**Notes on the c-theorem**

These notes are an addendum to my 2008 les Houches lecture notes.
Further references can be found in my 1988 les Houches notes, and in Komargodski, arXiv:1112.4583.
CFTs correspond to RG fixed points. If we perturb them by relevant operators we flow in the IR to other CFTs (including, possibly, the empty theory with only the identity operator.) Intermediate points between these correpond to QFTs with a mass scale. It would be useful to have a global picture of all these theories and the possible flows between them.
In 1986 Zamolodchikov proved a theorem which does this in 2 dimensions. It states that, in the space of all 2d renormalisable relativistic QFTs there exists a function $C(\{g\})$ of the coupling constants which is non-increasing along RG flows, which is stationary only at fixed points, where it equals the value of $c$ in the corresponding CFT.
His proof is rather straightforward. In any 2d relativistic QFT there are 3 independent components of the (symmetric) stress tensor, classified according to their spin: $T = T_{zz}$, $\overline{T} = T_{\bar{z}\bar{z}}$ and the trace $\Theta = T^{\mu}_{\mu}$. In a CFT $\Theta = 0$. Consider the 2-point functions, which have the form dictated by rotation symmetry,

$$\begin{aligned}
\langle T(z,\bar{z})T(0,0)\rangle &= F(z\bar{z})/z^4 \\
\langle\Theta(z,\bar{z})T(0,0)\rangle = \langle T(z,\bar{z})\Theta(0,0)\rangle &= G(z\bar{z})/z^3\bar{z} \\
\langle\Theta(z,\bar{z})\Theta(0,0)\rangle &= H(z\bar{z})/z^2\bar{z}^2
\end{aligned}$$

Conservation of the stress tensor implies

$$\partial^z T_{zz} + \partial^{\bar{z}} T_{\bar{z}z} \propto \partial_{\bar{z}} T + \tfrac{1}{4}\partial_z \Theta = 0$$

This gives

$$\begin{aligned}
\dot{F} + \tfrac{1}{4}(\dot{G} - 3G) &= 0 \\
\dot{G} - G + \tfrac{1}{4}(\dot{H} - 2H) &= 0
\end{aligned}$$

where $\dot{F} = z\bar{z}F'(z\bar{z})$, etc. Eliminating $G$ and defining $C = 2F - G - \frac{3}{8}H$ we see that

$$\dot{C} = -\frac{3}{4}H \le 0$$

Thus $C$ is a non-increasing function of $R = |z|$. On the other hand, being dimensionless, it should obey the RG equation

$$\left(R\frac{\partial}{\partial R} + \sum_j \beta_j(\{g\})\frac{\partial}{\partial g_j}\right) C(R,\{g\}) = 0$$

The function $C(\{g\}) = C(R_0, \{g\})$ for any fixed $R_0$ is therefore non-increasing along RG flows satisfying $\dot{g}_j = -\beta_j(\{g\})$. Moreover it is stationary only at fixed points when $\beta_j = 0$ and $\Theta = 0$, and at these points, since $F = c/2$ and $G = H = 0$, $C = c$. QED. (Notice that this argument also implies that scale invariance implies conformal invariance in 2d, at least for theories with a local stress tensor. This is much harder to argue in higher dimensions.)
Zamolodchikov's argument also has the integrated form

$$c_{UV} - c_{IR} = \tfrac{3}{4} \int H d(R^2)/R^2 = \frac{3}{4\pi} \int r^2 \langle \Theta(r)\Theta(0) \rangle d^2r \geq 0$$

If we perturb a CFT by a relevant term in the action $\lambda \int \Phi d^2 r$ it can be argued that $\Theta(r) = -2\pi\lambda(2 - x_\Phi)\Phi(r)$. This then gives a sum rule for non-critical theories which can be compared with experiment.
However a simple generalisation of Zamolodchikov's argument based on the 2-point function to $d > 2$ fails, because $T_{\mu\nu}$ has more independent components and there are too many unknown functions like $F, G, H$.
A key to progress is afforded by the observation that $c$ occurs in other places in a 2d CFT, in particular when it is coupled to a curved background metric. In that case the expectation value of the trace $\langle \Theta \rangle$ is non-vanishing, even in the CFT, because of the scale provided by the local curvature. In fact

$$\langle \Theta \rangle = -\frac{cR}{12}$$

where $R$ is the gaussian curvature, $= 1/(\rho_1 \rho_2)$ where $\rho_1$ and $\rho_2$ are the principal radii of curvature if the 2d surface is embedded in $\mathbf{R}^3$.
The fact that the coefficient should be proportional to $c$ can seen by perturbing about flat space: since $T_{\mu\nu}$ is the response to an infinitesimal change in the metric, in particular

$$\langle \delta T(z, \bar{z}) \rangle \propto c \int \frac{\delta g^{zz}(z', \bar{z}')}{(z - z')^4} d^2 z' + \cdots$$

Formally this looks like it depends only on $z$, but it is UV divergent and we need to introduce a cutoff $|z - z'| > a$. This means that $\partial_{\bar{z}} \langle \delta T(z, \bar{z}) \rangle \neq 0$, and so, to maintain conservation, we have to introduce a non-zero $\langle \Theta \rangle$. A more careful analysis then shows that we get the above when we express $R$ to first order in $\delta g_{\mu\nu}$.
The integrated form of Zamolodchikov's result may now be re-derived as follows. Introduce an additional scalar field called the dilaton $\tau$. In flat space we modify the perturbed CFT as follows

$$S = S_{CFT_{UV}} + \lambda \int \Phi(r) e^{(x_\Phi - 2)\tau} d^2 r$$

where, under a scale transformation $r^\mu \to e^b r^\mu$, $\Phi \to e^{-bx_\Phi}\Phi$ and $\tau \to \tau + b$, so the whole action is now scale invariant: $T^\mu_\mu|_{\text{tot}} = 0$. Note that if we expand around some constant value of $\tau$ (wlog $\tau = 0$) the first order term couples to $\Theta|_{\text{flat space}}$ as above.

Before we introduced the dilaton, the theory flowed from a CFT with central charge $c_{UV}$ to one with central charge $c_{IR}$. But now the theory is conformal all the way. So, adding the dilaton must give an additional central charge $\Delta c = c_{UV} - c_{IR}$.
In curved space the modification becomes

$$\lambda \int \Phi(r) e^{(x_\Phi - 2)\tau} \sqrt{g} d^2 r$$

and scale invariance now shows up as invariance under Weyl transformations of the metric:

$$g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu}, \qquad \tau \to \tau + \sigma$$

But we know that this should not be completely invariant, there should be an additional anomaly $\Theta = -(\Delta c/12)R$. This means that there must be a term in the action to reproduce this, such that

$$\delta S_{\text{anom}}/\delta\sigma = (\Delta c/24\pi)R$$

A first guess would be

$$S_{\text{anom}} = (\Delta c)/24\pi) \int \tau R \sqrt{g} d^2 r$$

but this isn't quite right because $R$ changes under a Weyl transformation: in fact $\delta R \propto \partial^2 \sigma$. The correct form is

$$S_{\text{anom}} = (\Delta c/24\pi) \int (\tau R + (\partial\tau)^2) \sqrt{g} d^2 r$$

The remarkable thing is that this extra term survives when we go back to flat space, and its coefficient is fixed by $\Delta c$.
Let's see where this term arises in flat space if we integrate out the other massive fields in the infrared. We saw that $\tau$ acts as a source for $\Theta$. Thus

$$\langle e^{\int \tau \Theta d^2 r} \rangle = 1 + \tfrac{1}{2} \int \int \tau(r)\tau(r') \langle \Theta(r)\Theta(r') \rangle d^2 r d^2 r' + \cdots$$

If we now make a gradient expansion of the $\tau$-field, we see that the $(\partial\tau)^2$ term couples to $\int r^2 \langle \Theta(r)\Theta(0) \rangle d^2 r$. Thus we find, after inserting all the factors, that

$$\Delta c = (3/4\pi) \int r^2 \langle \Theta(r)\Theta(0) \rangle d^2 r \geq 0$$

as before.
The useful thing about this approach is that it may be generalised to higher (even) dimensions. For example, in $d = 4$ the coupling to the dilaton is

$$\lambda \int \Phi(r) e^{(x_\phi - 4)\tau} \sqrt{g} d^4 r$$

However in 4-dimensional curved space there are two possible independent local invariants with the right dimensions to contribute to $\langle \Theta \rangle$:

$$\langle \Theta \rangle = cW^2 - aE_4$$

where

$$
\begin{aligned}
W^2 &= W_{\mu\nu\lambda\sigma}W^{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \tfrac{1}{3}R^2 \qquad \text{(Weyl tensor)}^2 \\
E_4 &= R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \qquad \text{(Euler density)}
\end{aligned}
$$

The anomalous part of the action therefore starts off with

$$S_{\text{anom}} = \int \tau(\Delta c W^2 - \Delta a E_4)\sqrt{g}d^4r + \cdots$$

But $E_4$ is not Weyl invariant (although $W^2$ is). The extra terms which need to be added are (up 4 derivatives)[1]

$$-\Delta a \int \left[4(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R)\partial_\mu\tau\partial_\nu\tau - 2(\partial\tau)^4\right]\sqrt{g}d^4r$$

In addition there is a non-universal kinetic term

$$f^2 \int e^{-2\tau}(\partial\tau)^2\sqrt{g}d^4r$$

where $f$ has the dimensions of mass. Once again we see that a term $\propto \Delta a \int(\partial\tau)^4 d^4r$ remains when we go back to flat space. However it appears difficult to relate this to the 4-point function of $\Theta$ in Euclidean space. Instead Komargodski and Schwimmer argued that in Minkowski space this gives the low-energy elastic dilaton-dilaton scattering amplitude

$$\mathcal{A}(s,t,u) \sim \frac{\Delta a}{f^4}(s^2 + t^2 + u^2)$$

and then argued, on the basis of dispersion relations, that $\Delta a \geq 0$, so that

$$a_{UV} \geq a_{IR}$$

Thus for $d = 4$ there is an $a$-theorem but not necessarily a $c$-theorem (and indeed there are counter examples to the latter.)

[1] We have also ignored terms which vanish by the equation of motion $\partial^2\tau \propto (\partial\tau)^2$.