Boundary Conformal Field Theory*

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Boundary conformal field theory (BCFT) is simply the study of conformal field theory (CFT) in domains with a boundary. It gains its significance because, in some ways, it is mathematically simpler: the algebraic and geometric structures of CFT appear in a more straightforward manner; and because it has important applications: in string theory in the physics of open strings and D-branes, and in condensed matter physics in boundary critical behavior and quantum impurity models.

In this article, however, I describe the basic ideas from the point of view of quantum field theory, without regard to particular applications nor to any deeper mathematical formulations.

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1 Review of CFT

1.1 Stress tensor and Ward identities

Two-dimensional CFTs are massless, local, relativistic renormalized quantum field theories. Usually they are considered in imaginary time, i.e. on two-dimensional manifolds with euclidean signature. In this article the metric is also taken to be euclidean, although the formulation of CFTs on general Riemann surfaces is also of great interest, especially for string theory. For the time being the domain is the entire complex plane.

Heuristically the correlation functions of such a field theory may be thought of as being given by the euclidean path integral, that is, as expectation values of products of local densities with respect to a Gibbs measure $Z^{-1}\,e^{-S_E(\{\psi\})}[d\psi]$, where the $\{\psi(x)\}$ are some set of fundamental local fields, S_E is the euclidean action, and the normalization factor Z is the partition function. Of course, such an object is not in general well-defined, and this picture should be seen only as a guide to formulating the basic principles of CFT which can then be developed into a mathematically consistent theory.

In two dimensions, it is useful to use so-called complex coordinates $z = x^1 + ix^2$, $\bar z = x^1 - ix^2$. In CFT there are local densities $\phi_j(z,\bar z)$, called primary fields, whose correlation functions transform covariantly under conformal mappings $z \to z' = f(z)$:

$$\langle\phi_1(z_1,\bar z_1)\phi_2(z_2,\bar z_2)\ldots\rangle = \prod_i f'(z_j)^{h_j}\overline{f}'(z_j)^{\bar h_j}\langle\phi_1(z_1',\bar z_1')\phi_2(z_2',\bar z_2')\ldots\rangle\,, \quad (1)$$

where $(h_j,\bar h_j)$ (usually real numbers, not complex conjugates of each other) are called the *conformal weights* of ϕ_j. These local fields can in general be normalized so that their two-point functions have the form

$$\langle\phi_j(z_j,\bar z_j)\phi_k(z_k,\bar z_k)\rangle = \delta_{jk}/(z_j - z_k)^{2h_j}(\bar z_j - \bar z_k)^{2\bar h_j}\,. \quad (2)$$

They satisfy an algebra known as the operator product expansion (OPE)

$$\phi_i(z_1,\bar z_1)\cdot\phi_j(z_2,\bar z_2) = \sum_k c_{ijk}(z_1-z_2)^{-h_i-h_j+h_k}(\bar z_1-\bar z_2)^{-\bar h_i-\bar h_j+\bar h_k}\phi_k(z_1,\bar z_1)+\cdots\,, \quad (3)$$

which is supposed to be valid when inserted into higher-order correlation functions in the limit when $|z_1 - z_2|$ is much less than the separations of all the other points. The ellipses denote the contributions of other non-primary scaling fields to be described below. The structure constants c_{ijk}, along with the conformal weights, characterize the particular CFT.

An essential role is played by the energy-momentum tensor, or, in euclidean field theory language, the stress tensor $T^{\mu\nu}$. Heuristically, it is defined as the response of the partition function to a local change in the metric:

$$T^{\mu\nu}(x) = -(2\pi)\,\delta \ln Z/\delta g_{\mu\nu}(x) \tag{4}$$

(the factor of 2π is included so that similar factors disappear in later equations).

The symmetry of the theory under translations and rotations implies that $T^{\mu\nu}$ is conserved, $\partial_\mu T^{\mu\nu} = 0$, and symmetric. Scale invariance implies that it is also traceless $\Theta \equiv T^\mu_\mu = 0$. It should be noted that the vanishing of the trace of the stress tensor for a scale invariant classical field theory does not usually survive when quantum corrections are taken into account: indeed $\Theta \propto \beta(g)$, the renormalization group (RG) beta-function. A quantum field theory is thus only a CFT when this vanishes, that is at an RG fixed point.

In complex coordinates the components $T_{z\bar z} = T_{\bar z z} = 4\Theta$ vanish, while the conservation equations read

$$\partial_{\bar z} T_{zz} = \partial_z T_{\bar z\bar z} = 0\,. \tag{5}$$

Thus correlators of $T(z) \equiv T_{zz}$ are locally analytic (in fact, globally meromorphic) functions of z, while those of $\overline{T}(\bar z) \equiv T_{\bar z\bar z}$ are anti-analytic. It is this property of analyticity which makes CFT tractable in two dimensions.

Since an infinitesimal conformal transformation $z \to z + \alpha(z)$ induces a change in the metric, its effect on a correlation function of primary fields, given by (1), may also be expressed through an appropriate integral involving an insertion of the stress tensor. This leads to the *conformal Ward identity:*

$$\int_C \langle T(z) \prod_j \phi_j(z_j, \bar z_j)\rangle\, \alpha(z) dz = \sum_j \left(h_j \alpha'(z_j) + \alpha(z_j)(\partial/\partial z_j)\right) \langle \prod_j \phi_j(z_j,\bar z_j)\rangle\,, \tag{6}$$

where C is a contour encircling all the points $\{z_j\}$. (A similar equation hold for the insertion of $\overline{T}$.) Using Cauchy's theorem, this determines the first few terms in the OPE of T with any primary density:

$$T(z)\cdot\phi_j(z_j,\bar z_j) = \frac{h_j}{(z-z_j)^2}\phi(z_j,\bar z_j) + \frac{1}{z-z_j}\partial_{z_j}\phi(z_j,\bar z_j) + O(1)\,. \tag{7}$$

The other, regular, terms in the OPE generate new scaling fields, which are not in general primary, called descendants. One way of defining a density to be primary is by the condition that the most singular term in its OPE with T is a double pole.

The OPE of T with itself has the form

$$T(z)\cdot T(z_1)=\frac{c/2}{(z-z_1)^4}+\frac{2}{(z-z_1)^2}T(z_1)+\cdots\,. \tag{8}$$

The first term is present because $\langle T(z)T(z_1)\rangle$ is non-vanishing, and must take the form shown, with c being some number (which cannot be scaled to unity, since the normalization of T is fixed by its definition) which is a property of the CFT. It is known as the *conformal anomaly number* or the *central charge*. This term implies that T is not itself primary. In fact under a finite conformal transformation $z\to z'=f(z)$

$$T(z)\to f'(z)^2T(z')+\tfrac{c}{12}\{z',z\}\,, \tag{9}$$

where $\{z',z\}=(f'''f'-\frac{3}{2}f''^2)/f'^2$ is the Schwartzian derivative.

1.2 Virasoro algebra

As with any quantum field theory, the local fields can be realized as linear operators acting on a Hilbert space. In ordinary QFT, it is customary to quantize on a constant time hypersurface. The generator of infinitesimal time translations is the hamiltonian $\hat H$, which itself is independent of which time slice is chosen, because of time translational symmetry. It is also given by the integral over the hypersurface of the time-time component of the stress tensor. In CFT, because of scale invariance, one may instead quantize on fixed circle of a given radius. The analog of the hamiltonian is the dilatation operator $\hat D$, which generates scale transformations. Unlike $\hat H$, the spectrum of $\hat D$ is usually discrete, even in an infinite system. It may also be expressed as an integral over the radial component of the stress tensor

$$\hat D=\frac{1}{2\pi}\int_0^{2\pi}r\,\hat T_{rr}\,rd\theta=\frac{1}{2\pi i}\int_C z\,\hat T(z)dz-\frac{1}{2\pi i}\int_C\bar z\,\hat{\overline{T}}(\bar z)d\bar z\equiv\hat L_0+\hat{\overline{L}}_0\,, \tag{10}$$

where, because of analyticity, C can be any contour encircling the origin. This suggests that one define other operators

$$\hat L_n\equiv\frac{1}{2\pi}\int_C z^{n+1}\hat T(z)dz\,, \tag{11}$$

and similarly the $\hat{\bar{L}}_n$. From the OPE (8) then follows the Virasoro algebra $\mathcal{V}$

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}\,, \tag{12}$$

with an isomorphic algebra $\overline{\mathcal{V}}$ generated by the $\hat{\bar{L}}_n$.

In radial quantization there is a vacuum state $|0\rangle$. Acting on this with the operator corresponding to a scaling field gives a state $|\phi_j\rangle \equiv \hat{\phi}_j(0,0)|0\rangle$ which is an eigenstate of $\hat{D}$: in fact

$$\hat{L}_0|\phi_j\rangle = h_j|\phi_j\rangle\,, \qquad \hat{\bar{L}}_0|\phi_j\rangle = \bar{h}_j|\phi_j\rangle\,. \tag{13}$$

From the OPE (7) one sees that $|L_n\phi_j\rangle \propto \hat{L}_n|\phi_j\rangle$, and, if ϕ_j is primary, $\hat{L}_n|\phi_j\rangle = 0$ for all $n \geq 1$.

The states corresponding to a given primary field, and those generated by acting on these with all the $\hat{L}_n$ with $n < 0$ an arbitrary number of times, form a highest weight representation of $\mathcal{V}$. However, this is not necessarily irreducible. There may be *null vectors*, which are linear combinations of states at a given level which are themselves annihilated by all the $\hat{L}_n$ with $n > 0$. They exist whenever h takes a value from the Kac table:

$$h = h_{r,s} = \frac{(r(m+1)-sm)^2-1}{4m(m+1)}\,, \tag{14}$$

with the central charge parametrized as $c = 1 - 6/(m(m+1))$, and r, s are non-negative integers. These null states should be projected out, giving an irreducible representation $\mathcal{V}_h$.

The full Hilbert space of the CFT is then

$$\mathcal{H} = \bigoplus_{h,\bar{h}} n_{h,\bar{h}}\, \mathcal{V}_h \otimes \overline{\mathcal{V}}_{\bar{h}}\,, \tag{15}$$

where the non-negative integers $n_{h,\bar{h}}$ specify how many distinct primary fields of weights $(h,\bar{h})$ there are in the CFT.

The consistency of the OPE (3) with the existence of null vectors leads to the fusion algebra of the CFT. This applies separately to the holomorphic and antiholomorphic sectors, and determines how many copies of $\mathcal{V}_c$ occur in the fusion of $\mathcal{V}_a$ and $\mathcal{V}_b$:

$$\mathcal{V}_a \odot \mathcal{V}_b = \sum_c N_{ab}^c \mathcal{V}_c\,, \tag{16}$$

where the N_{ab}^c are non-negative integers.

A particularly important subset of all CFTs consists of the *minimal models*. These have rational central charge $c = 1 - 6(p-q)^2/pq$, in which case the fusion algebra closes with a finite number of possible values $1 \leq r \leq q$, $1 \leq s \leq p$ in the Kac formula (14). For these models, the fusion algebra takes the form

$$\mathcal{V}_{r_1,s_1} \odot \mathcal{V}_{r_2,s_2} = \sum_{r=|r_1-r_2|}^{r_1+r_2-1\,'} \sum_{s=|s_1-s_2|}^{s_1+s_2-1\,'} \mathcal{V}_{r,s}\,, \tag{17}$$

where the prime on the sums indicates that they are to be restricted to the allowed intervals of r and s.

There is an important theorem which states that the only *unitary* CFTs with $c < 1$ are the minimal models with $p/q = (m+1)/m$, where m is an integer ≥ 3.

1.3 Modular Invariance

The fusion algebra limits which values of $(h, \bar{h})$ might appear in a consistent CFT, but not which ones actually occur, i.e. the values of the $n_{h,\bar{h}}$. This is answered by the requirement of modular invariance on the torus. First consider the theory on an infinitely long cylinder, of unit circumference. This is related to the (punctured) plane by the conformal mapping $z \to (1/2\pi)\ln z \equiv t + ix$. The result is a QFT on the circle $0 \leq x < 1$, in imaginary time t. The generator of infinitesimal time translations is related to that for dilatations in the plane:

$$\hat{H} = 2\pi\hat{D} - \frac{\pi c}{6} = 2\pi(\hat{L}_0 + \hat{\bar{L}}_0) - \frac{\pi c}{6}\,, \tag{18}$$

where the last term comes from the Schwartzian derivative in (9). Similarly, the generator of translations in x, the total momentum operator, is $\hat{P} = 2\pi(\hat{L}_0 - \hat{\bar{L}}_0)$.

A general torus is, up to a scale transformation, a parallelogram with vertices $(0, 1, \tau, 1+\tau)$ in the complex plane, with the opposite edges identified. We can make this by taking a cylinder of unit circumference and length Im, τ, twisting the ends by a relative amount $\mathrm{Re}\,\tau$, and sewing them together. This

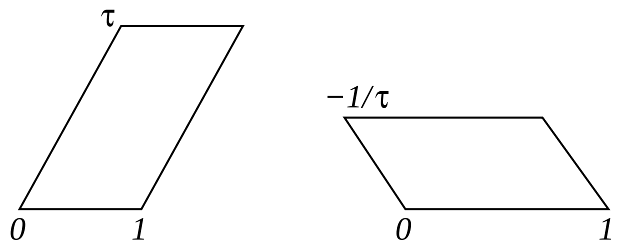

Figure 1: Two equivalent parametrizations of the same torus.

means that the partition function of the CFT on the torus can be written as

$$Z(\tau,\bar\tau) = \mathrm{Tr}\, e^{-(\mathrm{Im}\,\tau)\hat H + i(\mathrm{Im}\,\tau)\hat P} = \mathrm{Tr}\, q^{\hat L_0 - c/24}\, \bar q^{\hat{\bar L}_0 - c/24}\,, \tag{19}$$

using the above expressions for $\hat H$ and $\hat P$ and introducing $q \equiv e^{2\pi i\tau}$.

Through the decomposition (15) of $\mathcal{H}$, the trace sum can be written as

$$Z(\tau,\bar\tau) = \sum_{h,\bar h} n_{h,\bar h}\, \chi_h(q)\, \chi_{\bar h}(q)\,, \tag{20}$$

where

$$\chi_h(q) \equiv \mathrm{Tr}_{\mathcal{V}_h}\, q^{\hat L_0 - c/24} = \sum_N d_h(N)\, q^{h-(c/24)+N} \tag{21}$$

is the character of the representation of highest weight h, which counts the degeneracy $d_h(N)$ at level N. It is purely an algebraic property of the Virasoro algebra, and its explicit form is known in many cases.

All of this would be less interesting were it not for the observation that the parametrization of the torus through τ is not unique. In fact the transformations $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$ give the same torus (see Fig. 1). Together, these operations generate the modular group $\mathrm{SL}(2,\mathbf{Z})$, and the partition function $Z(\tau,\bar\tau)$ should be invariant under them. T-invariance is simply implemented by requiring that $h-\bar h$ is an integer, but the S-invariance of the right hand side of (20) places highly nontrivial constraints on the $n_{h,\bar h}$. That this can be satisfied at all relies on the remarkable property of the characters that they transform linearly under S:

$$\chi_h(e^{-2\pi i/\tau}) = \sum_{h'} S_h^{h'}\, \chi_{h'}(e^{2\pi i\tau})\,. \tag{22}$$

This follows from applying the Poisson sum formula to the explicit expressions for the characters, which are related to Jacobi theta-functions. In many cases (for example, the minimal models) this representation is finite-dimensional, and the matrix $\mathbf{S}$ is symmetric and orthogonal. This means that one can immediately obtain a modular invariant partition function by forming the diagonal sum

$$Z=\sum_h \chi_h(q)\chi_h(\bar q)\,, \tag{23}$$

so that $n_{h,\bar h}=\delta_{h\bar h}$. However, because of various symmetries of the characters, other modular invariants are possible: for the minimal models (and some others) these have been classified. Because of an analogy of the results with the classification of semisimple Lie algebras, the diagonal invariants are called the A-series.

2 Boundary CFT

In any field theory in a domain with a boundary, one needs to consider how to impose a set of consistent boundary conditions. Since CFT is formulated independently of a particular set of fundamental fields and a lagrangian, this must be done in a more general manner. A natural requirement is that the off-diagonal component $T_{\parallel\perp}$ of the stress tensor parallel/perpendicular to the boundary should vanish. This is called the conformal boundary condition. If the boundary is parallel to the time axis, it implies that there is no momentum flow across the boundary. Moreover, it can be argued that, under the RG, any uniform boundary condition will flow into a conformally invariant one. For a given bulk CFT, however, there may be many possible distinct such boundary conditions, and it is one task of BCFT to classify these.

To begin with, take the domain to be the upper half plane, so that the boundary is the real axis. The conformal boundary condition then implies that $T(z)=\overline{T}(\bar z)$ when z is on the real axis. This has the immediate consequence that correlators of $\overline{T}$ are those of T, analytically continued into the lower half plane. The conformal Ward identity, *c.f.* (7), now reads

$$\begin{aligned}\langle T(z)\prod_j\phi_j(z_j,\bar z_j)\rangle \;&=\; \sum_j\Bigg(\frac{h_j}{(z-z_j)^2}+\frac{1}{z-z_j}\partial_{z_j}\\ &\quad+\frac{\bar h_j}{(\bar z-\bar z_j)^2}+\frac{1}{\bar z-\bar z_j}\partial_{\bar z_j}\Bigg)\langle\prod_j\phi_j(z_j,\bar z_j)\rangle\,. \end{aligned}\tag{24}$$

In radial quantization, in order that the Hilbert spaces defined on different hypersurfaces be equivalent, one must choose semicircles centered on some point on the boundary, conventionally the origin. The dilatation operator is now

$$\hat{D} = \frac{1}{2\pi i}\int_S z\,\hat{T}(z)dz - \frac{1}{2\pi i}\int_S \bar{z}\,\hat{\overline{T}}(\bar{z})d\bar{z}\,, \tag{25}$$

where S is a semicircle. Using the conformal boundary condition, this can also be written as

$$\hat{D} = \hat{L}_0 = \frac{1}{2\pi i}\int_C z\,\hat{T}(z)dz\,, \tag{26}$$

where C is a complete circle around the origin. As before, one may similarly define the $\hat{L}_n$, and they satisfy a Virasoro algebra.

Note that there is now only one Virasoro algebra. This is related to the fact that conformal mappings which preserve the real axis correspond to real analytic functions. The eigenstates of $\hat{L}_0$ correspond to *boundary operators* $\hat{\phi}_j(0)$ acting on the vacuum state $|0\rangle$. It is well-known that in a renormalizable QFT operators at the boundary require a different renormalization from those in the bulk, and this will in general lead to a different set of conformal weights. It is one of the tasks of BCFT to determine these, for a given allowed boundary condition.

However, there is one feature unique to boundary CFT in two dimensions. Radial quantization also makes sense, leading to the same form (26) for the dilation operator, if the boundary conditions on the negative and positive real axes are different. As far as the structure of BCFT goes, correlation functions with this mixed boundary condition behave as though a local scaling field were inserted at the origin. This has led to the term 'boundary condition changing (bcc) operator', but it must be stressed that these are not local operators in the conventional sense.

3 The annulus partition function

Just as consideration of the partition function on the torus illuminates the bulk operator content $n_{h,\bar{h}}$, it turns out that consistency on the annulus helps classify both the allowed boundary conditions, and the boundary operator content. To this end, consider a CFT in an annulus formed of a rectangle of unit width and height δ, with the top and bottom edges identified (see

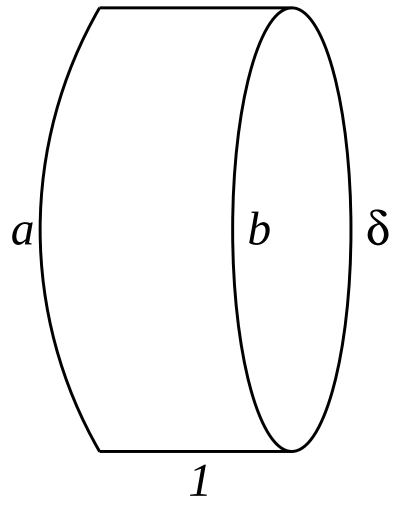

Figure 2: The annulus, with boundary conditions a and b on either boundary.

Fig. 2). The boundary conditions on the left and right edges, labelled by $a, b, \ldots$, may be different. The partition function with boundary conditions a and b on either edge is denoted by $Z_{ab}(\delta)$.

One way to compute this is by first considering the CFT on an infinitely long strip of unit width. This is conformally related to the upper half plane (with an insertion of boundary condition changing operators at 0 and ∞ if $a \neq b$) by the mapping $z \to (1/\pi)\ln z$. The generator of infinitesimal translations along the strip is

$$\hat{H}_{ab} = \pi\hat{D} - \pi c/24 = \pi\hat{L}_0 - \pi c/24\,. \tag{27}$$

Thus for the annulus

$$Z_{ab}(\delta) = \operatorname{Tr} e^{-\delta\,\hat{H}_{ab}} = \operatorname{Tr} q^{\hat{L}_0 - \pi c/24}\,, \tag{28}$$

with $q \equiv e^{-\pi\delta}$. As before, this can be decomposed into characters

$$Z_{ab}(\delta) = \sum_h n_h^{ab}\,\chi_h(q)\,, \tag{29}$$

but note that now the expression is linear. The non-negative integers n_{ab}^h give the operator content with the boundary conditions (ab): the lowest value of h with $n_{ab}^h > 0$ gives the conformal weight of the bcc operator, and the others give conformal weights of the other allowed primary fields which may also sit at this point.

On the other hand, the annulus partition function may be viewed, up to an overall rescaling, as the path integral for a CFT on a circle of unit circumference, being propagated for (imaginary) time δ^{-1}. From this point of view, the partition function is no longer a trace, but rather the matrix element of $e^{-\hat{H}/\delta}$ between *boundary states*:

$$Z_{ab}(\delta) = \langle a|e^{-\hat{H}/\delta}|b\rangle\,. \tag{30}$$

Note that $\hat{H}$ is the same hamiltonian that appears in (18), and the boundary states lie in $\mathcal{H}$, (15).

How are these boundary states to be characterized? Using the transformation law (9) the conformal boundary condition applied to the circle implies that $L_n = \overline{L}_{-n}$. This means that any boundary state $|B\rangle$ lies in the subspace satisfying

$$\hat{L}_n|B\rangle = \hat{\overline{L}}_{-n}|B\rangle\,. \tag{31}$$

Moreover, because of the decomposition (15) of $\mathcal{H}$, $|B\rangle$ is also some linear superposition of states from $\mathcal{V}_h\otimes\overline{\mathcal{V}}_{\bar{h}}$. This condition can therefore be applied in each subspace. Taking $n=0$ in (31) constrains $\bar{h}=h$. For simplicity, consider only the diagonal CFTs with $n_{h,\bar{h}}=\delta_{h,\bar{h}}$. It can then be shown that the solution of (31) is unique and has the following form. The subspace at level N of $\mathcal{V}_h$ has dimension $d_h(N)$. Denote an orthonormal basis by $|h,N;j\rangle$, with $1\leq j\leq d_h(N)$, and the same basis for $\overline{\mathcal{V}}_h$ by $\overline{|h,N;j\rangle}$. The solution to (31) in this subspace is then

$$|h\rangle\rangle \equiv \sum_{N=0}^{\infty}\sum_{j=1}^{d_h(N)} |h,N;j\rangle\otimes\overline{|h,N;j\rangle}\,. \tag{32}$$

These are called Ishibashi states. Matrix elements of the translation operator along the cylinder between them are simple:

$$\langle\langle h'|e^{-\hat{H}/\delta}|h\rangle\rangle \tag{33}$$

$$= \sum_{N'=0}^{\infty}\sum_{j'=1}^{d_{h'}(N')}\sum_{N=0}^{\infty}\sum_{j=1}^{d_h(N)} \langle h',N';j'|\otimes\overline{\langle h',N';j'|}e^{-(2\pi/\delta)(\hat{L}_0+\hat{\overline{L}}_0-c/12)}$$

$$|h,N;j\rangle\otimes\overline{|h,N;j\rangle} \tag{34}$$

$$= \delta_{h'h} \sum_{N=0}^{\infty} \sum_{j=1}^{d_h(N)} e^{-(4\pi/\delta)\left(h+N-(c/24)\right)} = \delta_{h'h}\, \chi_h\left(e^{-4\pi/\delta}\right). \tag{35}$$

Note that the characters which appear are related to those in (29) by the modular transformation S.

The physical boundary states satisfying (29), sometimes called the Cardy states, are linear combinations of the Ishibashi states:

$$|a\rangle = \sum_h \langle\langle h|a\rangle\, |h\rangle\rangle\,. \tag{36}$$

Equating the two different expressions (29,30) for Z_{ab}, and using the modular transformation law (22) and the linear independence of the characters gives the (equivalent) conditions:

$$n_{ab}^h = \sum_{h'} S_{h'}^h \langle a|h'\rangle\rangle\langle\langle h'|b\rangle\,; \tag{37}$$

$$\langle a|h'\rangle\rangle\langle\langle h'|b\rangle = \sum_h S_h^{h'} n_{ab}^h\,. \tag{38}$$

These are called the Cardy conditions. The requirements that the right hand side of (37) should give a non-negative integer, and that the right hand side of (38) should factorize in a and b, give highly nontrivial constraints on the allowed boundary states and their operator content.

For the diagonal CFTs considered here (and for the nondiagonal minimal models) a complete solution is possible. It can be shown that the elements S_0^h of $\mathbf{S}$ are all non-negative, so one may choose $\langle\langle h|\tilde{0}\rangle = (S_0^h)^{1/2}$. This defines a boundary state

$$|\tilde{0}\rangle \equiv \sum_h (S_0^h)^{1/2} |h\rangle\rangle\,, \tag{39}$$

and a corresponding boundary condition such that $n_{00}^h = \delta_{h0}$. Then, for each $h' \neq 0$, one may define a boundary state

$$\langle\langle h|\tilde{h'}\rangle \equiv S_{h'}^h/(S_0^h)^{1/2}\,. \tag{40}$$

From (37), this gives $n_{h'0}^h = \delta_{h'h}$. For each allowed h' in the torus partition function, there is therefore a boundary state $|\tilde{h'}\rangle$ satisfying the Cardy conditions. However, there is a further requirement:

$$n_{h'h''}^h = \sum_\ell \frac{S_\ell^h S_{h'}^\ell S_{h''}^\ell}{S_0^\ell} \tag{41}$$

should be a non-negative integer. Remarkably, this combination of elements of $\mathbf{S}$ occurs in the Verlinde formula, which follows from considering consistency of the CFT on the torus. This states that the right hand side of (41) is equal to the fusion algebra coefficient $N^h_{h'h''}$. Since these are non-negative integers, the consistency of the above ansatz for the boundary states is consistent.

We conclude that, at least for the diagonal models, there is a bijection between the allowed primary fields in the bulk CFT and the allowed conformally invariant boundary conditions. For the minimal models, with a finite number of such primary fields, this correspondence has been followed through explicitly.

3.0.1 Example

The simplest example is the diagonal $c = \frac{1}{2}$ unitary CFT corresponding to $m = 3$. The allowed values of the conformal weights are $h = 0, \frac{1}{2}\frac{1}{16}$, and

$$\mathbf{S} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \tag{42}$$

from which one finds the allowed boundary states

$$|\tilde{0}\rangle = \tfrac{1}{\sqrt{2}}|0\rangle\rangle + \tfrac{1}{\sqrt{2}}|\tfrac{1}{2}\rangle\rangle + \tfrac{1}{2^{1/4}}|\tfrac{1}{16}\rangle\rangle\,; \tag{43}$$

$$|\tilde{\tfrac{1}{2}}\rangle = \tfrac{1}{\sqrt{2}}|0\rangle\rangle + \tfrac{1}{\sqrt{2}}|\tfrac{1}{2}\rangle\rangle - \tfrac{1}{2^{1/4}}|\tfrac{1}{16}\rangle\rangle\,; \tag{44}$$

$$|\tilde{\tfrac{1}{16}}\rangle = |0\rangle\rangle - |\tfrac{1}{2}\rangle\rangle\,. \tag{45}$$

The nontrivial part of the fusion algebra of this CFT is

$$\mathcal{V}_{\frac{1}{16}} \odot \mathcal{V}_{\frac{1}{16}} = \mathcal{V}_0 + \mathcal{V}_{\frac{1}{2}} \tag{46}$$

$$\mathcal{V}_{\frac{1}{16}} \odot \mathcal{V}_{\frac{1}{2}} = \mathcal{V}_{\frac{1}{16}} \tag{47}$$

$$\mathcal{V}_{\frac{1}{2}} \odot \mathcal{V}_{\frac{1}{2}} = \mathcal{V}_0\,, \tag{48}$$

$$\tag{49}$$

from which can be read off the boundary operator content $n^h_{\tilde{h}} = 1$ and $n^0_{\tilde{\frac{1}{16}}\tilde{\frac{1}{16}}} = n^{\frac{1}{2}}_{\tilde{\frac{1}{16}}\tilde{\frac{1}{16}}} = n^{\frac{1}{2}}_{\tilde{\frac{1}{16}}\tilde{\frac{1}{16}}} = n^{\frac{1}{16}}_{\tilde{\frac{1}{2}}\tilde{\frac{1}{16}}} = 1$.

The $c = \frac{1}{2}$ CFT is known to describe the continuum limit of the critical Ising model, in which spins $s = \pm 1$ are localized on the sites of a regular lattice. The above boundary conditions may be interpreted as the continuum limit of the lattice boundary conditions $s = 1$, free and $s = -1$ respectively. Note there is a symmetry of the fusion rules which means that one could equally well have inverted the ordering of this correspondence.

4 Other topics

4.1 Boundary entropy

The partition function on annulus of length L and circumference β can be thought of as the quantum statistical mechanics partition function for a 1d QFT in an interval of length L, at temperature β^{-1}. It is interesting to consider this in the thermodynamic limit when $\delta = L/\beta$ is large. In that case, only the ground state of $\hat{H}$ contributes in (30), giving

$$Z_{ab}(L,\beta) \sim \langle a|0\rangle\langle 0|b\rangle\, e^{\pi cL/6\beta}\,, \tag{50}$$

from which the free energy $F_{ab} = -\beta^{-1}\ln Z_{ab}$ and the entropy $\mathcal{S}_{ab} = -\beta^2(\partial F_{ab}/\partial\beta)$ can be obtained. The result is

$$\mathcal{S}_{ab} = (\pi c/3\beta)L + s_a + s_b + o(1)\,, \tag{51}$$

where the first term is the usual extensive contribution. The other two pieces $s_a \equiv \ln(\langle a|0\rangle)$ and $s_b \equiv \ln(\langle b|0\rangle)$ may be identified as the *boundary entropy* associated with the corresponding boundary states. A similar definition may be made in massive QFTs. It has been shown that, analogously to the statement of Zamoldochikov's c-theorem in the bulk, the boundary entropy is a non-increasing function along boundary RG flows, and is stationary only for conformal boundary states.

4.2 Bulk-boundary OPE

The boundary Ward identity (24) has the implication that, from the point of view of the dependence of its correlators on z_j and $\bar{z}_j$, a primary field

$\phi_j(z_j, \bar{z}_j)$ may be thought of as the product of two local fields which are holomorphic functions of z_j and $\bar{z}_j$ respectively. These will satisfy OPEs as $|z_j - \bar{z}_j| \to 0$, with the appearance of primary fields on the right hand side being governed by the fusion rules. These fields are localized on the real axis: they are the boundary operators. There is therefore a kind of bulk-boundary OPE:

$$\phi_j(z_j, \bar{z}_j) = \sum_k d_{jk} (\operatorname{Im} z_j)^{-h_j - \bar{h}_j + h_k} \phi_k^b(\operatorname{Re} z_j)\,, \tag{52}$$

where the sum on the right hand side is in principle over all the boundary fields consistent with the boundary condition, and the coefficients d_{jk} are analogous to the OPE coefficients in the bulk. As before, they are non-vanishing only if allowed by the fusion algebra: a boundary field of conformal weight h_k is allowed only if $N_{h_j \bar{h}_j}^{h_k} > 0$.

For example, in the $c = \frac{1}{2}$ CFT, the bulk operator with $h = \bar{h} = \frac{1}{16}$ goes over into the boundary operator with $h = 0$, or that with $h = \frac{1}{2}$, depending on the boundary condition. The bulk operator with $h = \bar{h} = \frac{1}{2}$, however, can only go over into the identity boundary operator with $h = 0$ (or a descendent thereof.)

The fusion rules also apply to the boundary operators themselves. The consistency of these with bulk-boundary and bulk-bulk fusion rules, as well as the modular properties of partition functions, was examined by Lewellen.

4.3 Extended algebras

CFTs may contain other conserved currents apart from the stress tensor, which generate algebras (Kac-Moody, superconformal, W-algebras) which extend the Virasoro algebra. In BCFT, in addition to the conformal boundary condition, it is possible (but not necessary) to impose further boundary conditions relating the holomorphic and antiholomorphic parts of the other currents on the boundary. It is believed that all rational CFTs can be obtained from Kac-Moody algebras via the coset construction. The classification of boundary conditions from this point of view is fruitful and also important for applications, but is beyond the scope of this article.

4.4 Stochastic Loewner evolution

In recent years, there has emerged a deep connection between BCFT and conformally invariant measures on curves in the plane which start at a boundary of a domain. These arise naturally in the continuum limit of certain statistical mechanics models. The measure is constructed dynamically as the curve is extended, using a sequence of random conformal mappings called stochastic Loewner evolution (SLE). In CFT, the point where the curve begins can be viewed as the insertion of a boundary operator. The requirement that certain quantities should be conserved in mean under the stochastic process is then equivalent to this operator having a null state at level two. Many of the standard results of CFT correspond to an equivalent property of SLE.

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References

[1] P. di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory* (Springer, 1999.)

[2] J. Cardy, *Conformal invariance and surface critical behavior*, Nucl. Phys. B **240**, 514-532, 1984.

[3] J. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, Nucl. Phys. B **324**, 581, 1989.

[4] E. Verlinde, *Fusion rules and modular transformations in 2D conformal field theory*, Nucl. Phys. B **300**, 360, 1988.

[5] V.B. Petkova and J.-B. Zuber, *Conformal boundary conditions and what they teach us*, lectures given at the Summer School and Conference on

Nonperturbative Quantum Field Theoretic Methods and their Applications, August 2000, Budapest, Hungary, hep-th/0103007.

[6] I. Affleck, *Boundary condition changing operators in conformal field theory and condensed matter physics*, Nucl.Phys.Proc.Suppl. **58**, 35, 1997.

[7] D. Friedan, A. Konechny, *On the boundary entropy of one-dimensional quantum systems at low temperature*, Phys. Rev. Lett. **93**, 030402, 2004.

[8] D. C. Lewellen, *Sewing constraints for conformal field theories on surfaces with boundaries*, Nucl. Phys. B **372**, 654, 1992.

[9] For reviews of SLE, see W. Werner, *Random planar curves and Schramm-Loewner evolutions*, to appear (Springer Lecture Notes), math.PR/0303354; G. Lawler, *Conformally invariant processes in the plane*, in preparation, http://www.math.cornell.edu/˜lawler/book.ps. For a review aimed at theoretical physicists, see W. Kager and B. Nienhuis, *A guide to stochastic Loewner evolution and its applications*, J. Stat. Phys. **115**, 1149 (2004).