Functions of a Complex Variable: Theory and Applications

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Books No book covers the material of this course at just the right level. The basic concepts are covered in several books on "Mathematics for Physicists & Engineers", for example that of Riley. Three higher-level references are: *Introduction to Complex Analysis*, by H. A. Priestley (OUP); *Complex Variables & Applications*, by R. V. Churchill, J. W. Brown & R. F. Verhay (ISC); *Complex Variables with Applications* by A. D. Wunsch (Addison-Wesley).

Conventions: In these notes, the symbol $\exists$ means "there exists", the symbol $\forall$ means "for all" and the symbol $\in$ means "in".

1 Analyticity

This course is about complex-valued functions of a complex varable. But we won't be dealing with all possible kinds of function. Consider, for example, the following two very different functions:

(i)

$$f(z) = \frac{z-2}{z-1} \quad \text{(not defined at } z = 1\text{);}$$

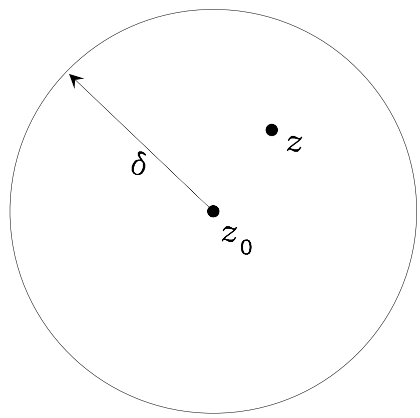
(ii)

$$f(z) = \begin{cases} 0 & \text{if } \Re\mathrm{e}(z)/\Im\mathrm{m}(z) = \dfrac{n}{m} \text{ for integer } n, m; \\ 1 & \text{if } \Re\mathrm{e}(z)/\Im\mathrm{m}(z) = \text{irrational} \end{cases}$$

We wish to exclude functions of the second, jumpy kind. To this end first define the **limit** of a complex function:

Def:

$\lim_{z \to z_0} f(z) = f_0 \Leftrightarrow$ given any $\epsilon > 0$, $\exists\ \delta > 0$ s.t. $|z - z_0| < \delta$ $\Rightarrow |f(z) - f_0| < \epsilon$.



Now we define **continuity**:

Def:

$f(z)$ is continuous at $z = z_0$ iff

(i) $f(z)$ is defined in a neighbourhood of z_0;

(ii) $\lim_{z \to z_0} f(z)$ exists and equals $f(z_0)$.

Def:

f is continuous in a domain D iff f is continuous at all $z \in D$.

1.2 Simple consequences of continuity

Theorem 1: *A continuous function of a continuous function is continuous.*

Theorem 2: *A rational function [e.g. $(az+b)/(cz+d)$] is continuous except at its singularities [e.g. where $cz + d = 0$].*

Example:

$(z-2)/(z-1)$ is not continuous at $z = 1$, or in any domain that contains $z = 1$, but is continuous in any domain that excludes $z = 1$.

1.3 Behaviour at infinity

Def:

$$\lim_{z\to\infty} f(z) = f_0 \quad \text{iff} \quad \lim_{z\to 0} f(1/z) = f_0.$$

It proves helpful to add an additional point "∞" to the complex plane. The location of this point is established by specfying its neighbourhoods:

Def:

For any $\delta \geq 0$, the set of points satisfying $|z| > \delta$ forms a neighbourhood of ∞.

Note:

With ∞ added, the complex "plane" is actually a *sphere* called the **extended complex plane** or **Gauss plane**; see Appendix A for details. In order to make some functions, for example $z^{1/2}$ or $\ln z$ into single-valued functions of z, one has to cut and sew this sphere into some very odd surfaces. We shall try to avoid these pathologies, some examples of which may be examined in Appendix A.

1.4 Differentiability

Def:

$$f \text{ is } \textbf{analytic} \text{ at } z_0 \text{ iff } f'(z_0) \equiv \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \quad \text{exists.} \qquad (\ddagger)$$

Notes:

(i) A real function is called analytic at x_0 iff it has Taylor series around x_0 that is convergent for sufficiently small $|x - x_0|$. We shall see that the *apparently* weak condition (‡) ensures that f has a convergent Taylor series too.

(ii) Analytic functions are sometimes called **holomorphic** or **monogenic**. The name monogenic says that these functions beget unique derivative functions. We shall see later why these are holomorphic, i.e. "whole-shape" functions.

Def:

f is analytic in a domain D if it is analytic at each $z \in D$. The limit (‡) then defines the **derivative function** $f'(z)$.

Theorem 3: *If f is analytic at z_0, then it is continuous at z_0.*

Proof: Let

$$\lim_{z\to z_0}\left[\frac{f(z)-f(z_0)}{z-z_0}\right] = f' \quad \text{and consider}$$

$$g(z) \equiv \frac{f(z)-f(z_0)}{z-z_0} - f'.$$

Clearly,

$$\lim_{z\to z_0} g(z) = 0 \quad \text{and} \quad 0 = \lim_{z\to z_0}\{[f'+g(z)](z-z_0)\} = \lim_{z\to z_0}[f(z)-f(z_0)],$$

so $\lim_{z\to z_0} f(z) = f(z_0)$ as required.◁

Theorem 4: *If f and g are analytic in D, then so are $f+g$ and fg. Furthermore, f/g is analytic except where $g = 0$.*

Proof: (For a typical part.)

$$fg(z) - fg(z_0) = g(z)[f(z) - f(z_0)] + f(z_0)[g(z) - g(z_0)].$$

So

$$\lim_{z\to z_0}\Big[\frac{fg(z) - fg(z_0)}{z - z_0}\Big] = \lim_{z\to z_0} g(z) \times \lim_{z\to z_0}\Big[\frac{f(z) - f(z_0)}{z - z_0}\Big] + \lim_{z\to z_0} f(z_0) \times \lim_{z\to z_0}\Big[\frac{g(z) - g(z_0)}{z - z_0}\Big] \quad \text{exists.}\triangleleft$$

A necessary *but not sufficient* condition for f to be analytic at z_0 is that $\lim_{r\to 0}\{[f(z_0 + re^{i\theta}) - f(z_0)]/re^{i\theta}\}$ be independent of θ. This condition leads to:

Theorem 5: *If $f(z) \equiv u(x, y) + iv(x, y)$ is analytic in D, then u and v are differentiable and*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \qquad \text{(C-R)}$$

Proof: Writing $re^{i\theta} = z - z_0$ we have

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{e^{-i\theta}}{r}\big\{[u(x_0 + r\cos\theta, y_0 + r\sin\theta) - u(x_0, y_0)] + i[v(x_0 + r\cos\theta, y_0 + r\sin\theta) - v(x_0, y_0)]\big\}.$$

Now taking $\theta = 0$ and letting $r \to 0$, we find

$$f'(z_0) = \lim_{r\to 0}\frac{1}{r}\big\{[u(x_0 + r, y_0) - u(x_0, y_0)] + i[v(x_0 + r, y_0) - v(x_0, y_0)]\big\}.$$

It follows that the real and imaginary parts of this limit separately tend to definite limits, which we recognize to be $\partial u/\partial x$ and $\partial v/\partial x$. Thus

$$f'(z_0) = \Big[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big]_{(x_0,y_0)}.$$

Similarly, if we set $\theta = \frac{\pi}{2}$, we obtain

$$f'(z_0) = -i\Big[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\Big]_{(x_0,y_0)} = \Big[\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\Big]_{(x_0,y_0)}.$$

Equating the real and imaginary parts of these expressions we obtain the Cauchy-Riemann conditions (C-R).◁

Note:

The following counterexample shows that satisfaction of the condition that $\lim_{r\to 0}\{[f(z_0 + re^{i\theta}) - f(z_0)]/re^{i\theta}\}$ be idependent of θ does not guarantee that f is analytic:

$$\text{Let } f(z) = \begin{cases} \dfrac{xy^2(x + iy)}{x^2 + y^4} & \text{for } z \neq 0; \\ 0 & \text{for } z = 0 \end{cases}.$$

Then holding $t \equiv \tan\theta$ constant as $z \to 0$, we have

$$\frac{f(z) - f(0)}{z} = \frac{x^4t^2(1 + it)}{x^2 + t^4x^4}\Big/(x + itx) \to 0 \quad \text{for all } t.$$

But if we send $z \to 0$ along the porabola $x = y^2$ we find

$$\frac{f(z) - f(0)}{z} = \frac{y^4(y^2 + iy)}{2y^4(y^2 + iy)} \to \frac{1}{2},$$

so $f'(0)$ does not exist.

Examination of $\partial u/\partial x$ and $\partial v/\partial y$ for the function of the last example shows that although the C-R conditions are satisfied at $z = 0$, they are not satisfied in a neighbourhood of 0. This is at the root of the difficulty, for we have:

Theorem 6: *The necessary and sufficient condition that $f = u + iv$ be analytic in a domain D is that in D u and v have continuous partial derivatives $u_x \equiv \partial u/\partial x$ etc which satisfy the C-R conditions.*

Proof: A theorem from the theory of functions of real variables states that when a function u has continuous partial derivatives, then $\exists$ quantities $\xi_u(x_0, y_0, \Delta x, \Delta y)$ and $\eta_u(x_0, y_0, \Delta x, \Delta y)$ such that

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \xi_u \Delta x + \eta_u \Delta y,$$

and $\xi_u,\ \eta_u \to 0$ as $\Delta x,\ \Delta y \to 0$. Given this theorem it is straightforward to show that when the C-R conditions hold,

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_0) + \frac{(\xi_u + \xi_v)\Delta x + (\eta_u + \eta_v)\Delta y}{\Delta x + i\Delta y}.$$

Then one may go on to show that given any $\epsilon > 0$, it is possible to choose $\delta > 0$ such that $|\Delta z| < \delta$ implies that Δx and Δy are both small enough that the term in the last equation involving ξ and η symbols is smaller than ϵ. Hence the limit $\Delta z \to 0$ of the left of the equation exists and is equal to $u_x + iv_x$.◁

1.4 Common functions of z

Many common functions f of the real variable x are defined by giving a rule for calculating f through algebraic operations on x, e.g. $f(x) = x^2 - 1$, or $f(x) = e^x \equiv \sum_n x^n/n!$. Replacing x with the complex variable z we obtain from each such rule a complex function $f(z)$. If $f(x)$ is differentiable, this complex extension is analytic:

$$f(z) = u + iv \equiv f(x + iy) \quad \Rightarrow \quad \begin{array}{l} u_x + iv_x = \partial f/\partial x = f'(x + iy) \\ u_y + iv_y = \partial f/\partial y = f'(x + iy)i \end{array}$$

$$\Rightarrow \quad u_y + iv_y = i(u_x + iv_x)$$

Hence $u_x = v_y$, $u_y = -v_x$ and f is analytic.

Examples:

(i) $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$ from the series.

(ii) $e^{\ln z} = z = r(\cos\theta + i\sin\theta) = e^{\ln r}e^{i\theta} = e^{(\ln r + i\theta)} \Rightarrow \ln z = \ln r + i\theta$. Since we can always add $2m\pi$ to θ, $\ln z$ is infinitely many-valued; the Riemann surface (see Appendix A) required to make $\ln z$ single-valued is a spiral around $z = 0$. The **principal value** of $\ln z$ is defined to be that for which $-\pi < \theta \le \pi$.

(iii) $\cos(iz) = \cosh z$ and $\sin(iz) = i\sinh z$ from the series.

Note:

Many continuous functions of z, for example $f(z) = x + 2iy$, are not obtainable by substituting $z = x + iy$ into a real function. The following argument shows these functions are not analytic. Extend x and y to complex variables in their own right and change variables to $z_1 = x + iy$ and $z_2 = x - iy$. Then $x = \frac{1}{2}(z_1 + z_2)$, $y = \frac{1}{2i}(z_1 - z_2)$ and $f(z) = u(x, y) + iv(x, y)$ is a function of z_1 and z_2 with $\partial f/\partial z_2 \neq 0$. But by the chain rule

$$0 \neq \frac{\partial f}{\partial z_2} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial z_2} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial z_2} + i\Big(\frac{\partial v}{\partial x}\frac{\partial x}{\partial z_2} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial z_2}\Big)$$
$$= \tfrac{1}{2}\Big[\Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big) + i\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big)\Big].$$

Hence such a function must violate at least one of the C-R conditions, and it cannot be analytic.

Example:

Find the analytic function whose imaginary part is

$$v(x, y) = (y \cos x + x \sin x)e^{-y} \qquad \text{and vanishes at the origin.}$$

Elegant solution: e^{-y} must come from $e^{iz} = (\cos x + i \sin x)e^{-y}$. To get the desired y and x factors we multiply by z:

$$\Im\mathrm{m}(z)e^{iz} = (y \cos x + x \sin x)e^{-y} \quad \text{as required.}$$

Plodding solution: By the C-R conditions,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^{-y}(\cos x - y \cos x - x \sin x).$$

Integrating w.r.t. x we have

$$\begin{aligned} u &= e^{-y}[(1 - y) \sin x - (-x \cos x + \sin x)] + Y(y) \\ &= e^{-y}(x \cos x - y \sin x) + Y(y), \end{aligned}$$

where $Y(y)$ is some function of y. Similarly,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -e^{-y}(-y \sin x + \sin x + x \cos x).$$

Integrating w.r.t. y we now obtain $u = -e^{-y}[(1 + y) \sin x - (\sin x + x \cos x)] + X(x)$. Equating our two expressions for u, we see that $X = Y =$ constant. Since the required function vanishes at $z = 0$, the constant must be zero and $f = ze^{iz}$.

2 Solutions of $\nabla^2\Phi = 0$

The electrostatic forces around a charged body and certain kinds of incompressible fluid flow near solid bodies are governed by Laplace's equation $\nabla^2\Phi = 0$.[1] In general the operator ∇^2 involves all three spatial coordinates (x, y, z). Sometimes it is a sufficient approximation to neglect the gradient of Φ w.r.t. one of the coordinates, say z. We have

Theorem 7: *If $f = u + iv$ is analytic in D, then $\nabla^2 u = \nabla^2 v = 0$ in D, where* $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.
Proof: By the C-R conditions, $u_x = v_y$ and $u_y = -v_x$. Hence differentiating w.r.t x, $u_{xx} = v_{yx} = -u_{yy}$ and similarly for v_{xx}.$\triangleleft$

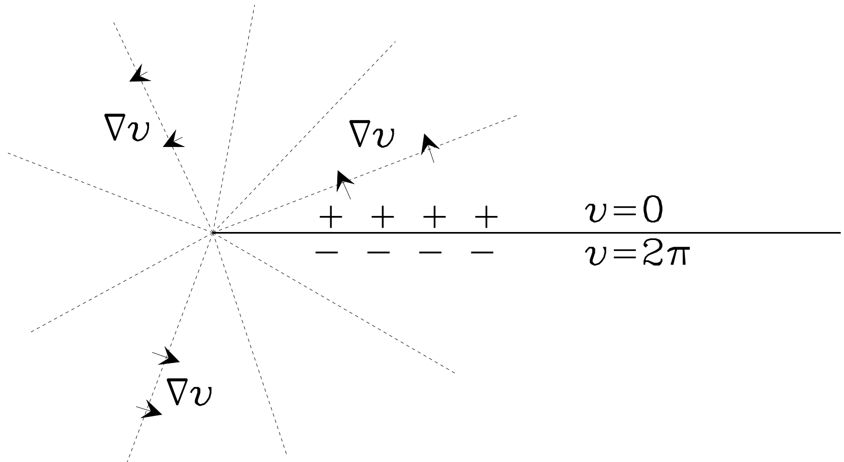
Thus there is an easy way of generating solutions to the two-dimensional Laplace equation: think of a real differentiable function, make it complex, extract the real and imaginary parts and you've got not one, but two solutions of $\nabla^2\Phi = 0$! The main difficulty is finding some physical problem to which to apply these solutions.

The nature of the solutions of $\nabla^2\Phi = 0$ derived from a complex function f is determined by the points at which f is *not* analytic, for these contain the key information about the system of charges (or walls etc) that generates the solution. Some examples will clarify this point.

[1] For an entertaining account of the flow of of "dry water" see Chapter 40 of the Feynman Lectures on Physics.

Examples:

(i) $f(z) = \ln z \quad \Rightarrow \quad u = \frac{1}{2}\ln(x^2 + y^2),\ v = \theta.$ u is the potential of a line of charge along $x = y = 0$, and v is the potential of an infinite polarized sheet $y = 0,\ x > 0$. The force-field of this strip is $\nabla v = \nabla \arctan(y/x) = (-y\hat{\mathbf{i}} + x\hat{\mathbf{j}})/(x^2 + y^2)$:

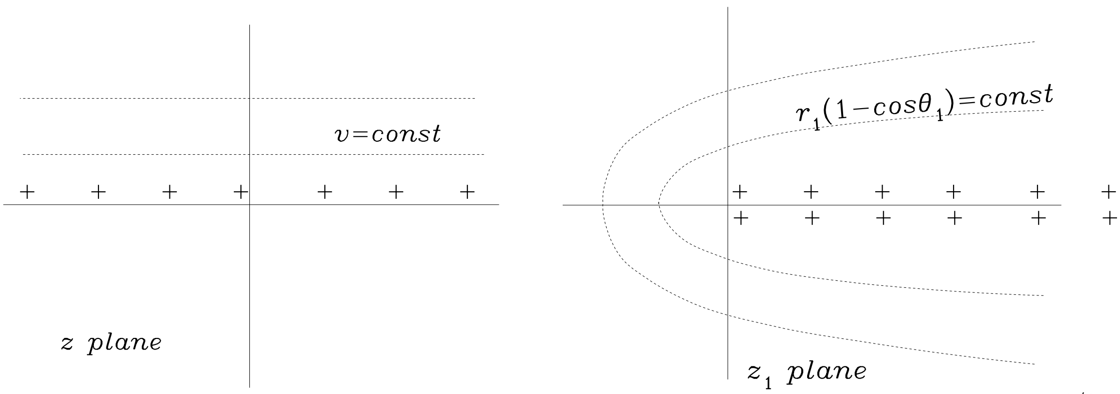


(ii)

$$f(z) = \begin{cases} a + z & \text{for } \Im\mathrm{m}(z) > 0 \\ a - z & \text{for } \Im\mathrm{m}(z) < 0 \end{cases} \quad \Rightarrow \quad \begin{matrix} u = \Re\mathrm{e}(a) + x, & v = \Im\mathrm{m}(a) + y \\ u = \Re\mathrm{e}(a) - x, & v = \Im\mathrm{m}(a) - y \end{matrix}.$$

v is proportional to the potential of an infinite charged plate in the plane $y = 0$.

It is possible to turn these rather tame examples into worthwhile solutions by using *another* analytic function to map a boring boundary, such as that of the last example, into a more interesting shape. For example, if we write $z_1 = z^2$, the upper half of the z-plane is mapped into the entire z_1 plane, and thus the boundary $y = 0$ of the upper half of the z-plane is wrapped round onto itself:



Since an analytic function of an analytic function is analytic, $f_1(z_1) \equiv f[z(z_1)] = a + z_1^{1/2}$ is analytic throughout the part of the z_1-plane that corresponds to $\Im\mathrm{m}(z) > 0$, i.e. to all the z_1-plane except the real line $(0 \le x_1, y_1 = 0)$. Hence $\nabla^2 u_1 = \nabla^2 v_1 = 0$ except on the positive real z_1-axis. The imaginary part of f_1 is, by definition, equal to the imaginary part of f at the point $z = z_1^{1/2}$ in the z-plane that corresponds to z_1. Hence v_1 must be constant along contours that wrap around the positive x_1-axis. Specifically,

$$v_1 = \Im\mathrm{m}(a + z_1^{1/2}) = \Im\mathrm{m}(a) + \sqrt{\tfrac{1}{2}r_1(1 - \cos\theta_1)}, \quad \text{where} \quad z_1 = r_1 e^{i\theta_1}.$$

This gives the contours of the potential around the edge of a charged semi-inifinite plate.

Note:

A complex function gives us two vector fields, namely (u_x, u_y) and (v_x, v_y). These are orthogonal by the C-R conditions: $(u_x, u_y) \cdot (v_x, v_y) = u_x v_x + u_y v_y = 0$. Thus the field lines in the last example are parallel to the lines $u_1 = \Re\mathrm{e}(a) + \sqrt{\frac{1}{2}r_1(1 + \cos\theta_1)} = \text{constant}.$

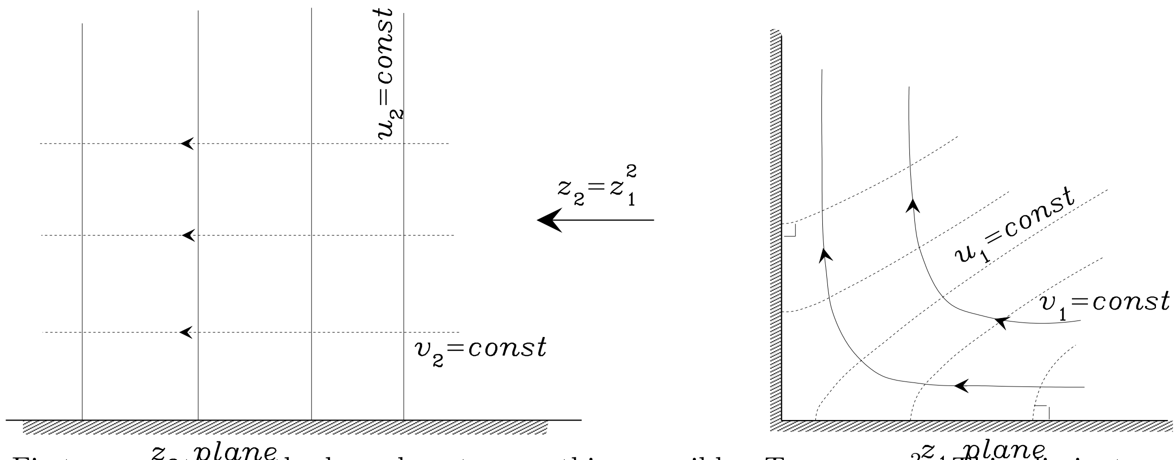
In deriving the example just given, we have put the cart before the horse. In real life one starts with a given physical boundary configuration away from which one wishes to solve Laplace's equation. Let this run along the curve $z_1(t)$ with t a real parameter. Then:

step 1 is to find a function $z_2(z_1)$ that maps the boundary $z_1(t)$ into a simple curve, for example the real z_2-axis;

step 2 is to generate, either by inspection or by solving the C-R equations, an analytic function $f(z_2)$ whose real (or imaginary) part is constant on the transformed boundary $z_2[z_1(t)]$. The real (or imaginary) part of $f_1(z_1) \equiv f[z_2(z_1)]$ will now be the desired potential.

Example:

We wish to find the irrotational flow of an incompressible fluid around a right-angle corner. We write the fluid velocity $\mathbf{V} = \nabla u_1(x, y)$. The incompressibility of the fluid requires $0 = \nabla \cdot \mathbf{V} = \nabla^2 u_1$. Near the boundary the flow velocity must be parallel to the boundary. Hence contours of u_1 must cut the boundary at right angles, and if $f_1 = u_1 + iv_1$ is analytic, contours of v_1, which are parallel to $\mathbf{V}$, run parallel to the boundary. Thus



First we must map the boundary to something sensible. Try $z_2 = z_1^2$. This eliminates the bend and makes it obvious that the desired function is $f(z_2) = z_2$ since the imaginary part is then constant on lines of fixed y_2 as required. The desired potential u_1 is now the real part of $f_1(z_1) = f[z_2(z_1)] = z_2(z_1) = z_1^2 = (x_1^2 - y_1^2) + 2ix_1y_1$. Thus the fluid flows along the hyperbolae $v_1 = 2x_1y_1 = \text{constant}$. These run perpendicular to the equipotential hyperbolae $(x_1^2 - y_1^2) = \text{constant}$.

3 Complex Integrals

A smooth curve or **contour** is specified by a complex-valued function $z(t)$ of the real variable t that has a continuous derivative $\dot{z} \equiv \dfrac{\mathrm{d}z}{\mathrm{d}t}$ such that $|\dot{z}|^2 \neq 0$; the point t on the curve "never stops moving". The curve is **closed** iff for some t_1, t_2 with $t_1 < t_2$ we have $z(t_1) = z(t_2)$. The curve is a **simple closed curve** if t_1 and t_2 are the smallest and largest values of t, and $z(t) \neq z(t_1)$ for $t_1 < t < t_2$. By convention, unless explicitly stated to the contrary, closed curves are to be traversed in the anticlockwise sense. It is intuitively obvious (but difficult to prove) that a simple closed curve divides the complex plane into two regions, one of which (the "exterior") contains ∞ (Jordan's theorem).

Def:

A domain D is **simply connected** iff every closed curve in D contains no boundary point of D in its interior.

Given a curve $\gamma \equiv \{z(t) = x(t) + iy(t),\, t \in [0, t_0)\}$[2] we define

$$\int_\gamma f(z)\,\mathrm{d}z = \lim_{N\to\infty} \sum_{n=1}^{N-1} f(\overline{z}_n)\,\Delta z_n$$

where

$$\overline{z}_n = \tfrac{1}{2}\Big[z\Big(\frac{nt_0}{N-1}\Big) + z\Big(\frac{(n-1)t_0}{N-1}\Big)\Big] \quad \text{etc;}$$
$$\Delta z_n = z\Big(\frac{nt_0}{N-1}\Big) - z\Big(\frac{(n-1)t_0}{N-1}\Big) \quad \text{etc.}$$

The limit involved in this definition exists when f is continuous and γ is smooth. Our next goal is to prove Cauchy's theorem, which opens up an extraordinary range of valuable results. But first we need a standard result from the theory of real variables:

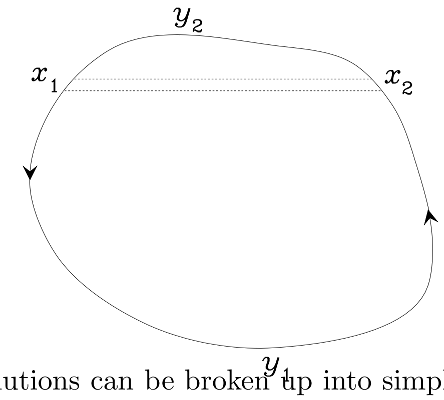
Lemma (Green): *If in a simply-connected domain D, the real-valued function $u(x, y)$ has continuous partial derivatives u_x and u_y, then*

$$\iint_D u_x\,\mathrm{d}x\mathrm{d}y = \oint_\gamma u\,\mathrm{d}y \quad \textit{and} \quad \iint_D u_y\,\mathrm{d}x\mathrm{d}y = -\oint_\gamma u\,\mathrm{d}x,$$

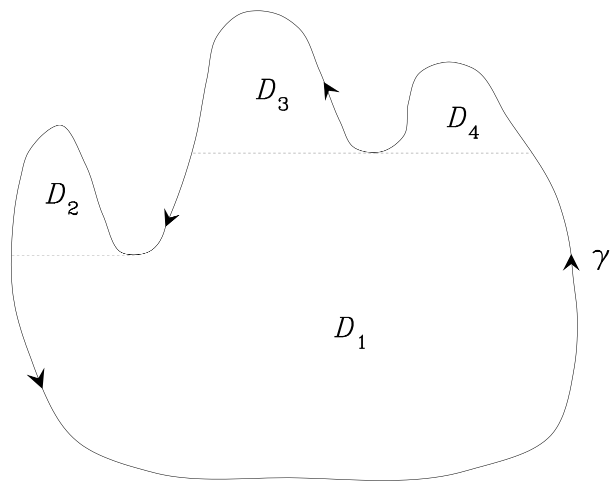
where the line integrals are in an anticlockwise sense around the smooth boundary γ of D.

Proof: We first assume that there are only two values of y at which $\dot{y} = 0$, namely the top, y_2, and the bottom, y_1, of D. Thus, when we divide D into strips parallel to the x-axis, each has two ends $x_2(y) > x_1(y)$. Consequently,

$$\begin{aligned}\iint_D u_x\,\mathrm{d}x\mathrm{d}y &= \int_{y_1}^{y_2} [u(x_2(y), y) - u(x_1(y), y)]\,\mathrm{d}y\\ &= \int_{y_1}^{y_2} u(x_2(y), y)\,\mathrm{d}y + \int_{y_2}^{y_1} u(x_1(y), y)\,\mathrm{d}y\\ &= \oint_\gamma u(x, y)\,\mathrm{d}y.\end{aligned}$$



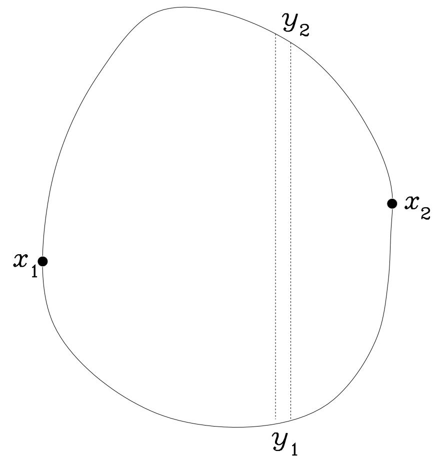
A ragged domain on whose boundary $\dot{y} = 0$ has many solutions can be broken up into simple sub-domains D_n:



[2] The real interval $[0, x)$ is defined to be the set of numbers x' such that $0 \le x' < x$; thus square brackets imply $\le$ and round brackets $<$.

The first part of the lemma applies to each sub-domain and its boundary. Hence the integral over the surface of D, which is just the sum of the integrals over each D_n, is equal to the integrals $\oint u\,\mathrm{d}y$ around the boundary of each D_n. Parts of these boundaries that are not part of the boundary of D do not contribute to the line integrals since on them $\mathrm{d}y = 0$. Hence the line integral around D is equal to the sum of the integrals around the D_n, and we have established the first part of the lemma.

Similarly, if $\dot{x} = 0$ has only two solutions, $x_2 > x_1$, each strip parallel to the y-axis has ends at $y_2 > y_1$ and



$$\begin{aligned}\iint_D u_y\,\mathrm{d}y\mathrm{d}x &= \int_{x_1}^{x_2} [u(x, y_2(x)) - u(x, y_1(x))]\,\mathrm{d}x \\ &= -\Big[\int_{x_2}^{x_1} u(x, y_2(x))\,\mathrm{d}x + \int_{x_1}^{x_2} u(x, y_1(y))\Big]\,\mathrm{d}x \\ &= -\oint_\gamma u(x, y)\,\mathrm{d}x,\end{aligned}$$

where the circuit is in the anticlockwise sense. The generalization of the second part of the Lemma to a general domain is now trivial.◁

Theorem 8 (Cauchy): *If $f = u + iv$ is analytic in a simply connected domain D, and γ is a simple closed curve in D, then*

$$\oint_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof: Since f is analytic, $u_x = v_y$ and $u_y = -v_x$. Thus by Green's lemma

$$\begin{aligned}\oint_\gamma f(z)\,\mathrm{d}z &= \oint_\gamma (u\,\mathrm{d}x - v\,\mathrm{d}y) + i\oint_\gamma (v\,\mathrm{d}x + u\,\mathrm{d}y) \\ &= \iint_D (-u_y - v_x)\,\mathrm{d}x\mathrm{d}y + i\iint_D (-v_y + u_x)\,\mathrm{d}x\mathrm{d}y \\ &= 0.◁\end{aligned}$$

Corollary: *For analytic f the integral*

$$F(z_1) \equiv \int_{z_0}^{z_1} f(z)\,\mathrm{d}z \qquad \text{(I)}$$

is independent of the path of integration from z_0 to z_1.

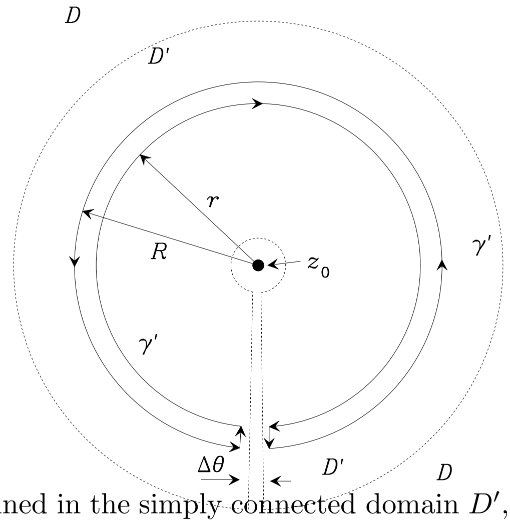
Proof: Let the curves γ_1 and γ_2 give answers, F_1 and F_2 respectively. Then integrating along γ_1 from z_0 to z_1 and then along γ_2 from z_1 to z_0 we obtain the value $F_1 - F_2$ for the integral of the an analytic function around a closed path. By Cauchy's theorem this is zero, so $F_1 = F_2$.◁

By this corollary we can deform the curve along which an integral is to be evaluated, providing only that the deformation never causes any part of the curve to leave the domain D of analyticity of the integrand.

We shall see (Morera's theorem in Appendix B) that the integral $F(z_1)$ defined by (I) is an analytic function of z_1 such that $F'(z) = f(z)$. Hence it is called the **indefinite integral** of f.

Theorem 9 (Cauchy's formula): *If f is analytic in the simply connected domain D, and γ is any simple closed curve in D about $z_0 \in D$, then*

$$f(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-z_0}\mathrm{d}z. \qquad (\mathrm{C}_0)$$



Proof: Consider the integral

$$I' = \oint_{\gamma'} \frac{f(z)}{z-z_0}\mathrm{d}z,$$

where γ' is as shown in the figure. γ' is contained in the simply connected domain D', which is in its turn contained in D. Since z_0 lies outside D', the integrand of I' is analytic throughout D'. Hence $I' = 0$

The integral I' is made up of three parts: (i) a nearly complete circle of radius R; (ii) a nearly complete circle of radius r; (iii) two radially directed sections. Since the integrand is continuous everywhere except at $z = z_0$, in the limit in which the angle $\Delta\theta$ becomes arbitrarily small, the first and second parts of I' tend to anticlockwise and clockwise integrals around complete circles, and the radially directed portions of the integral make equal and opposite contributions. Hence I' can be zero only if the two circular integrals are equal in magnitude and opposite in sign. Thus deforming the outer contour into the originally given contour γ we have

$$I \equiv \oint_{\gamma} \frac{f(z)}{z-z_0}\mathrm{d}z = \oint_{|z-z_0|=r} \frac{f(z)}{z-z_0}\mathrm{d}z, \qquad (\dagger)$$

where both integrals are to be taken in the anticlockwise sense. Hence

$$I = \oint_{|z-z_0|=r} \frac{f(z)-f(z_0)}{z-z_0}\mathrm{d}z + f(z_0)\oint_{|z-z_0|=r} \frac{\mathrm{d}z}{z-z_0}. \qquad (\ddagger)$$

Now f is continuous at z_0, so given any ϵ we can find $r > 0$ such that $|f(z) - f(z_0)| < \epsilon$ for all $|z - z_0| \le r$. With r thus chosen,

$$\Big|\oint_{|z-z_0|=r} \frac{f(z)-f(z_0)}{z-z_0}\mathrm{d}z\Big| < \oint_{|z-z_0|=r} \frac{|f(z)-f(z_0)|}{|z-z_0|}|\mathrm{d}z|$$
$$< \epsilon \oint_{|z-z_0|=r} \frac{|\mathrm{d}z|}{|z-z_0|} = 2\pi\epsilon.$$

Furthermore, writing $z = z_0 + re^{i\theta}$, we have

$$\oint_{|z-z_0|=r} \frac{\mathrm{d}z}{z-z_0} = \oint \frac{ire^{i\theta}}{re^{i\theta}}\mathrm{d}\theta = 2\pi i \quad \text{for any } r.$$

Hence in the limit $r \to 0$ the first integral in (‡) tends to zero, and the second integral becomes equal to $2\pi i f(z_0)$. The theorem now follows from (†).◁

Notes:

(i) We have shown that $\oint_{|z-z_0|=r} \mathrm{d}z/(z-z_0) = 2\pi i$ independent of r. But (†) shows that I is independent of r. Hence it follows that the first integral in (‡) is exactly zero, independent of r.

(ii) (C_0) is a very remarkable result for it says that the value taken by an analytic function at any point in the complex plane is determined by the values it takes on any curve around that point. One can understand this by regarding the C-R conditions as coupled partial differential equations for u and v; evidently the solution to these p.d.e's in D is uniquely specified by giving u and v on the boundary of D.

We can use Cauchy's formula to differentiate f by integrating a suitable multiple of it:

Theorem 10: *If f is analytic in D, then*

$$f'(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)\,\mathrm{d}z}{(z-z_0)^2}, \qquad (\mathrm{C}_1)$$

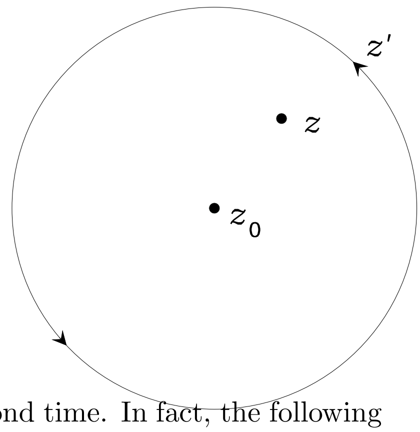
where the integral is to be taken on any simple closed curve in D around z_0.

Proof: By Cauchy's formula

$$\begin{aligned}\frac{f(z)-f(z_0)}{z-z_0} &= \frac{1}{2\pi i(z-z_0)}\Big[\oint \frac{f(z')}{z'-z}\,\mathrm{d}z' - \oint \frac{f(z')}{z'-z_0}\,\mathrm{d}z'\Big]\\ &= \frac{1}{2\pi i(z-z_0)}\oint \frac{f(z')(z-z_0)}{(z'-z)(z'-z_0)}\,\mathrm{d}z'\\ &= \frac{1}{2\pi i}\Big[\oint \frac{f(z')}{(z'-z_0)^2}\,\mathrm{d}z' + \oint \frac{f(z')(z-z_0)}{(z'-z)(z'-z_0)^2}\,\mathrm{d}z'\Big], \qquad (\dagger)\end{aligned}$$

where the integral is around a circle enclosing both z and z_0. But

$$\begin{aligned}\Big|\oint \frac{f(z')(z-z_0)}{(z'-z)(z'-z_0)^2}\,\mathrm{d}z'\Big| &< |z-z_0|\oint \frac{|f(z')||\mathrm{d}z'|}{|z'-z||z'-z_0|^2}\\ &< |z-z_0|\oint \frac{|f(z')|r\,\mathrm{d}\theta}{|re^{i\theta}+z_0-z|r^2},\end{aligned}$$



where $z'-z_0 = re^{i\theta}$. For finite r this last integral is finite. Thus letting $z\to z_0$ we show that the second integral in (†) is zero. But the first integral in (†) is well behaved (even unvarying!) in this limit. Therefore the left side of (†) also has a well-behaved limit as $z\to z_0$, and the theorem is established.◁

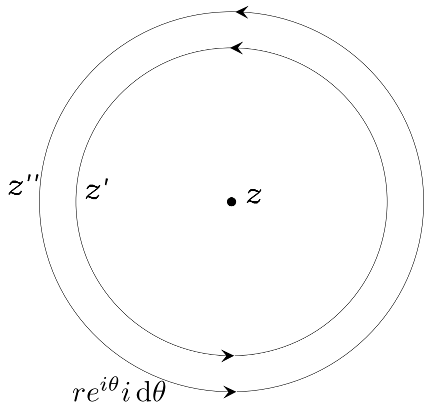
We can now use the expression (C_1) to differentiate f a second time. In fact, the following shows that we can differentiate f as many times as we please—and the result can be expressed as a single simple integral:

Theorem 11: *An analytic f in a simply connected domain D can be differentiated arbitrarily often. Furthermore,*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)\,\mathrm{d}z}{(z-z_0)^{n+1}} \quad (n=0,1,\dots). \qquad (\mathrm{C}_n)$$

Proof: Let the formula be valid for some index n—by Theorem (10) it is certainly valid for $n=1$. Then $f^{(n)}$ is analytic in D and we use (C_1) to differentiate it:

$$\begin{aligned}f^{(n+1)}(z) &= \frac{1}{2\pi i}\oint \frac{f^{(n)}(z')\,\mathrm{d}z'}{(z'-z)^2}\\ &= \frac{n!}{(2\pi i)^2}\oint \frac{\mathrm{d}z'}{(z'-z)^2}\oint \frac{f(z'')\mathrm{d}z''}{(z''-z')^{n+1}} \qquad (\dagger)\\ &= \frac{(n+1)!}{2\pi i}\oint \frac{\mathrm{d}z''f(z'')}{(z''-z)^{n+2}}\,\frac{(z''-z)^{n+2}}{2\pi i(n+1)}\oint \frac{\mathrm{d}z'}{(z'-z)^2(z''-z')^{n+1}},\end{aligned}$$



Here we have used (C_n) and have taken advantage of the finite integration paths of z' and z'' to reverse the order of integration. The integration paths are indicated in the figure. The only singularity of the inner integrand interior to the z' contour is that at $z' = z$. Hence we may deform the z' contour until it is a circle of small radius r around z. Writing $z' = z + re^{i\theta}$ we have

$$\begin{aligned}\oint \frac{\mathrm{d}z'}{(z'-z)^2(z''-z')^{n+1}} &= \frac{1}{(z''-z)^{n+1}} \int_0^{2\pi} \frac{re^{i\theta} i\,\mathrm{d}\theta}{r^2 e^{2i\theta}[1 - re^{i\theta}/(z''-z)]^{n+1}} \\ &= \frac{i}{(z''-z)^{n+1}} \int_0^{2\pi} \frac{\mathrm{d}\theta}{re^{i\theta}} \left[1 + (n+1)\frac{re^{i\theta}}{z''-z} + \cdots\right].\end{aligned}$$

Only the second term in the infinite sum survives the integral over θ. Thus

$$\oint \frac{\mathrm{d}z'}{(z'-z)^2(z''-z')^{n+1}} = \frac{2\pi i(n+1)}{(z''-z)^{n+2}}$$

and on substituting this expression into (†) the validity of (C_{n+1}) follows from the validity of (C_n), and the theorem is established.◁

Theorem 11 leads to:

Theorem 12 (Cauchy's inequality): *If f is analytic in D, then $\exists$ numbers F and r such that for all $z \in D$*

$$|f^{(n)}(z)| < \frac{Fn!}{r^n} \quad (n = 0, 1, \ldots). \tag{CI$_n$}$$

Proof: Take $F \geq$ the largest value attained by $|f|$ on some circle of radius r around z_0, and then apply (C_n).◁

Thus by performing an appropriate integral we can differentiate an analytic function arbitrarily often. Furthermore, we obtain an estimate of the size of all the derivatives. These very remarkable results open up a wide field of possibilities. Indeed, there is almost no limit to the number of of beautiful results that follow easily from (C_n); you will find three specimens in Appendix B. Life being short and an outbreak of theoremophobia being to be feared, we now turn without delay to those applications which lead to power series. These enable us to evaluate integrals, transforms and much else.

4 Power Series Expansions

Theorem 13 (Maclaurin expansion): *If f is analytic in D and the circle $|z| < r_0$ lies in D, then*

$$f(z) = f(0) + f'(0)z + \frac{1}{2!}f''(0)z^2 + \cdots.$$

Proof: We have

$$\frac{1-s^n}{1-s} = 1 + s + s^2 + \cdots + s^{n-1}$$

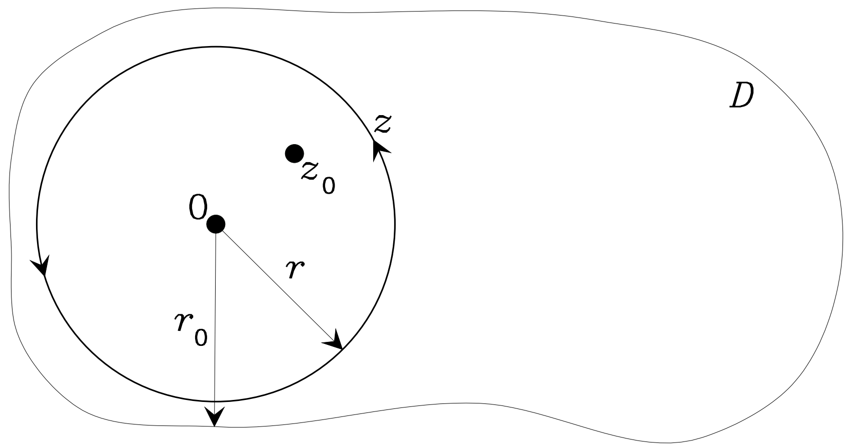
so

$$\frac{1}{1-s} = 1 + s + s^2 + \cdots + s^{n-1} + \frac{s^n}{1-s}.$$

Setting $s = z_0/z$ we obtain

$$\frac{1}{z-z_0} = \frac{1}{z}\frac{1}{1-z/z_0} = \frac{1}{z} + \frac{z_0}{z^2} + \frac{z_0^2}{z^3} + \cdots + \frac{z^{n-1}}{z^n} + \frac{(z_0/z)^n}{z-z_0}. \tag{M}$$

We substitute this expression into (C_0) and integrate around a circle of radius $r < r_0$ that lies in D and encloses z_0:



$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\oint \frac{f(z)}{z-z_0}\,\mathrm{d}z \\ &= \frac{1}{2\pi i}\oint f(z)\frac{\mathrm{d}z}{z} + \frac{z_0}{2\pi i}\oint f(z)\frac{\mathrm{d}z}{z^2} + \cdots \\ &\qquad \cdots + \frac{z_0^{n-1}}{2\pi i}\oint f(z)\frac{\mathrm{d}z}{z^n} + \frac{z_0^n}{2\pi i}\oint f(z)\frac{\mathrm{d}z}{z^n(z-z_0)} \\ &= f(0) + f'(0)z_0 + \frac{1}{2!}f''(0)z_0^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}z_0^{n-1} + R_n, \end{aligned} \tag{†}$$

where R_n is the last term in the third line of (†). Since f is analytic in D, $|f|$ takes a maximum value on the circle of interest, say $|f| \le F$. Thus

$$|R_n| \le F\oint \left|\frac{z_0}{r}\right|^n \frac{|\mathrm{d}z|}{|z-z_0|},$$

which tends to zero as $n \to \infty$ because $|z_0/r| < 1$ by hypothesis.◁

Theorem 14 (Taylor expansion): *If f is analytic in D and the circle $|z - z_0| < r_0$ lies within D, then*

$$f(z) = \sum_{n=0}^{\infty} f^{(n)}(z_0)\frac{(z-z_0)^n}{n!} \tag{T}$$

for $|z - z_0| < r_0$.

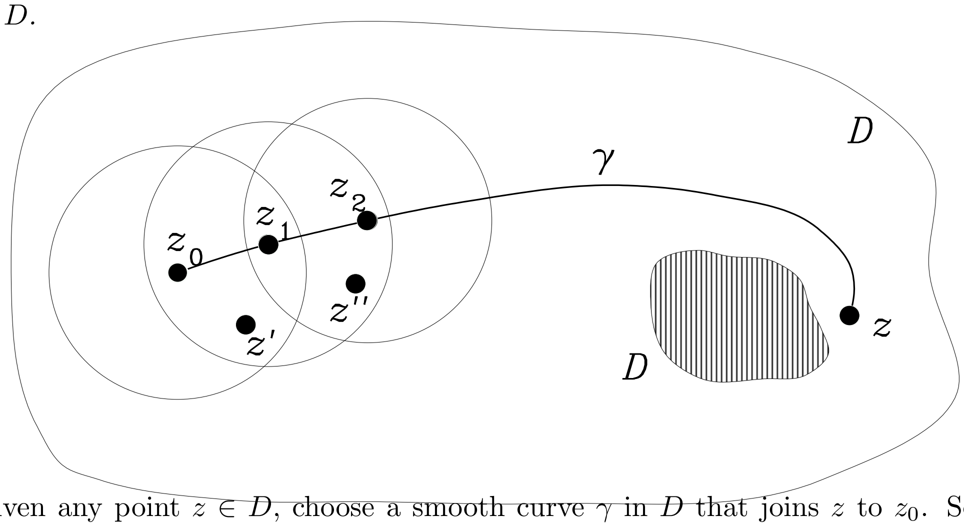
Proof: Apply the Maclaurin expansion to $g(z') \equiv f(z_0 + z')$.◁

Note:

Theorem 14 is a much stronger result than one can obtain for real functions, and indeed first gives one insight into the convergence interval of the expansion of a real function. For example, the series for $1/(1+x^2) = 1 - x^2 + x^4 - \cdots$ converges only for $|x| < 1$ although the function itself is well behaved for all x. The reason for this is that $1/(1+z^2)$ is analytic only inside the circle $|z| = 1$ on which lie the poles $z = \pm i$.

Analytic functions are sometimes called holomorphic because the whole structure of such a function is determined by its behaviour near any given point in the domain of analyticity:

Theorem 15: *If f is analytic in D and if $f^{(n)}(z_0) = 0$ for $n = 1, 2, \ldots$ at $z_0 \in D$, then f is constant in D.*



Proof: Given any point $z \in D$, choose a smooth curve γ in D that joins z to z_0. Select a point z_1 on γ as shown in the figure and use Taylor's expansion for $f(z_1)$ in powers of $z_1 - z_0$ to show that $f(z_1) = f(z_0)$. Then pick another point z' that lies both in the circle of radius r_0 about z_0 and also lies within the circle of convergence of the Taylor series centred on z_1. From the Taylor series centred on z_0 we have $f(z') = f(z_0)$, but expanding about z_1 we obtain

$$f(z_0) = f(z') = f(z_1) + f'(z_1)(z' - z_1) + \tfrac{1}{2} f''(z_1)(z' - z_1)^2 + \cdots$$

Since $f(z_1) = f(z_0)$ it now follows that $0 = f'(z_1) = f''(z_1) = \cdots$. Hence $f(z_2) = f(z_1) = f(z_0)$ and so on down γ until we reach the given point z.◁

Corollary: *If f and g are analytic in D and $g^{(n)}(z_0) = f^{(n)}(z_0)$ for $n = 0, 1, 2, \ldots$ at $z_0 \in D$, then $f = g$ in D.*

Proof: Apply the last theorem to $h \equiv g - f$.◁

4.1 Analytic continuation

The following result shows that the functions with which we are familiar are really functions of a complex variable; we know them only from the colourless shadows which they cast on the real line.

Theorem 16: *If F is analytic in D and if $f(z) = 0$ for z on a curve γ in D, then $f(z) = 0$ for all $z \in D$.*

Proof: Since f is analytic, we may calculate the derivatives of f at some point z_0 on γ by taking limits of f and its derivatives as z on γ tends to z_0. Clearly this process leads to the result $f^{(n)} = 0$ for $n = 1, 2, \ldots$ The Theorem now follows from Theorem 18.◁

Corollary: *There is at most one complex analytic function that reduces to a given function on the real line.*

Proof: If there were two functions, say f and g, we could apply the last theorem to $f - g$ to show that $f = g$ everywhere.◁

The procedure given by Theorem 18 for working one's way around a singularity is all very well in principle, but it is distinctly tedious in practice. Fortunately one can often obtain a single expansion that is valid for an entire annular region even when the function is not analytic somewhere inside the annulus.

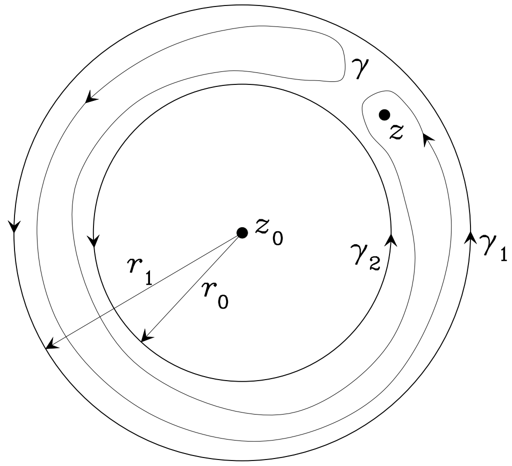
Theorem 17 (Laurent expansion): *If f is analytic in the annulus $r_0 < |z - z_0| < r_1$, then*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad \text{where} \quad a_n \equiv \frac{1}{2\pi i} \oint \frac{f(z')\,\mathrm{d}z'}{(z - z_0)^{n+1}}, \tag{L}$$

the integral being taken along any closed curve around z_0 that is contained in the specified annulus.

Proof: We have

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z')\,\mathrm{d}z'}{z'-z},$$



where the contour is as shown. As in the proof of Cauchy's formula, we then deduce that

$$f(z) = \frac{1}{2\pi i}\Big[\oint_{\gamma_1} \frac{f(z')\,\mathrm{d}z'}{z'-z} - \oint_{\gamma_2} \frac{f(z')\,\mathrm{d}z'}{z'-z}\Big]. \qquad (\sharp)$$

On γ_1, $|z'-z_0| > |z-z_0|$ so we may expand

$$\frac{1}{z'-z} = \frac{1}{(z'-z_0)-(z-z_0)} = \frac{1}{z'-z_0}\frac{1}{1-s} \quad \text{where} \quad s \equiv \frac{z-z_0}{z'-z_0},$$

as a power series in s as in the proof of the Maclaurin expansion. Hence

$$\frac{1}{2\pi i}\oint_{\gamma_1} \frac{f(z')\,\mathrm{d}z'}{z'-z} = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_{\gamma_1} \frac{f(z')\,\mathrm{d}z'}{(z'-z_0)^{n+1}}. \qquad (\dagger)$$

Similarly, on γ_2, $|z'-z_0| < |z-z_0|$ so we may expand

$$\frac{1}{z'-z} = \frac{-1}{(z-z_0)-(z'-z_0)} = \frac{-1}{z-z_0}\frac{1}{1-s} \quad \text{where} \quad s \equiv \frac{z'-z_0}{z-z_0},$$

and then

$$\frac{-1}{2\pi i}\oint_{\gamma_2} \frac{f(z')\,\mathrm{d}z'}{z'-z} = \frac{1}{2\pi i}\sum_{m=0}^{\infty}\frac{1}{(z-z_0)^{m+1}} \oint_{\gamma_2} f(z')(z'-z_0)^m\,\mathrm{d}z'. \qquad (\ddagger)$$

The integrals on the right sides of (†) and (‡) have integrands that are analytic throughout the annulus. Therefore we can deform the contours involved from γ_1 and γ_2 to any contour γ that is contained in the annulus and encloses z_0. Then changing the variable of summation in (‡) from m to $n \equiv -(m+1) = -1, -2, \ldots$, and assembling (♯), (†) and (‡) the relations (L) follow.◃

The Laurent expansion enables us to classify singularities of f. If when we expand f about z_0 we find that $a_{-m} = 0$ for all $m > M > 0$ but $a_{-M} \neq 0$, we say that f has a **pole of order** M at z_0. If $M = 1$, we call this a **simple pole**. If for any M there exists $a_{-m} \neq 0$ with $m > M$, we say that f has an **essential singularity** at z_0.

Examples:

(i) $f(z) = (z^2+1)^{-1}$ has simple poles at $z = \pm i$.

(ii) $f(z) = (z+1)^{-2}$ has a pole of order 2 at $z = -1$.

(iii) $f(z) = \exp(1/z) = w^{-3}(1+w^2)$, where $w \equiv 1/z$, has an essential singularity at $z = 0$ and a pole of order 3 at $z = \infty$.

5 Contour Integration

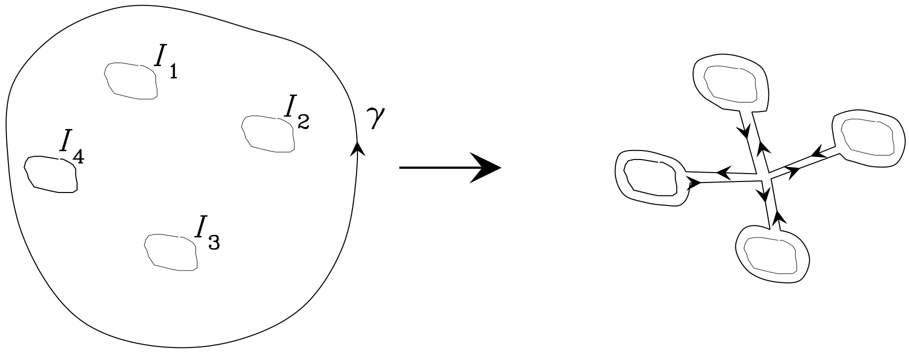
5.1 Residues

Theorem 18: *If f is analytic in a domain D which contains a finite number of islands I_j around each one of which a finite annulus, centre z_j of points in D can be circumscribed, and γ is a simple closed curve that lies wholly in D, then*

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_j R_j,$$

where R_j is the coefficient a_{-1} in the Laurent expansion (L) of f around z_j and the summation is over those values of j for which (i) I_j lies in the interior of γ, and (ii) f is not analytic throughout I_j.

Proof: We may distort the path of integration until it runs in an all-but closed circle around each I_j contained in γ:



Thus the integral around γ is the sum of integrals on circles around the I_j. But setting $z = z_j + re^{i\theta}$, these integrals are by (L)

$$\begin{aligned}\oint f(z)\,\mathrm{d}z &= \int_0^{2\pi} \sum_n a_n r^n e^{in\theta} re^{i\theta} i\,\mathrm{d}\theta \\ &= a_{-1}(2\pi i).\end{aligned}$$

Adding the contribution from each island in γ and noting by Taylor's expansion (T) that $a_{-1} = 0$ for any island through which f is analytic, we obtain the desired result.◁

Def:

$R_j = a_{-1}$ is called the **residue** of the j^{th} singularity.

Theorem 18 provides a powerful method of evaluating *definite* integrals of real functions: we first express the integral as a closed contour integral of a complex function (or possibly as the real or imaginary part of such an integral); then we find the singularities of the complex integrand that are contained in the chosen contour, and evaluate the associated residues.

We distinguish two basic types of problem:

5.2 Integrals over a finite range

Examples:

(i)

$$I \equiv \int_0^\pi \frac{\mathrm{d}\theta}{a - b\cos\theta} \quad (a > b > 0).$$

The integrand is even in θ, so

$$I = \tfrac{1}{2}\int_0^{2\pi} \frac{\mathrm{d}\theta}{a - b\cos\theta} = \tfrac{1}{2}\int_0^{2\pi} \frac{\mathrm{d}\theta}{a - \frac{1}{2}b(e^{i\theta} + e^{-i\theta})}.$$

Set $z = e^{i\theta}$. Then

$$I = \oint \frac{-i\,\mathrm{d}z}{2az - bz^2 - b} = \frac{1}{b}\oint \frac{i\,\mathrm{d}z}{(z-z_-)(z-z_+)},$$

where the integral is to be taken around the unit circle and $z_\pm = a/b \pm \sqrt{(a/b)^2 - 1}$ are the roots of the denominator of the integrand. Only the smaller root z_- is contained in the unit circle. If we write $z = z_- + \delta$, the integrand becomes

$$\frac{i}{\delta(z_- - z_+ + \delta)} = \frac{i}{\delta(z_- - z_+)}\big(1 - \frac{\delta}{z_- - z_+} + \cdots\big).$$

Hence the residue is $R = i/(z_- - z_+) = -i/[2\sqrt{(a/b)^2 - 1}]$ and the integral is $I = (2\pi i R)/b = \pi/\sqrt{a^2 - b^2}$.

(ii)

$$I \equiv \int_0^\pi \frac{\mathrm{d}\theta}{(a - b\cos\theta)^2}.$$

We proceed as before to conclude that

$$I = \frac{2}{ib^2}\oint \frac{z\,\mathrm{d}z}{(z-z_-)^2(z-z_+)^2}$$

around the unit circle. Expanding the integrand, this becomes

$$\frac{z_- + \delta}{\delta^2[(z_- - z_+) + \delta]^2} = \frac{z_- + \delta}{\delta^2(z_- - z_+)^2}\Big[1 - \frac{2\delta}{z_- - z_+} + \cdots\Big],$$

so the residue is $R = (z_- - z_+)^{-2} - 2z_-(z_- - z_+)^{-3} = -(z_- + z_+)(z_- - z_+)^{-3}$, whence $I = (2/ib^2)(2\pi i R) = \pi a/(a^2 - b^2)^{3/2}$.

5.3 Integrals over an infinite range

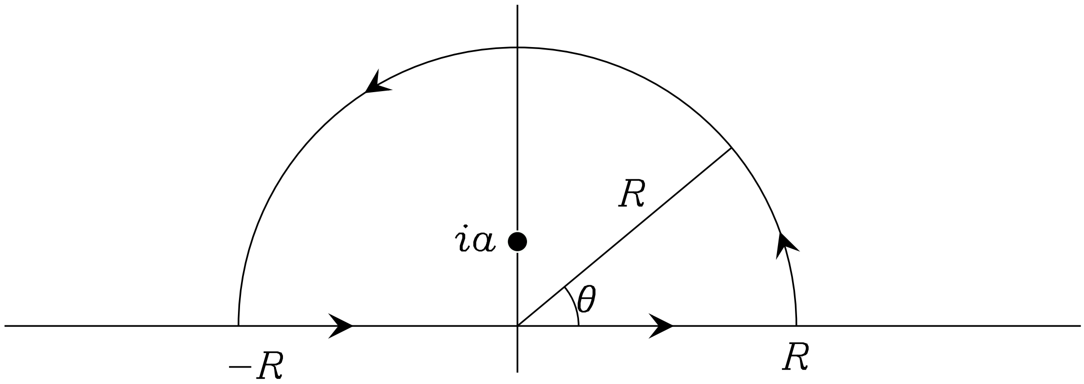
Our definition of a contour integral is valid only when the contour has finite length. However, we can often give meaning to integration over an infinite contour by defining this as the limit of corresponding integrals over a series of finite contours of ever increasing length. This limiting process is often an important part of the logic by which we convert a real integral into a contour integral.

Examples:

(i)

$$I \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2 + a^2}.$$

Now by definition, $I = \lim_{R\to\infty} \int_{-R}^{R} \frac{\mathrm{d}x}{x^2 + a^2}$. Suppose we define I' to be the integral of the same integrand but around the closed contour



The absolute value of the contribution to I' from the integral along the semi-circle is certainly less than

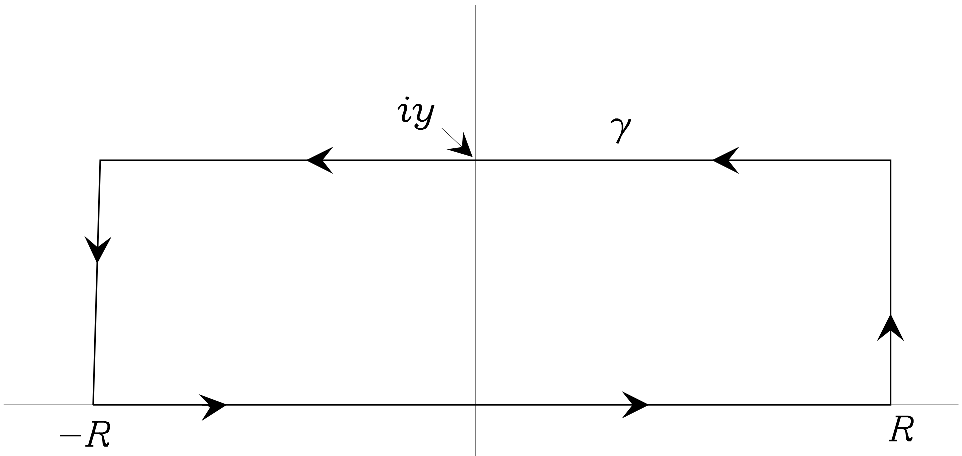
$$\int_0^{\pi} \frac{R\,\mathrm{d}\theta}{R^2+a^2},$$

which tends to zero as $R\to\infty$. Hence I' tends to I as $R\to\infty$. By the residue theorem, $I' = 2\pi i/(2ia)$, so $I=\pi/a$ as one might have shown by elementary means.[3]

(ii)

$$I \equiv \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,\mathrm{d}x \quad (0<a<1)$$

Consider the effect of integrating $e^{az}/(1+e^z)$ around the contour



The integral along the bottom stretch tends to the desired integral. Along the top the integrand is

$$\frac{e^{iay}e^{ax}}{1+e^{iy}e^x}.$$

So if we set $y=2\pi$, the integral along the top becomes $(-e^{2\pi ia})$ times the desired integral. Finally, the left and right ends contribute to the contour integral less than $2\pi e^{-aR}$ and $2\pi e^{(a-1)R}$, respectively, and may be neglected. The only singularity of the integrand which lies lies in γ is that at $z=i\pi$. Writing $z=i\pi+\delta$, the integrand becomes

$$\begin{aligned}\frac{e^{ia\pi}e^{\delta a}}{1-e^{\delta}} &= -e^{ia\pi}\frac{(1+\delta a+\cdots)}{(\delta+\delta^2/2!+\cdots)}\\ &= -\frac{e^{ia\pi}}{\delta}(1+\delta a+\cdots)\Big[1-\Big(\frac{\delta}{2!}+\cdots\Big)+\cdots\Big].\end{aligned}$$

Hence $R=-e^{i\pi a}$, so $I(1-e^{2\pi ia}) = 2\pi i(-e^{i\pi a})$ and

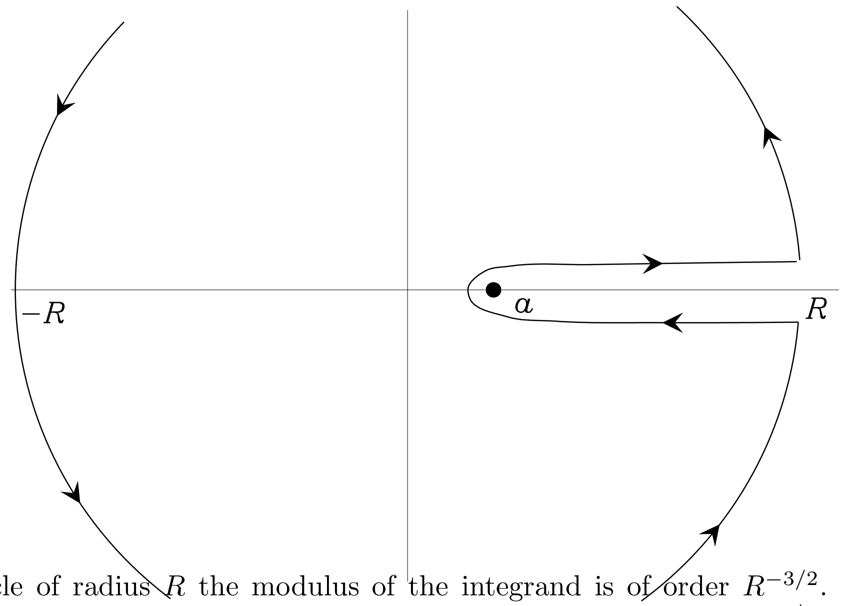
$$I = \frac{2\pi i}{e^{i\pi a}-e^{-i\pi a}} = \frac{\pi}{\sin \pi a}.$$

(iii)

$$I \equiv \int_a^{\infty} \frac{\sqrt{x-a}}{x^2+b^2}\,\mathrm{d}x.$$

In the integrand replace x with the complex variable z and consider integrating this integrand around the closed contour γ

[3] In this example we could equally well have completed the contour by a semi-circle below the real axis. We should then have had to think carefully about signs, however.



On the circle of radius R the modulus of the integrand is of order $R^{-3/2}$. Hence, the contribution to the entire contour integral from this circle is of order $R^{-1/2}$ and becomes negligible in the limit $R \to \infty$. Furthermore, to the right of $z = a$ both the radical in the integrand and dz change sign as one crosses the real axis. Thus

$$I = \tfrac{1}{2} \oint_\gamma \frac{(z-a)^{1/2}}{z^2+b^2}\,\mathrm{d}z.$$

The integrand has poles at $z = \pm ib$. Writing $z = ib + \delta$, we have

$$\begin{aligned}\frac{(z-a)^{1/2}}{z^2+b^2} &= \frac{(z-a)^{1/2}}{(z-ib)(z+ib)} = \frac{(ib-a+\delta)^{1/2}}{\delta(2ib+\delta)}\\ &= \frac{(ib-a)^{1/2}}{2ib\delta}\Big(1+\tfrac{1}{2}\frac{\delta}{ib-a}-\cdots\Big)\Big(1-\frac{\delta}{2ib}+\cdots\Big).\end{aligned}$$

Thus the required residue is

$$\frac{(ib-a)^{1/2}}{2ib}.$$

Similarly, the residue of the pole at $z = -ib$ is $(-ib-a)^{1/2}/(-2ib)$, so the required integral is

$$\begin{aligned}I &= \tfrac{1}{2}\frac{2\pi i}{2ib}\big[(ib-a)^{1/2} - (-ib-a)^{1/2}\big]\\ &= \frac{\pi}{b}\Im\mathrm{m}[(ib-a)^{1/2}] = \frac{\pi}{b}(a^2+b^2)^{1/4}\sin[\tfrac{1}{2}\arctan(-b/a)].\end{aligned}$$

Other interesting examples require a lemma and a concept.

Lemma (Jordan): *Let f be analytic in the upper half-plane except for a finite number of poles, and let*

$$\lim_{R\to\infty} F(R) = 0 \quad \textit{where} \quad F(R) \equiv \sup_{|z|=R,\,\Im\mathrm{m}(z)>0} |f(z)|.$$

Then for $m > 0$

$$\lim_{R\to\infty}\Big|\int_{S(R)} e^{imz} f(z)\,\mathrm{d}z\Big| = 0,$$

where $S(R)$ is the semi-circle of radius R in the upper half-plane.

Proof:

$$1 \ge \frac{\sin\theta}{\theta} \ge \frac{2}{\pi} \quad \text{for} \quad 0 \le \theta \le \tfrac{1}{2}\pi.$$

Hence

$$\Big|\int_{S(R)} e^{imz} f(z)\,\mathrm{d}z\Big| \le \int_0^\pi e^{-mR\sin\theta} F(R)\,\mathrm{d}\theta \le 2RF(R)\int_0^{\pi/2} e^{-2mR\theta/\pi}\,\mathrm{d}\theta$$
$$= \frac{\pi F(R)}{m}\Big(1-e^{-mR}\Big) < \frac{\pi F(R)}{m},$$

which tends to zero with $F(R)$.◁

Def:

Suppose $f(z)$ has a pole at z_0 on the contour γ. Then let γ_δ be γ less points that satisfy $|z_0 - z| < \delta$. Then if

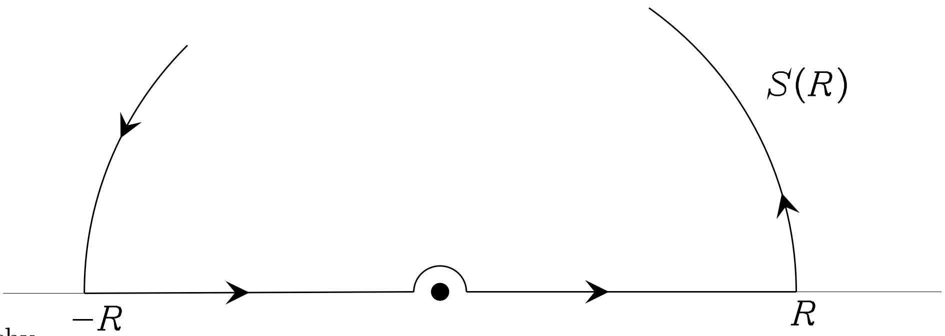
$$\lim_{\delta\to 0}\int_{\gamma_\delta} f(z)\,\mathrm{d}z$$

exists, it is called the **Cauchy principal value** of the integral of f over γ and is written $P\{\int_\gamma f(z)\,\mathrm{d}z\}$.

Example:

$$I \equiv \int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x.$$

Suppose we integrate e^{iz}/z around γ, where γ is along the real axis from $z = -R$ to $z = R$, and then around the semi-circle $S(R)$ in the upper half-plane. By Jordan's lemma the contribution of $S(R)$ to our integral becomes ignorable as $R \to \infty$. The only problem is that the new integrand has a pole at $z = 0$ where the original integrand was well behaved. We side-step the problem:



By Cauchy

$$I' \equiv \oint_{\gamma'} \frac{e^{iz}}{z}\,\mathrm{d}z = 0. \qquad (\dagger)$$

But

$$I' = \int_{-\infty}^{0-\delta} \frac{e^{ix}}{x}\,\mathrm{d}x + \int_{0+\delta}^{\infty} \frac{e^{ix}}{x}\,\mathrm{d}x + \int_\pi^0 e^{i\delta e^{i\theta}}\, i\,\mathrm{d}\theta, \qquad (\ddagger)$$

so as $\delta \to 0$ the imaginary part of I' yields

$$\int \frac{\sin x}{x}\,\mathrm{d}x = \pi.$$

The real parts of the first two integrals in (‡) tend to $-\infty$ and $+\infty$, respectively. However, (†) and (‡) show that for all δ they cancel. Hence

$$P\Big\{\int_{-\infty}^{\infty} \frac{\cos x}{x}\,\mathrm{d}x\Big\} = 0.$$

6 Fourier Transforms

If $f(x)$ is a periodic function with period L, we know that[4]

$$f(x') = \sum_n A_n e^{2n\pi i x'/L} \equiv \sum_{n=-\infty}^{\infty} \left[\frac{1}{L}\int_{-L/2}^{L/2} f(x) e^{-\frac{2n\pi i x}{L}}\,\mathrm{d}x\right] e^{\frac{2n\pi i x'}{L}}.$$

If we let $k_n = 2n\pi/L$, then this formula may be rewritten

$$\begin{aligned} f(x') &= \sum_{n=-\infty}^{\infty} \left[\frac{1}{L}\int_{-L/2}^{L/2} f(x) e^{-ik_n x}\,\mathrm{d}x\right] e^{ik_n x'} \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} (k_{n+1}-k_n)\left[\int_{-L/2}^{L/2} f(x) e^{-ik_n x}\,\mathrm{d}x\right] e^{ik_n x'}. \end{aligned}$$

If we now let $L \to \infty$, $k_{n+1} - k_n \to 0$ and the sum goes over into an integral over the continuous variable k. Thus if the period of f is infinite, i.e. if f is not periodic, we have

$$f(x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\left[\int_{-\infty}^{\infty} f(x) e^{-ikx}\,\mathrm{d}x\right] e^{ikx'}.$$

Thus it appears that we can recover a non-periodic function from the inner integral of the last equation. This leads to

Def:

The **Fourier transform** of $f(x)$ is

$$\widetilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-ikx}\,\mathrm{d}x. \qquad (\mathrm{F}_1)$$

Our argument concerning the limit $L \to \infty$ is unfortunately not rigorous. In particular, it gives no indication of what functions have Fourier transforms. One can see that this is not an academic question by attempting to evaluate the FT of the simple function $\{f(x) = -1 \text{ for } x < 0,\ f(x) = 1 \text{ for } x > 0\}$, which is just the $L \to \infty$ limit of the familiar square wave.

Fourier transforms are absolutely fundamental to innumerable branches of physics, so it is important to put them on a secure foundation. We have

Theorem 19 (Fourier integral): *If f is a function of bounded variation, and $|f|$ is integrable from $-\infty$ to ∞, then*

$$\lim_{\delta\to 0} \tfrac{1}{2}[f(x'+\delta) + f(x'-\delta)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{f}(k) e^{ikx'}\,\mathrm{d}k. \qquad (\mathrm{F}_2)$$

For the definition of a function of bounded variation and a proof of (F_2), see Appendix D.

As physicists we are usually happy to confine ourselves to continuous functions. For these the left side of (F_2) becomes identical with $f(x')$. Hence

$$f(x') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{f}(k) e^{ikx'}\,\mathrm{d}k \quad (\text{continuous } f) \qquad (\mathrm{F}_3)$$

[4] For the relationship of exponential Fourier series to the sine and cosine series, see Appendix C.

Notes:

(i) Comparing (F_1) and (F_3) we see that for continuous functions the FT operation and its inverse are *symmetrical*.

(ii) Since kx occurs in the argument of an exponential in these formulae, the dimensions of k must be the inverse of those of x: if x is distance, k is wavenumber, i.e. 2π/wavelength; if x is time, k is an angular frequency. The FT expresses functions of distance as superpositions of harmonic waves, and functions of time as superpositions of simple harmonic oscillations.

(iii) Fourier transforms are easily generalized to functions of several variables. For example, if $f = f(x,y)$ and $\iint |f(x,y)|\,\mathrm{d}x\mathrm{d}y$ over all space is finite, then

$$\widetilde{f}(k_x,k_y) = \frac{1}{2\pi}\iint f(x,y)e^{-i(k_x x + k_y y)}\,\mathrm{d}x\mathrm{d}y.$$

The space spanned by the vectors $\mathbf{k} \equiv (k_x,k_y)$ is called **reciprocal space**. We get an extra factor $(2\pi)^{-1/2}$ outside the integral for every extra independent variable.

6.1 Dirac delta function

In the proof of Theorem 23 in Appendix D it is shown that for continuous f

$$f(x') = \int_{-\infty}^{\infty} f(x)\frac{\sin[A(x'-x)]}{\pi(x'-x)}\,\mathrm{d}x + \frac{\epsilon(A)}{2\pi},$$

where as $A \to \infty$, $\epsilon(A) \to 0$ and the *integral* tends to a well defined limit $f(x')$. It is tempting to imagagine that the *integrand* goes to a limit

$$\delta(x'-x) = \lim_{A\to\infty}\frac{\sin[A(x'-x)]}{\pi(x'-x)} \tag{D$_1$}$$

such that

$$\int_{-\infty}^{\infty} f(x)\delta(x'-x)\,\mathrm{d}x = f(x') \quad \text{for all } f. \tag{D$_2$}$$

If we retain the narrow-minded view that a function is a rule that assigns to each point a definite number, the limit in (D_1) does not exist. But we can give δ, which is called the **Dirac delta function** (though it is not a function!) a meaning by interpreting δ as a declaration of intent: whenever we write a δ-function we intend eventually to integrate over its argument. This integral will be interpreted as a limit of integrals containing $\sin(Ax)/\pi x$ in place of $\delta(x)$. In practice we simply eliminate δ from our expression by means of (D_2).[5]

If we substitute (F_1) into (F_3) and reverse the order of integration without further ado, we find

$$f(x') = \int_{-\infty}^{\infty}\mathrm{d}x\, f(x)\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}e^{ik(x'-x)}.$$

Comparing this with (D_2) we say

$$\delta(x'-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x'-x)}\,\mathrm{d}k. \tag{F$_4$}$$

This manner of speaking is very convenient, but remember that we have stepped well outside the range of what can be understood in terms of the usual Riemann integral.

[5] This naive interpretation of the δ-function extends to mathematics the physicist's trick of introducing metaphysical quantities like force-fields and wave-functions which we propose to eliminate before we come up with a prediction for the experimentalists. A mathematically more satisfactory procedure is to extend the class of objects that our integration theory can handle to so-called *distributions*, of which δ is an example.

Note:

Like a Hindu god δ has many manifestations. For example, if J_n is the cylindrical Bessel function of order n, then

$$\delta(x' - x) = x \int_0^\infty J_n(kx) J_n(kx') k \, \mathrm{d}k.$$

This formula is the basis of the so-called **Hankel transforms**.

Theorem 20: *Let a be any constant. Then* $\delta(ax) = \frac{1}{a}\delta(x)$

Proof: Given any function $f(x)$ we have

$$\int f(x)\delta(ax)\,\mathrm{d}x = \frac{1}{a}\int f(u/a)\delta(u)\,\mathrm{d}u = \frac{1}{a}f(0).\triangleleft$$

Theorem 21: *Let g be a smooth function with a finite number of zeros x_k. Then* $\delta[g(x)] = \sum_k \frac{1}{g'(x_k)}\delta(x - x_k)$.

Proof: Clearly, there are no contributions to the integral $\int f(x)\delta[g(x)]\,\mathrm{d}x$ from intervals of x that do not contain a zero of g. Since g is smooth, there will be a neighbourhood of each zero x_k in which g is monotone. In each such interval we may change to g as the variable of integration, obtaining a contribution to the entire integral equal to

$$\int_{x \approx x_k} f[x(g)]\delta(g)\frac{\mathrm{d}g}{\mathrm{d}g/\mathrm{d}x}.$$

Evaluating each such integral completes the proof.◁

The following example illustrates the use of Fourier transforms and Dirac delta functions in a practical problem.

Example:

The mass per unit length ρ of an impurity in a very long pipe obeys the equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\Big(\frac{\partial \rho}{\partial x} - v\rho\Big),$$

where v is a constant. At time $t < 0$ the pipe is uncontaminated, but at $t = 0$ a mass m of impurity suddenly enters the pipe at $x = 0$. Find $\rho(x, t)$ for $t > 0$.

We transform the equation by multiplying through by $\frac{1}{\sqrt{2\pi}}e^{-ikx}$ and integrating over all x to find

$$\frac{\partial \widetilde{\rho}}{\partial t} = -k^2\widetilde{\rho} - ikv\widetilde{\rho}.$$

At each fixed k we solve this first-order differential equation in time. Thus

$$\widetilde{\rho}(k, t) = \widetilde{\rho}(k, 0)e^{-(k^2 + ikv)t}.$$

But the given initial condition may be expressed $\rho(x, 0) = m\delta(x)$, and transforming this we have

$$\widetilde{\rho}(k, 0) = \frac{m}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)\,e^{-ikx}\,\mathrm{d}x = \frac{m}{\sqrt{2\pi}}.$$

Combining the last two equations and applying the inverse transform we find

$$\begin{aligned}\rho(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\widetilde{\rho}(k,t)\,\mathrm{d}k\\ &= \frac{m}{2\pi}\int_{-\infty}^{\infty}\exp\left[-\left(k^2t+ik(vt-x)\right)\right]\mathrm{d}k\\ &= \frac{m}{2\pi}\exp\left[-\frac{(vt-x)^2}{4t}\right]\int_{-\infty}^{\infty}\exp\left\{-\left[k\sqrt{t}+\frac{i(vt-x)}{2\sqrt{t}}\right]^2\right\}\mathrm{d}k\end{aligned}$$

An appropriate change of variable transforms the integral in the last line of this equation into $\int e^{-K^2}\mathrm{d}K$ along a contour which runs parallel to the real K-axis. Hence the value of this integral is simply $\sqrt{\pi}$ and we conclude that

$$\rho(x,t) = \frac{m}{2\sqrt{\pi t}}\exp\left[-\frac{(vt-x)^2}{4t}\right].$$

This distribution is a spreading Gaussian hump of impurity, which moves down the pipe at speed v.

6.2 Fourier convolutions

Def:

If f and g are such that $|f|$ and $|g|$ are integrable from $-\infty$ to ∞, then the **Fourier convolution** $f\circ g$ of f and g is

$$f\circ g(x') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)g(x'-x)\,\mathrm{d}x. \qquad (F_5)$$

Notes:

(i) The factor $(2\pi)^{-1/2}$ in this formula is not standard. However, it simplifies many subsequent formulae.

(ii) Through the convolution operation every function g maps each other function f into a new function $f' = f\circ g$. This map is obviously linear. The function which generates the identity mapping is $\sqrt{2\pi}\delta$.

Theorem 22: $f\circ g = g\circ f$.

Proof: Eliminate x from the definition of $f\circ g$ in favour of $y\equiv(x'-x)$.◁

Theorem 23(Fourier Convolution): *If f and g are continuous functions with* FTs, *then*

$$\widetilde{f\circ g}(k) = \widetilde{f}(k)\times\widetilde{g}(k) \qquad (F_6)$$

i.e. the FT *of the convolution of two functions is the product of the* FTs *of the two functions.*

Proof: Taking the FT of $f\circ g$ and substituting $y\equiv x'-x$, we have

$$\begin{aligned}\widetilde{f\circ g}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}x\,f(x)e^{-ikx}\int_{-\infty}^{\infty}g(y)e^{-iky}\,\mathrm{d}y\\ &= \widetilde{f}(k)\widetilde{g}(k).◁\end{aligned}$$

Corollary: *If a and b are continuous functions with* FTs, *and $f(x) = a(x)b(x)$, then*

$$\widetilde{f} = \widetilde{a} \circ \widetilde{b}. \tag{F$_7$}$$

Proof: By the symmetry of the FT (F_1) and its inverse (F_3), we may write $k = x$, $f = \widetilde{a}$ and $g = \widetilde{b}$ in (F_5) to find

$$\widetilde{\widetilde{a} \circ \widetilde{b}}(x) = \widetilde{\widetilde{a}} \times \widetilde{\widetilde{b}} = a \times b(x) = f(x).$$

(F_7) follows when we transform both sides of this equation.◁

Convolution integrals occur in a great number of physical problems.

Examples:

(i) Light from stars is deflected by small rapidly fluctuating inhomogeneities in the Earth's atmosphere. Consequently light from even the most distant star forms an extended circular blob on a photographic plate at the focus of the telescope. One can measure the distribution of intensity in this blob and find the intensity $G(x, y; x_0, y_0)$ at (x, y) on the plate due to a source whose image should really be at (x_0, y_0). We might find

$$G(x, y; x_0, y_0) = K \times \exp\left[\frac{(x - x_0)^2 + (y - y_0)^2}{2\sigma^2}\right],$$

where K and σ are suitable constants. When the telescope is turned on Mars, the intensity $I(x, y)$ observed at (x, y) will be the sum of the intensities due to various rocks & c. Let $I_0(x_0, y_0)$ be the intensity that would be observed at (x_0, y_0) if there were no refraction in the atmosphere. Then

$$\begin{aligned} I(x, y) &= \iint I_0(x_0, y_0) G(x, y; x_0, y_0)\, \mathrm{d}x_0 \mathrm{d}y_0 \\ &= K \iint I_0(x_0, y_0) \exp\left[\frac{(x - x_0)^2 + (y - y_0)^2}{2\sigma^2}\right] \mathrm{d}x_0 \mathrm{d}y_0. \end{aligned}$$

Since the argument of the exponential depends only on the differences $(x - x_0)$ and $(y - y_0)$, I is the double convolution of I_0 with the point spread function G. The convolution theorem can be extended to yield

$$\widetilde{I}(k_x, k_y) = 2\pi \widetilde{I}_0(k_x, k_y) \widetilde{G}(k_x, k_y).$$

Since we know G, we can find $\widetilde{G}$ and hence obtain $\widetilde{I}_0 = \widetilde{I}/(2\pi\widetilde{G})$. Fourier inversion then yields the undistorted intensity I_0.

(ii) The electrostatic potential $\Phi(\mathbf{x}')$ at $\mathbf{x}'$ is related to the charge density $\rho(\mathbf{x})$ by

$$\Phi(\mathbf{x}') = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x})\, \mathrm{d}^3\mathbf{x}}{|\mathbf{x}' - \mathbf{x}|}.$$

This is a triple convolution of $\rho(\mathbf{x})$ and the **Green's function** $|\mathbf{x}|^{-1}$. Unfortunately both Φ and $|\mathbf{x}|^{-1}$ fall too slowly at large $\mathbf{x}$ to possess FTs. But if $\rho(\mathbf{x}) = 0$ for $|\mathbf{x}| > R$, say, and we are only interested in Φ for $|\mathbf{x}| < R'$, say ($R \leq R'$ any finite numbers), we can consider the potential Φ_1 that is generated by the Green's function $G(\mathbf{x} - \mathbf{x}') = |\mathbf{x} - \mathbf{x}'|^{-1}$ for $|\mathbf{x} - \mathbf{x}'| < 2R'$ and $G = 0$ otherwise. $\Phi(\mathbf{x}) = \Phi_1(\mathbf{x})$ for $|\mathbf{x}| < R'$, and both Φ_1 and G have FTs. Thus

$$\sqrt{2\pi} 2\epsilon_0 \widetilde{\Phi}_1(\mathbf{k}) = \widetilde{\rho}(\mathbf{k}) \widetilde{G}(\mathbf{k}).$$

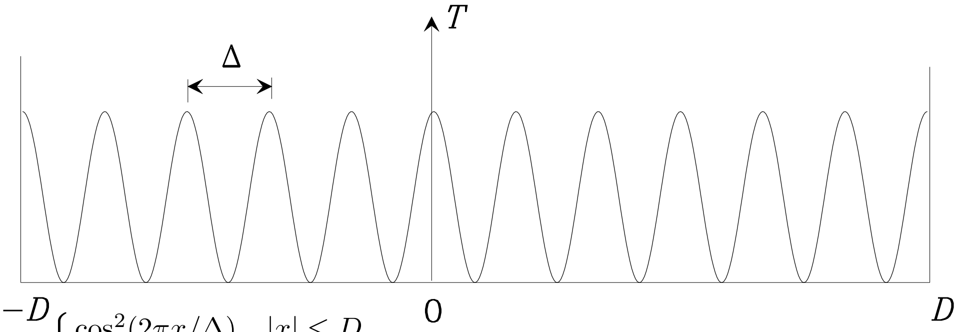
The transform $\widetilde{G}$ need be worked out just once and can then be repeatedly used to find the Φ generated by a constantly varying ρ.

A substantial part of the world's computer capacity is engaged in taking FTs (stricly the numerical cousin of the FT, the discrete FT) as a cheap device for evaluating convolution integrals in connection with these sorts of problems.

Not only do convolutions occur very naturally in many physical problems, but the convolution theorem enables us to Fourier transform functions in our heads.

Example:

The diffraction pattern of a grating is determined by the FT of the transmission $T(x)$ at distance x along the grating. A simple grating might have



i.e. $T(x) = \begin{cases} \cos^2(2\pi x/\Delta) & |x| \le D \\ 0 & |x| > D \end{cases}$. Now if $W(x)$ is defined to be 0 if $|x| > D$ and 1 otherwise,

$$T(x) = \tfrac{1}{4} W(x)\big(e^{4\pi i x/\Delta} + e^{-4\pi i x/\Delta} + 2\big).$$

Since by (F_1) & (F_4)

$$\widetilde{\exp\left(\tfrac{4\pi i x}{\Delta}\right)}(k) = \sqrt{2\pi}\,\delta(4\pi/\Delta - k) = \sqrt{2\pi}\,\delta(k - 4\pi/\Delta),$$

The corollary to Theorem 21 yields

$$\widetilde{T}(x) = \tfrac{1}{4}\sqrt{2\pi}\Big[\widetilde{W}(k) \circ \delta\Big(k - \frac{4\pi}{\Delta}\Big) + \widetilde{W}(k) \circ \delta\Big(k + \frac{4\pi}{\Delta}\Big)\Big] + \tfrac{1}{2}\widetilde{W}(k).$$

But

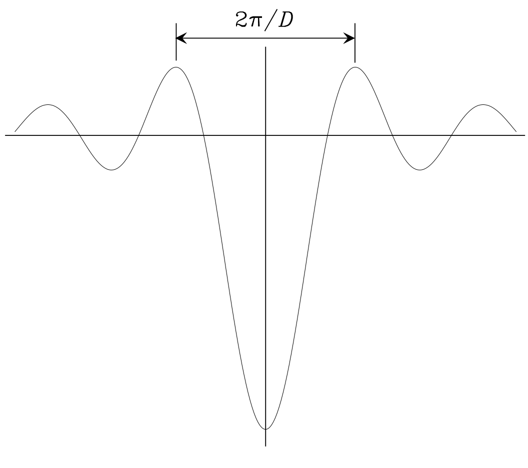
$$\begin{aligned} \sqrt{2\pi}\,\widetilde{W}(k) \circ \delta\Big(k - \frac{4\pi}{\Delta}\Big) &= \int \widetilde{W}(k')\delta\Big(k - \frac{4\pi}{\Delta} - k'\Big)\,\mathrm{d}k' \\ &= \widetilde{W}\Big(k - \frac{4\pi}{\Delta}\Big), \end{aligned}$$

so,

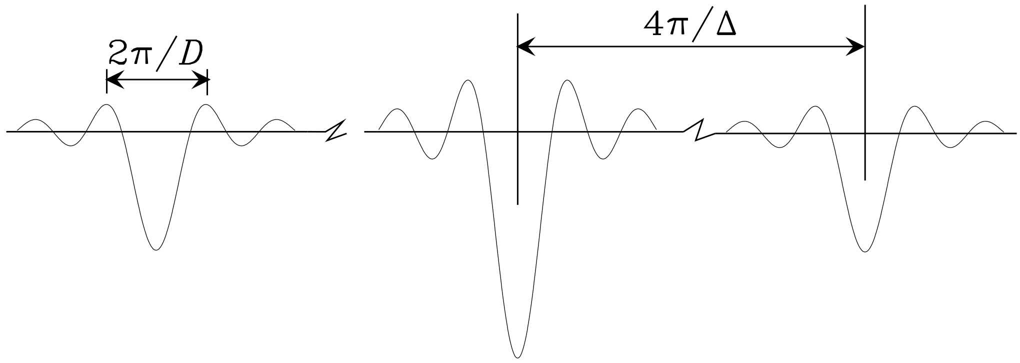
$$\widetilde{T}(k) = \tfrac{1}{4}\Big[\widetilde{W}\Big(k - \frac{4\pi}{\Delta}\Big) + W\Big(k + \frac{4\pi}{\Delta}\Big) + 2\widetilde{W}(k)\Big].$$

Thus $\widetilde{T}$ is the sum of suitably displaced copies of the FT of the window function W. Furthermore,

$$\widetilde{W}(k) = \frac{1}{\sqrt{2\pi}} \int_{-D}^{D} e^{-ikx}\,dx = -\sqrt{\frac{2}{\pi}}\,\frac{\sin(kD)}{k} =$$



and $\widetilde{T}(k) =$



With a little practice one can quickly construct the main features of the FTs of typical functions by representing those functions as products of functions whose FTs one knows. Notice that the large-scale structure of T is represented by the small-scale structure of $\widetilde{T}$, and *vice versa*. Hence the name reciprocal space for k-space.

7 Laplace Transforms

When $|f|$ is not integrable from $-\infty$ to ∞, f does not have an FT. But if f is any sane function such that $f(x) = 0$ for $x < 0$, the function $g_s \equiv e^{-sx} f(x)$ will have an FT provided we make s large enough. We have

$$\begin{aligned}\widetilde{g}_s(k) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x) e^{-(s+ik)x}\,\mathrm{d}x \\ &\equiv \frac{1}{\sqrt{2\pi}} \overline{f}(s+ik), \quad \text{say,}\end{aligned} \qquad (\dagger)$$

where we have defined a new function of a complex variable

$$\overline{f}(z) \equiv \int_0^\infty f(x) e^{-zx}\,\mathrm{d}x. \qquad (\mathrm{Lap}_1)$$

$\overline{f}$ is called the **Laplace transform** of f. The standard Fourier inversion formula (F_3) gives

$$\begin{aligned}f(x') = g_s(x') e^{sx'} &= \frac{e^{sx'}}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{ikx'} \widetilde{g}_s(k)\,\mathrm{d}k \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty e^{(s+ik)x'} \overline{f}(s+ik)\,\mathrm{d}k.\end{aligned}$$

If we write $z = s + ik$, this expression becomes

$$f(x') = \frac{1}{2\pi i} \int_{s-i\infty}^{s+i\infty} \overline{f}(z) e^{zx'}\,\mathrm{d}z \quad (x' > 0). \qquad (\mathrm{Lap}_2)$$

(Lap_1) and (Lap_2) play analogous roles to (F_1) and (F_3); the first converts functions into Laplace transforms and the second turns transforms back into the original functions. However, notice that these formulae, unlike (F_1) and (F_3), are not symmetrical. (Lap_2) is known as the **Bromwich integral** after Mr Bromwich, who introduced this expression in 1916.

Although Laplace transforms are intrinsically functions of a complex variable, by the theory of analytic continuation it is sufficient to specify a transform along the real line $z = s$. Consequently transforms are usually given as functions of the real variable s (or p).

Examples:

(Li) $\overline{1}(s) = \int_0^\infty e^{-sx}\,\mathrm{d}x = \frac{1}{s}$

(Lii)

$$\begin{aligned}\overline{x^n}(s) &= \int_0^\infty x^n e^{-sx}\,\mathrm{d}x = \Big[\frac{x^n e^{-sx}}{-s}\Big]_0^\infty + \frac{n}{s}\int_0^\infty x^{n-1}e^{-sx}\,\mathrm{d}x \\ &= \frac{n}{s}\overline{x^{n-1}}(s) = \frac{n!}{s^{n+1}}.\end{aligned}$$

(Liii)

$$\begin{aligned}\overline{x^\alpha}(s) &= \int_0^\infty x^\alpha e^{-sx}\,\mathrm{d}x = s^{(\alpha+1)}\int_0^\infty t^\alpha e^{-t}\,\mathrm{d}t \\ &\equiv s^{-(\alpha+1)}\Gamma(\alpha+1).\end{aligned}$$

[In a rational notation $\Gamma(\alpha+1) = \alpha!$].

(Liv)

$$\overline{e^\alpha f(x)} = \int_0^\infty e^{-(s-\alpha)x} f(x)\,\mathrm{d}x = \overline{f}(s-\alpha).$$

(Lv)

$$\text{(Li)} + \text{(Liv)} \quad \Rightarrow \quad \overline{e^{\alpha x}} = \frac{1}{s-\alpha}.$$

(Lvi)

$$\begin{aligned}\overline{\cos\omega x} &= \tfrac{1}{2}\big(\overline{e^{i\omega x}} + \overline{e^{-i\omega x}}\big) = \tfrac{1}{2}\Big(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\Big) = \frac{s}{s^2+\omega^2}; \\ \overline{\sin\omega x} &= \tfrac{1}{2i}\big(\overline{e^{i\omega x}} - \overline{e^{-i\omega x}}\big) = \tfrac{1}{2i}\Big(\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\Big) = \frac{\omega}{s^2+\omega^2}.\end{aligned}$$

(Lvii)

$$\begin{aligned}\overline{\frac{\mathrm{d}f}{\mathrm{d}x}} &= \int_0^\infty e^{-sx}\frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x = \big[fe^{-sx}\big]_0^\infty + s\int_0^\infty f(x)e^{-sx}\,\mathrm{d}x \\ &= -f(0) + s\overline{f}(s).\end{aligned}$$

(Lviii)

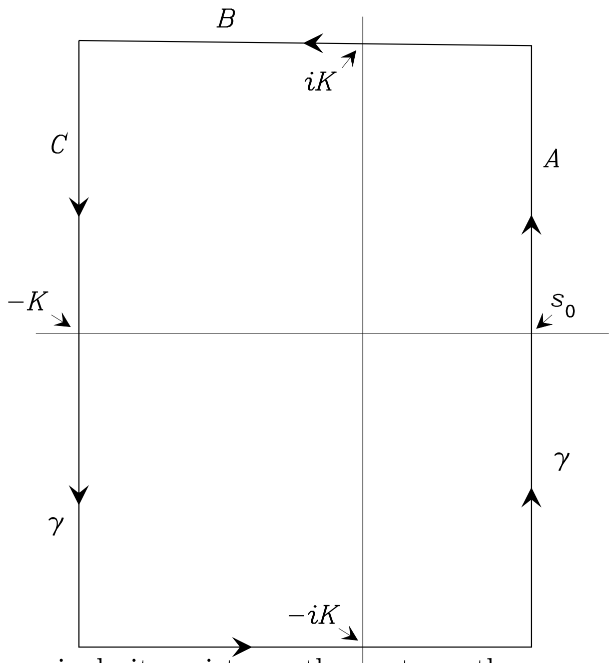
$$\overline{\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}} = -f'(0) + s\overline{\frac{\mathrm{d}f}{\mathrm{d}x}} = -f'(0) - sf(0) + s^2\overline{f}(s).$$

(Lix)

$$\begin{aligned}\overline{xf(x)} &= \int_0^\infty xf(x)e^{-sx}\,\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty f(x)e^{-sx}\,\mathrm{d}x \\ &= -\frac{\mathrm{d}\overline{f}}{\mathrm{d}s}.\end{aligned}$$

The condition for g_s to have an FT $(2\pi)^{-1/2}\overline{f}(z)$ is that $s = \Re\mathrm{e}(z)$ should be large enough that the integral in (Lap$_1$) exists. Clearly, if this condition is to be satisfied for some s_0, then it will be satisfied for all $s > s_0$. Hence f must be non-singular to the right of the line $\Re\mathrm{e}(z) = s_0$. It is not difficult to show that f is analytic in this region. A non-trivial f must have singularities to the left of $\Re\mathrm{e}(z) = s_0$ and these are inherited by the integrand in (Lap$_2$).

Consider the result of integrating the integrand of (Lap$_2$) around the contour



If we pick K such that no sigularity exists on the contour, then as we make K larger and larger, the exponential factor $e^{-Kx'}$ in the integrand makes the contribution C to the integral smaller and smaller. Thus in the limit $K \to \infty$ we may neglect this contribution to the integral. Similarly, the integral along the top is

$$B = \int_{s_0}^{-K} e^{sx'} e^{iKx'} \overline{f}(s + iK)\,\mathrm{d}s.$$

Hence

$$|B| \le \sup_{-K \le s \le s_0} \left|\overline{f}(s + iK)\right| \left|\frac{e^{-Kx'} - e^{s_0 x'}}{x'}\right|.$$

Hence if (as will usually prove to be the case) $\overline{f}(z) \to 0$ as $|z|$ becomes large in the upper half-plane $\Re\mathrm{e}(z) < s_0$, $|B| \to 0$ as $K \to \infty$. Under these circumstances $|D|$ also tends to zero, and thus

$$\begin{aligned} f(x') &= \frac{1}{2\pi i} \int_\gamma e^{zx'} \overline{f}(z)\,\mathrm{d}z \\ &= \sum_{\text{poles}} \text{Residue}\big[e^{zx'} \overline{f}(z)\big]. \end{aligned} \qquad (\text{Lap}_3)$$

Furthermore, if $\overline{f}(z)$ has only simple poles, this simplifies to

$$f(x') = \sum_{\text{poles}} e^{zx'} \times \text{Residue}\big[\overline{f}(z)\big].$$

Note that it does not matter how slowly $\overline{f}(z) \to 0$ for large $|z|$.

Examples:

(i) The LT of $\sin \omega x$ is $\omega/(s^2 + \omega^2)$. The corresponding complex function falls to zero at large $|z|$ and has simple poles at $z = \pm i\omega$ with residues $R_\pm = \pm\frac{1}{2i}$. Hence the Bromwich integral of this transform is

$$\tfrac{1}{2i} e^{i\omega x'} + \tfrac{1}{-2i} e^{-i\omega x'} = \sin \omega x' \quad \text{as expected.}$$

(ii) Consider the transform of e^{-sT}/s. The integrand in the Bromwich integral is in this case $s^{s(t-T)}/s$. If $t > T$ we can complete the Bromwich contour to the left as above. The only singularity, that at the origin, contributes 1 as its residue. Hence for $t > T$ the original function takes value 1. If $t < T$, we can complete the Bromwich contour to the right. Now there is no singularity inside the contour, so the Bromwich integral equals zero. Hence e^{-sT}/s is the transform of

$$f(t) = \begin{cases} 1 & \text{for } t > T \\ 0 & \text{otherwise} \end{cases} \quad \text{(Heaviside step function).}$$

7.1 Solving o.d.e's with Laplace transforms

Laplace transforms turn some linear differential equations into algebraic equations. Consider, for example,

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + y = (1-t)e^{-t} \quad \text{with} \quad y(0) = 1, \quad \left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_0 = 0.$$

We multiply the equation through by e^{-st} and integrate w.r.t t. Using (Lvii), (Lviii), L(iv) and (Lii) this yields

$$s^2\overline{y}(s) - 0 - s + 2(s\overline{y} - 1) + \overline{y} = \frac{1}{s+1} - \frac{1}{(s+1)^2}.$$

Hence

$$y(s) = \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{(s+1)^3} - \frac{1}{(s+4)^4}.$$

Examples (Lii) and (Liv) now imply

$$y(t) = e^{-t} + te^{-t} + \tfrac{1}{2}t^2e^{-t} + \tfrac{1}{6}t^3e^{-t}$$

as we might have shown by elementary methods.

Unfortunately, if the coefficients of the derivatives in an o.d.e. are not constants, the transformed equation is still not algebraic. For example,

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + ty = f(t) \quad \text{with} \quad y(0) = y'(0) = 0$$

leads to

$$s^2\overline{y} + 2s\overline{y} - \frac{\mathrm{d}\overline{y}}{\mathrm{d}s} = \overline{f}.$$

Nevertheless, we still might consider this a step forward since we now have to solve only a first-order equation. On the other hand, if in the original equation the coefficient of y or one of its derivatives had involved t^2, the resulting equation for $\overline{y}$ would have been second-order and would not have represented a significant improvement on the original equation.

7.2 Solving p.d.e's with Laplace transforms

Consider the problem of solving the diffusion equation

$$\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2} \quad \text{subject to} \quad \left\{ \begin{array}{l} \phi(0,t) = 0\ ;\ \phi(a,t) = 0 \quad (0 < a \neq m\pi) \\ \phi(x,0) = \sin x - 2\sin 2x \end{array} \right\}.$$

Multiplying the equation through by e^{-st} and integrating w.r.t. t, we have

$$-\phi(x,0) + s\overline{\phi}(x,s) = \frac{\partial^2\overline{\phi}}{\partial x^2}.$$

With the given condition for $\phi(x,0)$ this becomes

$$\overline{\phi}(x,s) = A(s)e^{\sqrt{s}x} + B(s)e^{-\sqrt{s}x} + \frac{\sin x}{s+1} - 2\frac{\sin 2x}{s+4}. \qquad (\dagger)$$

But transforming the given conditions for $\phi(0,t)$ and $\phi(a,t)$ we have $\overline{\phi}(0,s) = \overline{\phi}(a,s) = 0$, so setting $x = 0$ in (†) we infer $B = -A$. Setting $x = a$ in (†) now leads to

$$0 = 2A\sinh(a\sqrt{s}) + \frac{\sin a}{s+1} - 2\frac{\sin 2a}{s+4}.$$

Thus

$$\overline{\phi}(x,s) = \left(2\frac{\sin 2a}{s+4} - \frac{\sin a}{s+1}\right)\frac{\sinh(x\sqrt{s})}{\sinh(a\sqrt{s})} + \frac{\sin x}{s+1} - 2\frac{\sin 2x}{s+4}.$$

To recover $\phi(x,t)$ we need to find the Bromwich integral of this transform. From $0 < x < a$, it follows that $|\overline{\phi}(x,s)| \to 0$ as $|s| \to \infty$ in the half-plane $\Re\mathrm{e}(s) < 0$. Thus the Bromwich integral can be extended to a closed line-integral and evaluated in terms of the enclosed poles. There are potential singularities at $s = -1$, $s = -4$ and $s = -(m\pi/a)^2$, but since $a \neq m\pi$, the residues at $s = -1$ and $s = -4$ are zero. The residue at $s = -(m\pi/a)^2$ follows from

$$\begin{aligned}\sinh\left[a\sqrt{-\left(\frac{m\pi}{a}\right)^2 + \delta}\right] &= \sinh\left[im\pi\left(1 - \tfrac{1}{2}\delta\left(\frac{a}{m\pi}\right)^2 + \cdots\right)\right] \\ &\simeq -i\cos(m\pi)\sin\left(\frac{\delta a^2}{2m\pi}\right) \simeq i(-1)^{m+1}\frac{\delta a^2}{2m\pi}.\end{aligned}$$

Evaluating the rest of the transform at $s = -(m\pi/a)^2$ and applying (Lap$_3$), we find [since then $\sinh(x\sqrt{s}) = i\sin(m\pi x/a)$]

$$\begin{aligned}\phi(x,t) = \sum_{m=0}^{\infty} 2(-1)^{m+1}\frac{m\pi}{a^2}e^{-(m\pi/a)^2 t} \\ \times \sin\left(\frac{m\pi x}{a}\right)\left[\frac{2\sin 2a}{4-(m\pi/a)^2} - \frac{\sin a}{1-(m\pi/a)^2}\right]\end{aligned}$$

as we might have shown by developing the solution in normal modes.

7.3 Solving integral equations with Laplace transforms

We are all familiar with the importance of differential equations for physics; since Newton showed the way, we have come to consider it natural to formulate as differential equations the laws which link together myriads of phenomena into a common intellectual framework, relying on variation of boundary conditions to account for the diversity of the world we see around us. In effect, fundamental differential equations, such as Maxwell's equations, serve as general rules for the extraction of a particular function $p(x)$ (the prediction) from any given function $g(x)$ (the initial data). However, differential equations are by no means the only device known to mathematics for this purpose. Equally potent are **integral equations** of the form

$$g(x) = \int K(x,x')p(x')\,\mathrm{d}x'.$$

Thus it is interesting to investigate ways of finding a function p which generates a given function g when "folded" with a given **kernel** K according to this last formula.

There are two reasons why integral equations do not feature so largely as differential equations in undergraduate courses: (i) historically, fewer natural laws have been formulated in integral rather than differential form;[6] (ii) both analytically and numerically, integral equations are generally harder to solve than differential equations. One class of integral equations that can be readily solved, is that for which the kernel depends only on the difference $x-x'$ between its arguments and the range of integration over x' is such that

$$g(x)=\int_0^x K(x-x')p(x')\,\mathrm{d}x'.$$

The following is the concept that enables us to solve this kind of equation.

Def:

The **Laplace convolution** of f and g is

$$f\star g\equiv\int_0^t f(t')g(t-t')\,\mathrm{d}t'.$$

Theorem 24:

$$f\star g=g\star f.$$

Proof: Replace t' by $T\equiv t-t'$.◁

Theorem 25 (Laplace convolution):

$$\overline{f\star g}(s)=\overline{f}(s)\overline{g}(s).$$

Proof:

$$\begin{aligned}\overline{f\star g}(s)&=\int_0^\infty \mathrm{d}t\,e^{-st}\int_0^t f(t')g(t-t')\,\mathrm{d}t'\\&=\int_0^\infty e^{-st'}f(t')\,\mathrm{d}t'\int_{t'}^\infty e^{-s(t-t')}g(t-t')\,\mathrm{d}t\\&=\int_0^\infty e^{-st'}f(t')\,\mathrm{d}t'\int_0^\infty e^{-sT}g(T)\,\mathrm{d}T\\&=\overline{f}(s)\overline{g}(s).\triangleleft\end{aligned}$$

Examples:

(i) The general Abel integral equation for y in terms of x is defined to be

$$x(t)=\int_0^t\frac{y(t')\,\mathrm{d}t'}{(t-t')^\alpha}\quad\text{where}\quad 0<\alpha<1.$$

The right side of this equation is the Laplace convolution of $f(t')=y(t')$ and $g(t)=t^{-\alpha}$. Thus the LT of the left side is the product of the LTs of y and $t^{-\alpha}$:

$$\overline{x}(s)=\overline{y}(s)\times\overline{w^{-\alpha}}(s).$$

But by (Liii) $\overline{w^{-\alpha}}(s)=\Gamma(1-\alpha)s^{-(1-\alpha)}$, so

$$\begin{aligned}\overline{y}(s)&=\frac{\overline{x}(s)}{\Gamma(1-\alpha)s^{-(1-\alpha)}}\\&=\frac{s}{\Gamma(\alpha)\Gamma(1-\alpha)}\overline{x}(s)\Gamma(\alpha)s^{-\alpha}.\end{aligned}$$

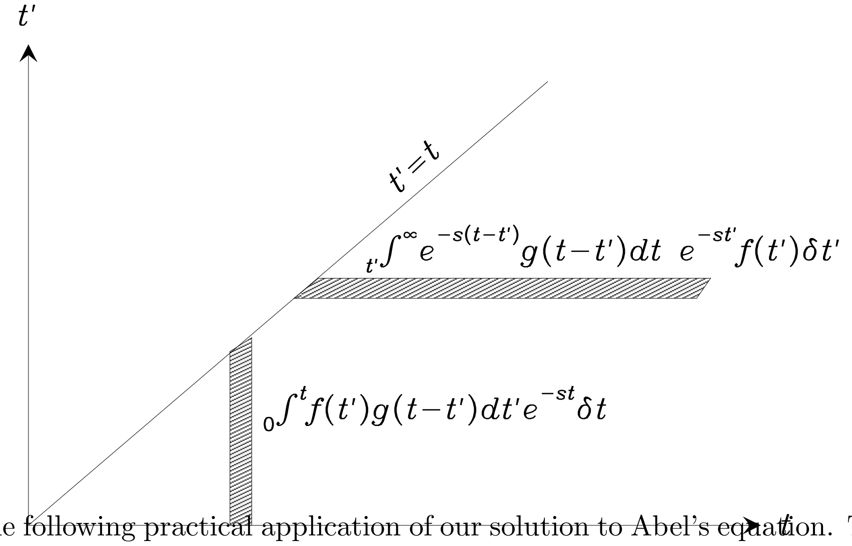
[6] The same law can often, but not always, be formulated in both forms.

Using the convolution theorem again we have

$$\overline{y}(s) = \frac{s}{\Gamma(\alpha)\Gamma(1-\alpha)} \overline{\left[\int_0^t \frac{x(t')}{(t-t')^{1-\alpha}}\,\mathrm{d}t'\right](s)}.$$

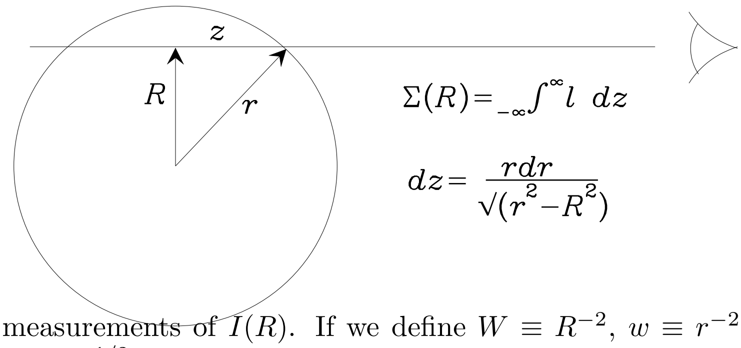
Using (Lvii) and performing the inverse transform, we now have

$$y(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{x(t')}{(t-t')^{1-\alpha}}\,\mathrm{d}t',$$



(ii) Consider the following practical application of our solution to Abel's equation. The surface brightness $I(R)$ due to a spherical star-cluster in in which the volume luminosity density is $j(r)$ is

$$I(R) = 2\int_R^\infty \frac{j(r)}{\sqrt{r^2-R^2}} r^2\,\mathrm{d}r$$



We wish to deduce $j(r)$ from measurements of $I(R)$. If we define $W \equiv R^{-2}$, $w \equiv r^{-2}$, $I_1(W) \equiv I(W^{-1/2})$ and $j_1(w) \equiv j(w^{-1/2})$, then we have

$$I_1(W)W^{-1/2} = \int_0^W \frac{(j_1(w)w^{-3/2})}{\sqrt{W-w}}\,\mathrm{d}w,$$

which we recognize as an Abel equation with $\alpha = \frac{1}{2}$. Now $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, so

$$\frac{j_1}{w^{3/2}} = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}w}\int_0^w \frac{I_1(W)}{\sqrt{W(w-W)}}\,\mathrm{d}W,$$

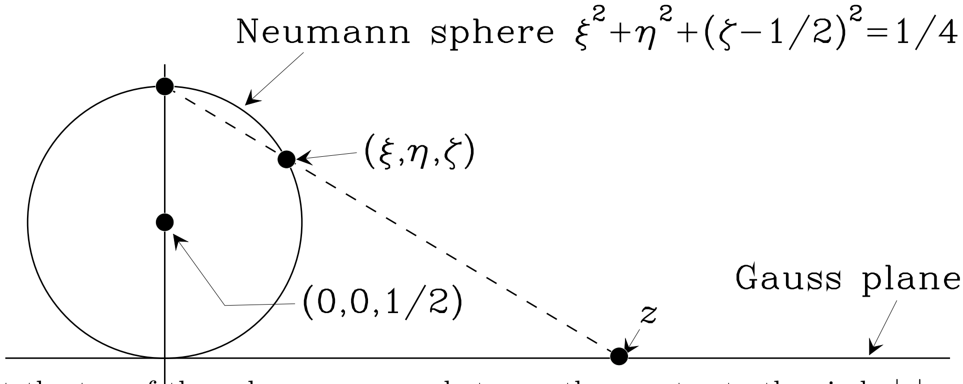
In terms of the original variables this is

$$j(r) = -\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^\infty \frac{r}{R}\frac{I(R)\,\mathrm{d}R}{\sqrt{R^2-r^2}}.$$

Appendix A: Structure of the Extended Complex Plane

(a) Neumann sphere

One can visualize the spherical nature of the complex plane with the aid of stereographic projection:



The point at the top of the sphere corresponds to ∞, the equator to the circle $|z| = 1$. In general

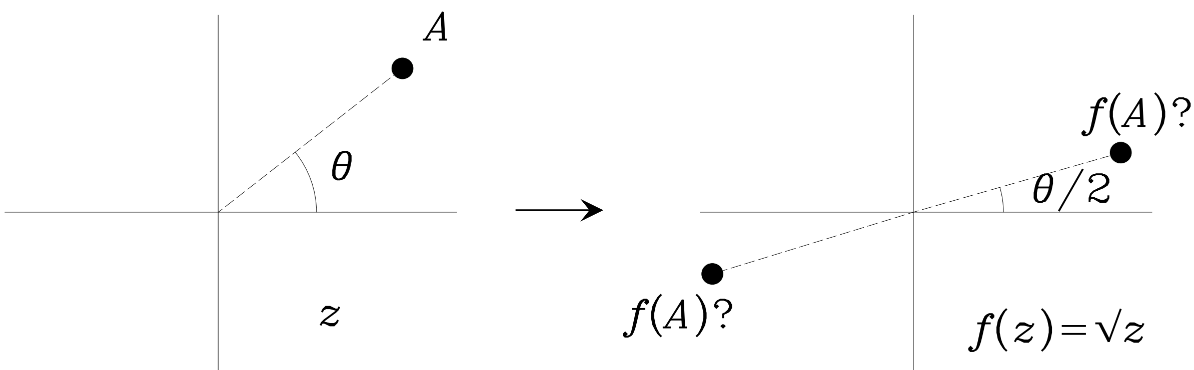
$$\xi = \frac{\Re e(z)}{1+|z|^2} \quad ; \quad \eta = \frac{\Im m(z)}{1+|z|^2} \quad ; \quad \zeta = \frac{|z|^2}{1+|z|^2}.$$

Each region of the complex plane is mapped **conformally** [i.e. angles preserved but subject to shrinkage by a factor $(1-\zeta)$] onto the surface of the Neumann sphere.

The transformation $z \to w \equiv 1/z$ turns the Neumann sphere upside-down.

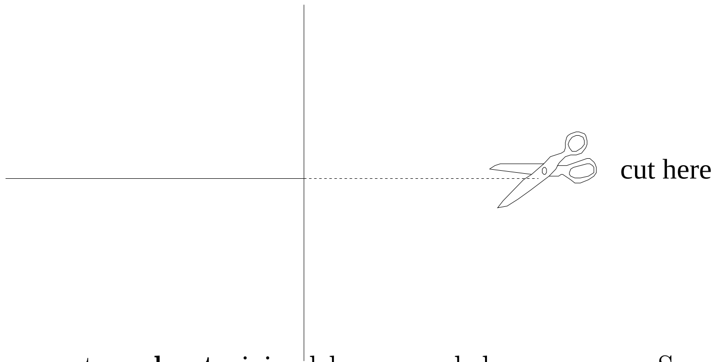
(b) Riemann surfaces

If a function $f(z)$ of the complex variable z is an assignment of a complex number to each point of the complex plane, what are we to make of $f(z) = z^{1/2}$? $z = re^{i\theta}$ implies $f = \sqrt{r}e^{i\theta/2}$, so increasing θ by 2π changes the sign of f.



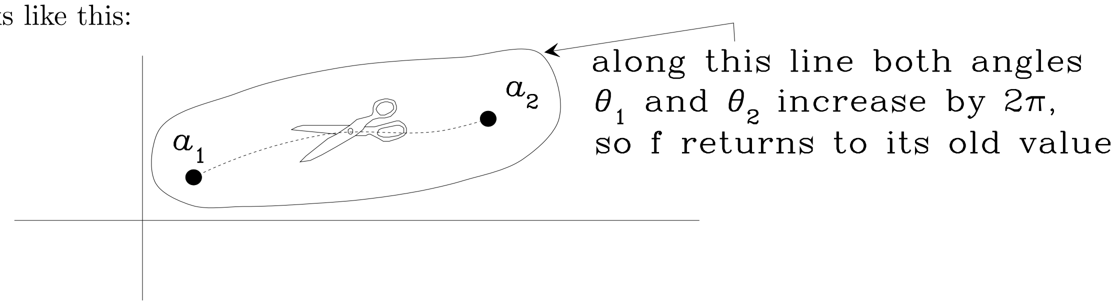
So in this case f would appear not to be a single-valued function of position on the complex plane.

Riemann's solution to this conundrum was to cut the plane, say along the positive real axis.

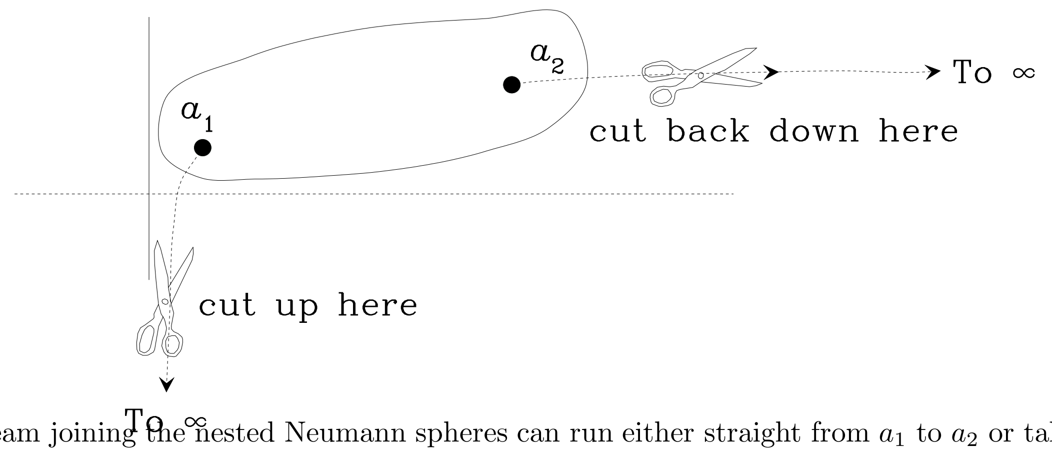


Then the "plane" becomes two **sheets** joined by up and down ramps. Suppose we start on the bottom sheet and move from a point on the positive real axis in the direction of increasing θ. At $\theta = 2\pi$ the up ramp takes us to the top sheet. When $\theta = 4\pi$ the down ramp returns us to the bottom sheet. On the Neumann sphere, Riemann's cut runs half way around the sphere from bottom to top. Thus Riemann's surgery turns the Neumann sphere into a pair of concentric spheres sewn together down a vertical seam.

Trickier functions than $f(z) = z^{1/2}$ require more elaborate needlework. Consdider, for example, $f(z) = \sqrt{(z - a_1)(z - a_2)} = \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2}$, where $z - a_k = r_k e^{i\theta_k}$. A suitable cut looks like this:



However, a perfectly satisfactory alternative would be the following:



i.e. the seam joining the nested Neumann spheres can run either straight from a_1 to a_2 or take the long route over the top.

Points where ramps joining different sheets of a Riemann surface end are called **branch points**.

Appendix B: Some Applications of ($\mathbf{C}_n$)

The following shows that every bona-fide analytic function has to freak out from time to time.

Theorem 26 (Liouville): *If $f(z)$ is analytic for all finite z and if $|f(z)| < F$ for every z, then $f =$ constant.*

Proof: We evaluate $f'(z_0)$ by evaluating the integral (C_1) around a circle of radius r. Evidently $|f'(z_0)| \leq F/r$, where r is as big as we please. Hence $f'(z_0) = 0 \quad \forall \quad z_0$, and f is constant.◁

Corollary: (fundamental theorem of algebra) *Every complex polynomial $P(z)$ of degree $n > 0$ has at least one root.*

Proof: Suppose $P(z)$ had no root. Then $1/P(z)$ would be analytic everywhere. Furthermore, it is easy to show that $\lim_{z\to\infty}[1/P(z)] = 0$. Hence we would have that $|1/P(z)| < F$ for some F and $1/P(z)$ would be constant. But this contradicts the statement that P is of degree greater than zero. Hence P must have a root.◁

The following shows that f drifts steadily and ineluctably towards its next singularity:

Theorem 27 (maximum modulus): *If f is analytic in D and on its boundary γ, then unless f is constant in D, $|f|$ attains its maximum value on γ and not in the interior of D.*

Proof: Suppose the interior point z_0 of D is such that $|f(z_0)|$ is at least as great as at any other point in D. Then put $z = z_0 + re^{i\theta}$ into Cauchy's formula (C_0). Dividing through by $f(z_0)$ we obtain

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(z)}{f(z_0)}\,\mathrm{d}\theta \tag{†}$$

Hence $1 \leq \frac{1}{2\pi}\int |f(z)/f(z_0)|\,\mathrm{d}\theta \leq 1$, where the second inequality follows from our hypothesis that $f(z_0) \geq f(z)$. Thus $f(z)/f(z_0) = e^{i\psi(\theta)}$ and (†) becomes

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \{\cos[\psi(\theta)] + i\sin[\psi(\theta)]\}\,\mathrm{d}\theta,$$

from which we infer $\cos[\psi(\theta)] = 1 \quad \forall \quad \theta$, and thus $f(z) = f(z_0)$ on the circle of radius r. It now follows easily that f must be constant throughout D.◁

Corollary: *If f is analytic in D and on its boundary γ, then $|f|$ cannot have a minimum in D other than $|f| = 0$.*

Proof: Apply the last theorem to $1/f$.◁

Note:

This last result helps us to locate zeros of an analytic function f, since we only have to show that at some point z_0, $|f(z_0)| < |f(z)|$ for any z on the boundary of a domain D to be sure that D contains a zero of f.

Cauchy's theorem has a converse:

Theorem 28 (Morera): *If f is continuous in a simply connected domain D and $\oint f(z)\,\mathrm{d}z = 0$ around every simple closed curve in D, then f is analytic in D.*

Proof: Choose some $z_1 \in D$. Then the integral

$$F(z) \equiv \int_{z_1}^{z} f(z')\,\mathrm{d}z'$$

defines a single-valued function F. Furthemore,

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0}\int_{z_0}^{z} [f(z') - f(z_0)]\,\mathrm{d}z'. \tag{‡}$$

If we choose to evaluate the integral along the straight line joining z_0 to z, we have

$$\begin{aligned}\left|\frac{1}{z - z_0}\int_{z_0}^{z}[f(z') - f(z_0)]\,\mathrm{d}z'\right| &\leq \frac{1}{|z - z_0|}\int_{z_0}^{z} |f(z') - f(z_0)|\,|\mathrm{d}z'| \\ &\leq \max_{z'\in(z,z_0)} |f(z') - f(z_0)|.\end{aligned}$$

But f is continuous, so as z approaches z_0 the quantity on the right of this inequality tends to zero. Hence when we let $z \to z_0$ in (‡), we deduce that F is analytic, with $F'(z_0) = f(z_0)$. The theorem now follows from our earlier demonstration (Theorem 11) that the derivative of an analytic function is itself analytic.◁

Appendix C: Fourier Series

Let $f(\theta)$ be a real-valued function of period 2π. Then it is plausible that f can be expressed in the form

$$f(\theta) = \sum_{n=-\infty}^{\infty} E_n e^{in\theta}, \tag{†}$$

where the E_n are suitable complex numbers. Multiplying through by $e^{-im\theta}$ and integrating over θ, we find

$$E_m = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-im\theta}\,\mathrm{d}\theta. \tag{‡}$$

If we are to have $f^*(\theta) = f(\theta)$ for all θ, we require $E_n^* = E_{-n}$. If we write

$$E_n = u_n + iv_n \quad (u_n, v_n \text{ real}), \text{ then } \quad \begin{array}{c} u_n = u_{-n} \\ v_n = -v_{-n} \end{array} \quad \text{and} \quad \begin{array}{c} u_n = \frac{1}{2}(E_n + E_{-n}) \\ v_n = \frac{1}{2i}(E_n - E_{-n}) \end{array}.$$

Substituting these results into (†), we find

$$\begin{aligned} f(\theta) &= \sum_{-\infty}^{\infty}(u_n + iv_n)e^{in\theta} \\ &= u_0 + \sum_{n=1}^{\infty}\big[u_n(e^{in\theta} + e^{-in\theta}) + iv_n(e^{in\theta} - e^{-in\theta})\big] \\ &= u_0 + \sum_{n=1}^{\infty}\big[B_n \cos n\theta + A_n \sin n\theta\big], \end{aligned}$$

where $B_n \equiv 2u_n$ and $A_n \equiv -2v_n$. The real and imaginary parts of equation (‡) now yield

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,\mathrm{d}\theta \quad \text{and} \quad A_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,\mathrm{d}\theta,$$

respectively.

Appendix D: Proof of Theorem 19

First we need a definition and a lemma.

Def:

A function $f(x)$ is of **bounded variation** in $[a, b]$ iff $\exists$ a number F such that for all possible subdivisions $a = x_0 < x_1 < \ldots < x_n = b$ of $[a, b]$,

$$|f(x_0) - f(x_1)| + |f(x_1) - f(x_2)| + \cdots + |f(x_{n-1}) - f(x_n)| \le F.$$

Clearly a function of bounded variation can have at most a finite number of discontinuities.

Lemma (Riemann): *If f is of bounded variation in $[a,b]$, then*

$$\lim_{A\to\infty}\left[\int_a^b f(x)\sin(Ax)\,\mathrm{d}x\right]=0.$$

The proof of this proposition is tedious. The general idea is that for large A nearly everywhere $f(x+\pi/A)\simeq f(x)$ and so contributions from neighbouring half-waves cancel more and more exactly as $A\to\infty$.

Proof of Theorem 19: From the definition (F_1) of an FT and the integrability of $|f|$ it follows that $\widetilde{f}(k)$ exists. If we substitute (F_1) into (F_2) we obtain a double integral I in the (x,k) plane. If I exists, it must differ from

$$I_A(x')\equiv\int_{-A}^{A}\mathrm{d}k\int_{-\infty}^{\infty}f(x)\,e^{ik(x'-x)}\,\mathrm{d}x$$

by an amount $\epsilon(A)$ that can be made as small as we please by choosing A sufficiently large. Since the inner integral is absolutely convergent, for any finite A we may interchange the order of integration:

$$\begin{aligned}I_A(x')&=\int_{-\infty}^{\infty}\mathrm{d}x\,f(x)\int_{-A}^{A}e^{ik(x'-x)}\,\mathrm{d}k\\&=2\int_{-\infty}^{\infty}f(x)\frac{\sin[A(x'-x)]}{x'-x}\,\mathrm{d}x.\end{aligned}\qquad(\dagger)$$

We break the range of integration up into three parts: $(-\infty,x'-\delta]$, $(x'-\delta,x'+\delta)$, $[x'+\delta,\infty)$, where $\delta>0$. In the first and last intervals $f(x)/(x'-x)$ is of bounded variation, so the integrals over these intervals tend to zero as $A\to\infty$ by Riemann's lemma. In terms of $y\equiv A(x-x')$ the integral over the central interval is

$$2\int_{-A\delta}^{A\delta}f\Big(x'+\frac{y}{A}\Big)\frac{\sin y}{y}\,\mathrm{d}y=2\int_0^{A\delta}\Big[f\Big(x'-\frac{y}{A}\Big)+f\Big(x'+\frac{y}{A}\Big)\Big]\frac{\sin y}{y}\,\mathrm{d}y.$$

Since f can have only a finite number of discontinuities, the quantity in $[\ldots]$ tends as $A\to\infty$ to the well-defined limit on the left of (F_2), and we know that $\int_0^\infty \sin y/y\,\mathrm{d}y=\pi/2$. Hence the double integral I that is obtained by substituting (F_1) into (F_2) exists, and the theorem follows.