fold ∞ of states formed by superposition. This is necessary e.g. polarisation of photons.

The dual of a ket is a bra vector $\langle B|$. The bra $\langle B|$ and the ket $|A\rangle$ have a scalar product $\langle B|A\rangle$

Lecture notes, University of Cambridge Maths Tripos Part III, 1967

3. Field Theoretical Approach to Critical Phenomena

E. Brézin, J. C. Le Guillou* and J. Zinn-Justin

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Centre d'Etudes Nucléaires de Saclay,
BP n° 2—91190 Gif-sur-Yvette, France*

E. Brézin *et al.* in *Phase Transitions and Critical Phenomena*, Vol. 6, edited by C. Domb and M.S. Green, 1976

1983: The conformal field theory revolution

Nuclear Physics B241 (1984) 333–380

INFINITE CONFORMAL SYMMETRY IN TWO-DIMENSIONAL QUANTUM FIELD THEORY

A A BELAVIN, A M POLYAKOV and A B ZAMOLODCHIKOV

L D Landau Institute for Theoretical Physics, Academy of Sciences, Kosygina 2, 117334 Moscow, USSR

Received 22 November 1983

We present an investigation of the massless, two-dimensional, interacting field theories Their basic property is their invariance under an infinite-dimensional group of conformal (analytic) transformations It is shown that the local fields forming the operator algebra can be classified according to the irreducible representations of Virasoro algebra, and that the correlation functions are built up of the "conformal blocks" which are completely determined by the conformal invariance Exactly solvable conformal theories associated with the degenerate representations are analyzed In these theories the anomalous dimensions are known exactly and the correlation functions satisfy the systems of linear differential equations

The BPZ paper, 1983

Nuclear Physics B240[FS12] (1984) 514–532

CONFORMAL INVARIANCE AND SURFACE CRITICAL BEHAVIOR

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Received 27 June 1984

Conformal invariance constrains the form of correlation functions near a free surface In two dimensions, for a wide class of models, it completely determines the correlation functions at the critical point, and yields the exact values of the surface critical exponents They are related to the bulk exponents in a non-trivial way For the Q-state Potts model ($0 \leqslant Q \leqslant 4$) we find $\eta_{\|} = 2/(3\nu - 1)$, and for the O(N) model ($-2 \leqslant N \leqslant 2$), $\eta_{\|} = (2\nu - 1)/(4\nu - 1)$

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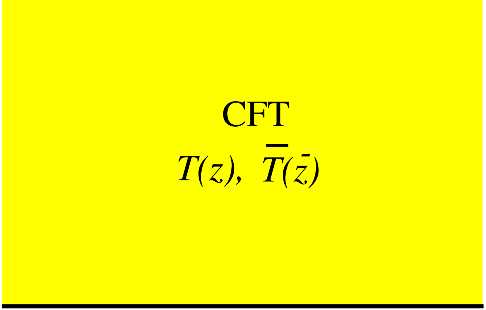
Boundary CFT, 1984

Boundary conditions in CFT



$$T_{xy} = 0 \quad \text{at the boundary} \quad \Rightarrow T(z) = \overline{T}(\bar{z})$$

- $\overline{T}(\bar{z})$ is the analytic continuation of $T(z)$
- boundary CFT is simpler than bulk CFT!

Classifying boundary conditions

Nuclear Physics B324 (1989) 581–596
North-Holland, Amsterdam

BOUNDARY CONDITIONS, FUSION RULES AND THE VERLINDE FORMULA

John L. CARDY

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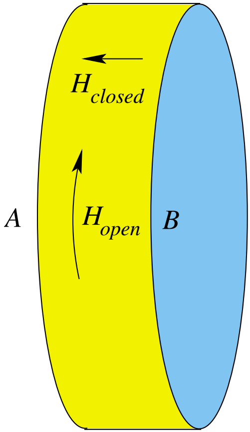
Received 27 February 1989

Boundary operators in conformal field theory are considered as arising from the juxtaposition of different types of boundary conditions. From this point of view, the operator content of the theory in an annulus may be related to the fusion rules. By considering the partition function in such a geometry, we give a simple derivation of the Verlinde formula.

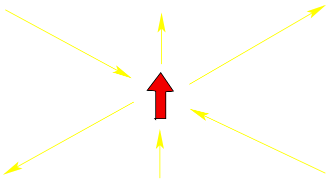
Annulus partition function



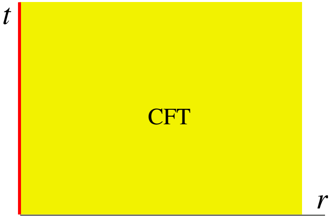
$$
\begin{aligned}
Z_{\text{annulus}} &= \operatorname{Tr} e^{-L\,\hat{H}^{AB}_{\text{open}}} \\
&= \langle A | e^{-W\,\hat{H}_{\text{closed}}} | B \rangle
\end{aligned}
$$

- for a rational CFT, this allows us to classify boundary states
- the states C that can propagate in H^{AB}_{open} are given by the fusion rules N^{C}_{AB} of the CFT
- in string theory, this is the consistency between the open and closed string sectors: the boundary states tell us how open strings can end, *eg* on D-branes

Quantum Impurity Problems [Affleck, Ludwig]

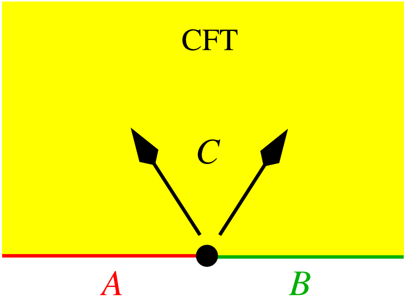


- localised impurity (*eg* magnetic moment) interacting with gapless bulk degrees of freedom (*eg* a Fermi sea of electrons)
- only S-waves matter: 1+1-dimensional problem $r > 0$



- impurity acts as boundary condition on CFT
- allowed conformal boundary conditions $\Rightarrow$ different possibilities for how the electrons screen the impurity

Changes in boundary conditions



- the fusion rules N_{AB}^{C} tell us which operators can sit at a point where the boundary conditions change (boundary condition changing operators) and which states can propagate from there into the bulk

J. Phys. A: Math. Gen. **25** (1992) L201–L206. Printed in the UK

LETTER TO THE EDITOR

Critical percolation in finite geometries

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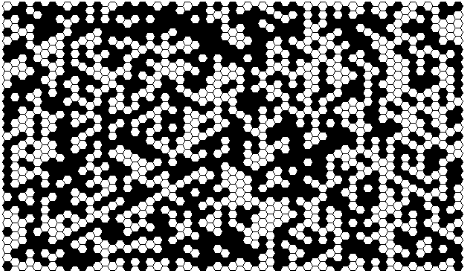
Received 25 November 1991

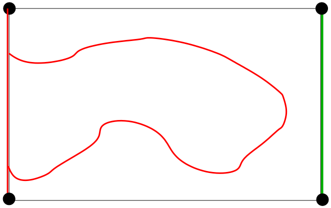
Abstract. The methods of conformal field theory are used to compute the crossing probabilities between segments of the boundary of a compact two-dimensional region at the percolation threshold. These probabilities are shown to be invariant not only under changes of scale, but also under mappings of the region which are conformal in the interior and continuous on the boundary. This is a larger invariance than that expected for generic critical systems. Specific predictions are presented for the crossing probability between opposite sides of a rectangle, and are compared with recent numerical work. The agreement is excellent.

Percolation crossing formula, 1991

Percolation



- each hexagon independently coloured black or white with probability p or $1-p$
- what is the probability P of a left-right crossing on only black hexagons as the lattice spacing $\to 0$?
- for $p > p_c = \frac{1}{2}$, $P \to 1$; for $p < p_c$, $P \to 0$
- at $p = p_c$ it is a non-trivial function of the shape of the rectangle



- colour each black cluster one of Q colours (red, green,...)
- impose red boundary conditions on the left, green on the right

$$1 - P = \lim_{Q \to 1} Z(\mathrm{red}|\mathrm{green})$$

- partition function $Z(\mathrm{red}|\mathrm{green})$ can be viewed as a correlation function of boundary condition changing operators:

$$\langle \phi_{(\mathrm{red}|\mathrm{free})}\, \phi_{(\mathrm{free}|\mathrm{green})}\, \phi_{(\mathrm{green}|\mathrm{free})}\, \phi_{(\mathrm{free}|\mathrm{red})} \rangle$$

- this satisfies a BPZ-type differential equation

The answer is

$$P = \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})^2}\,\eta^{1/3}{}_2F_1(\tfrac{1}{3},\tfrac{2}{3},\tfrac{4}{3};\eta)$$

where

$$\eta = ((1-k)/(1+k))^2 \quad \text{with} \quad \text{width/height} = K(1-k^2)/2K(k^2)$$

- this formula led to the mathematical development of Schramm-Loewner Evolution (SLE) [Schramm, 2000] and was finally proven rigorously by Smirnov [2001]

PRL **96,** 136801 (2006) PHYSICAL REVIEW LETTERS week ending 7 APRIL 2006

Time Dependence of Correlation Functions Following a Quantum Quench

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(Received 13 January 2006; published 3 April 2006)

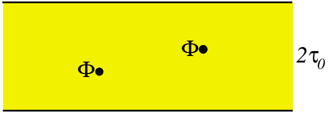
We show that the time dependence of correlation functions in an extended quantum system in d dimensions, which is prepared in the ground state of some Hamiltonian and then evolves without dissipation according to some other Hamiltonian, may be extracted using methods of boundary critical phenomena in $d + 1$ dimensions. For $d = 1$ particularly powerful results are available using conformal field theory. These are checked against those available from solvable models. They may be explained in terms of a picture, valid more generally, whereby quasiparticles, entangled over regions of the order of the correlation length in the initial state, then propagate classically through the system.

P. Calabrese + JC, 2006

- prepare an extended quantum system in a pure state $|\psi_0\rangle$
- for times $t > 0$, evolve the state *unitarily* with hamiltonian H
- how do correlation functions of local observables $\langle\Phi_1(x_1,t)\Phi_2(x_2,t)\cdots\rangle$ evolve?
- do they become stationary? If so what is the stationary state?
- we studied this in the case when

$$H = H_{\mathrm{CFT}} \quad \text{and} \quad |\psi_0\rangle = e^{-\tau_0 H}|B\rangle$$

where $|B\rangle$ is a conformally invariant boundary state



- in imaginary time:

$$\langle B|e^{-\tau_0 H}e^{itH}\,\Phi_1(x_1)\Phi_2(x_2)\cdots e^{-itH}e^{-\tau_0 H}|B\rangle$$
$$= \langle\Phi_1(x_1,\tau)\Phi_2(x_2,\tau)\cdots\rangle_{\text{slab}} \quad \text{with} \quad \tau\to\tau_0+it$$

- this can be computed by conformal mapping to the upper half plane, with the results:
  - correlations in region of length ℓ become stationary after a time $t=\ell/2v$
  - thermalisation: reduced density matrix is Gibbs ensemble

$$\rho_\ell \propto e^{-\beta H_{\text{CFT}}} \quad \text{where} \quad \beta = 4\tau_0$$

- quench from a more general state leads to a generalised Gibbs ensemble
- this suggests a more generally applicable physical picture

A Plea for Curiosity-Driven Theoretical Physics

- none of the above results were ever part of a formal research proposal
  - no ‘milestones’
  - no ‘beneficiaries’
  - no ‘post-doctoral training programme’
  - no box-ticking on some research assessment
- there was no scramble to post them on the arXiv
- they never appeared on the front of *Nature* or *Science*

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- they never appeared on the front of *Nature* or *Science*
- they grew out of pure scientific curiosity (in a way that I hope Dirac would have approved)
- current funding, research assessment and academic promotion procedures are stifling this central aspect of our subject!