Theoretical Physics Major Option MT 2002

Scattering Theory and Relativistic Quantum Mechanics I.J.R. Aitchison

Problem Set 1: Scattering Theory

1 The potential for 'hard sphere' scattering is $V(r) = \infty$ for $r < r_0$ and $V(r) = 0$ for $r > r_0$.
(i) Show that the s-wave phase shift $\delta_0(k)$ is given by $\delta_0(k) = -kr_0$, and calculate the total $\ell = 0$ cross section.

(ii) The 'reduced' radial wave function $u_\ell(r)$ is defined via $\psi(r, \theta, \phi) = R_\ell(r) Y_{\ell,m}(\theta, \phi)$, and $R_\ell(r) = u_\ell(r)/r$. Find the equation satisfied by u_ℓ, and show that the radial wave equation for $r > r_0$ and $\ell = 1$ is satisfied by the wave function

$$u_1(r) = C\left[\frac{\sin kr}{kr} - \cos kr + a\left(\frac{\cos kr}{kr} + \sin kr\right)\right]$$

where C and a are constants. Show that the definition of the p-wave phase shift $\delta_1(k)$ implies that $\tan \delta_1(k) = a$, and determine a from the condition satisfied by $u_1(r)$ at $r = r_0$.

Show that, as k approaches zero, $\delta_1(k)$ behaves like $(kr_0)^3$.

2 The scattering amplitude $f(\theta, k)$ has the partial wave expansion

$$f(\theta, k) = \frac{1}{k}\sum_\ell e^{i\delta_\ell} \sin \delta_\ell (2\ell + 1) P_\ell(\cos\theta)$$

where the symbols have their usual meaning.

(i)Show

(a) $\sigma_{tot} = \sum_\ell \sigma_\ell$ where $\sigma_\ell = \frac{4\pi}{k^2}\sin^2\delta_\ell.(2\ell+1)$

(b) $\sigma_{tot} = \frac{4\pi}{k}\mathrm{Im} f(0, k)$.

(ii) In the 'Fermi theory' of lepton-lepton scattering the s-wave amplitude $a_0 \equiv e^{i\delta_0}\sin\delta_0/k$ for the scattering process

$$\bar{\nu}_\mu + \mu^- \to \bar{\nu}_e + e^-$$

is given approximately by $a_0^F \approx G_F k$ where k is the centre of mass momentum and G_F is Fermi's constant, with the value (in units $\hbar = c = 1$)

$$G_F \approx \frac{10^{-5}}{M_p^2}$$

where M_p is the proton mass. Assuming that k in part (i) can also be taken to be the CM momentum (in units $c = \hbar = 1$) estimate the CM energy (neglecting lepton masses) at which unitarity is violated in the theory. What actually happens? (Note: $\hbar \approx 200$(MeV/c)fm).

3 The momentum-space Green function for the TISE with outgoing wave boundary conditions is

$$\langle \boldsymbol{k}' | \frac{1}{E - \hat{H}_0 + i\epsilon} | \boldsymbol{k}'' \rangle = \frac{1}{E - \frac{\hbar^2 (k'')^2}{2m} + i\epsilon} \delta(\boldsymbol{k}' - \boldsymbol{k}''),$$

where $E = \frac{\hbar^2 k^2}{2m}$. Show that the corresponding coordinate space Green function is given by

$$\langle \boldsymbol{r} | \frac{1}{E - \hat{H}_0 + i\epsilon} | \boldsymbol{r}' \rangle = -\frac{m}{2\pi\hbar^2} \frac{e^{ik|\boldsymbol{r} - \boldsymbol{r}'|}}{|\boldsymbol{r} - \boldsymbol{r}'|}.$$

4 L_x is a linear ordinary differential operator and $\psi(x)$ satisfies the equation $L_x \psi(x) = g(x)$ where $g(x)$ is a known function. The functions $\phi_n(x)$ form a complete orthonormal set of eigenfunctions of L_x, with eigenvalues λ_n:

$$L_x \phi_n(x) = \lambda_n \phi_n(x).$$

Expanding $\psi(x)$ as a linear combination of the $\phi_n(x)$ via

$$\psi(x) = \sum_n a_n \phi_n(x),$$

show that

$$a_m = \frac{1}{\lambda_m} \int \phi_m^*(y) g(y) dy.$$

Deduce that the Green function for the equation $L_x \psi = g$ must be

$$G(x, y) = \sum_n \frac{\phi_n(x) \phi_n^*(y)}{\lambda_n} = \sum_n \frac{\langle x|n\rangle \langle n|y\rangle}{\lambda_n}.$$

Verify that this is correct by applying the operator L_x to $G(x, y)$.

Use this approach to find the coordinate space Green function for the TISE (you may quote any result you like from question 3).

5 (i) In first order time-dependent perturbation theory, the transition probability per unit time for a particle in the momentum eigenstate $| \boldsymbol{k}_i \rangle$ to scatter, via the time-independent potential $V(r)$, into the momentum eigenstate $| \boldsymbol{k}_f \rangle$ is

$$\int 2\pi \mid \langle \boldsymbol{k}_f \mid V \mid \boldsymbol{k}_i \rangle \mid^2 \delta(E_i - E_f) \frac{d^3\mathbf{k}_f}{(2\pi)^3}$$

where units $\hbar = 1$ are used. Writing

$$d^3\mathbf{k}_f = \mid \mathbf{k}_f \mid^2 d \mid \mathbf{k}_f \mid d\Omega \quad \text{and} \quad E_f = \mathbf{k}_f^2/2m$$

show that $d^3\mathbf{k}_f = m \mid \mathbf{k}_f \mid dE_f d\Omega$, and hence show that the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2} \mid \langle f \mid V \mid i \rangle \mid^2 .$$

(ii) Show that

$$\int e^{i\boldsymbol{q}\cdot\boldsymbol{r}} \frac{e^{-r/a}}{r} d^3\boldsymbol{r} = \frac{4\pi}{\boldsymbol{q}^2 + a^{-2}}.$$

By considering the limit $a \to \infty$, derive the Rutherford scattering cross section formula

$$\left(\frac{d\sigma}{d\Omega}\right)_R = \frac{\hbar^2 c^2 \alpha^2}{16 E^2 \sin^4 \theta/2}$$

using Born Approximation for the Coulomb potential for two charges, one $+e$ and one $-e$; here $E = \frac{1}{2}mv^2$ where v is the speed of the incoming particle and m is its mass; also, α is the fine structure constant and θ is the scattering angle (ie the angle between the incident wave vector and the final wave vector).