Quantum Field Theory Michaelmas Term 2009 Prof. J. Cardy

Problem Set 1

Although everyone in the class is welcome to try these problems, only the papers of students in Theoretical Physics will be marked. They should be put in my box in Theoretical Physics by **Monday November 2 at noon.** They will be reviewed at a Problems Class (to be arranged) which everyone in the class is welcome to attend. Any questions before the due date should be directed to me at `j.cardy1@physics.ox.ac.uk`
Parts of questions marked † may be found slightly harder. However anyone seriously interested in learning QFT should try to tackle them.

1. A one-dimensional simple harmonic oscillator has the action $S = \int(\frac{1}{2}(\dot{q}^2 - \frac{1}{2}\omega^2 q^2)dt$. This looks like the action for a QFT in zero space dimension and one time dimension.

   (a) Use this observation, and what we went through in class, to write down an expression for the the time-ordered product $\langle 0|\mathbf{T}[\hat{q}(t_1)\hat{q}(t_2)]|0\rangle$. Check that when $t_1 = t_2$ this agrees with the result for $\langle 0|\hat{q}^2|0\rangle$ using the ground state wave-function.

   (b) Now work out the same quantity in imaginary time. Use it to find the equilibrium correlation function $\langle y(x_1)y(x_2)\rangle$ of an infinitely long classical stretched string, with string tension σ, at temperature T, and in a confining potential $V(y) = \frac{1}{2}\omega y^2$, where $y(x)$ is the transverse displacement at the point x on the string. †What happens in the limit $\omega \to 0$?

2. In the spin-wave theory of a classical XY ferromagnet, the local magnetisation has the form $\vec{S}(x) = (\cos\phi(x), \sin\phi(x))$ and the energy is $E[\phi] = J\int(\partial\phi)^2 d^d x$. In order to compute the correlation function $\langle\vec{S}(x_1)\cdot\vec{S}(x_2)\rangle = \langle\cos(\phi(x_1) - \phi(x_2))\rangle$ we have to evaluate the euclidean path integral

   $$\frac{1}{Z}\int[d\phi]\,\cos(\phi(x_1) - \phi(x_2))\,e^{-\beta E[\phi]}$$

   Use the formula derived in class for $Z[J]$ with a source $J(x) = i\delta(x - x_1) - i\delta(x - x_2)$ to get an explicit expression for this correlation function. †Show that its behaviour as $|x_1 - x_2| \to \infty$ is qualitatively different in dimensions $d = 1, 2$ and 3.

3. What are the propagators (in momentum space) of the euclidean field theories with the following free actions? (You should be able to write these down more or less by inspection. There may be more than one propagator if there is more than one type of field.)

   (a) $S = \frac{1}{2}\int[(\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_z^2\phi)^2]dxdydz$ (note the ∂^2 in the last term)

   (b) $S = \frac{1}{2}\int[(\partial\phi_1)^2 + (\partial\phi_2)^2 + m_1^2\phi_1^2 + m_2\phi_2^2 + 2\mu\phi_1\phi_2]d^dx$

   (c)† $S = \int[\frac{1}{2}(\phi^*\partial_\tau\phi - \phi\partial_\tau\phi^*) + (\nabla\phi^*)\cdot(\nabla\phi)]d\tau d^Dx$ (ϕ is complex)

4. What are the vertices, in momentum space, corresponding to the following interaction terms in a scalar field theory? Draw the way they would appear at lowest order in a tree diagram, give the numerical factor, and label any lines as appropriate.

   a) $\lambda(\phi^{*2}\phi + \phi^*\phi^2)$ (ϕ is complex); (b) $\lambda\cos\phi$; (c)† $\lambda\phi(\partial_\mu\phi)(\partial^\mu\phi)$.

5. Consider a QFT with two real scalar fields ϕ and Φ and a lagrangian density

$$\tfrac{1}{2}((\partial\phi)^2 + m^2\phi^2) + \tfrac{1}{2}((\partial\Phi)^2 + M^2\Phi^2) + \tfrac{1}{2}\lambda\phi^2\Phi$$

   Write down the Feynman rules for this theory (be careful to use a different sort of line for the propagators of different fields). Draw the tree and one loop diagrams that contribute to the correlation functions $\langle\phi\phi\rangle$, $\langle\Phi\Phi\rangle$ and $\langle\phi\phi\Phi\rangle$.

   † Evaluate as far as you can the irreducible 1-loop diagram contributing to $\langle\phi\phi\Phi\rangle$. To simplify, you can assume that the external momenta (p_1, p_2, p_3) are all zero, but you should work in arbitrary dimension d. For what range of d is the diagram finite?