Theoretical Physics Major Option HT 2003

Scattering Theory and Relativistic Quantum Mechanics I.J.R. Aitchison

Problem Set 2: Relativistic Quantum Mechanics

1 A 4-D Green function for the scalar wave equation is defined (with $c = 1$) by

$$\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2\right) G(\boldsymbol{r}-\boldsymbol{r}', t-t') = \delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t').$$

(i) Show that the Fourier transform of G with respect to the variable $t - t'$, namely

$$\text{(A)} \qquad \overline{G}(\boldsymbol{r}-\boldsymbol{r}', \omega) = \int G(\boldsymbol{r}-\boldsymbol{r}', t-t')e^{i\omega(t-t')}d(t-t'),$$

satisfies the equation

$$(\boldsymbol{\nabla}^2 + \omega^2)\overline{G}(\boldsymbol{r}-\boldsymbol{r}', \omega) = -\delta(\boldsymbol{r}-\boldsymbol{r}').$$

Show further that the Fourier transform of $\overline{G}$ with respect to the variable $\boldsymbol{r}-\boldsymbol{r}'$, namely

$$\text{(B)} \qquad \overline{\overline{G}}(\boldsymbol{p}, \omega) = \int \overline{G}(\boldsymbol{r}-\boldsymbol{r}', \omega)e^{-i\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}')}d^3(\boldsymbol{r}-\boldsymbol{r}'),$$

is given by $\overline{\overline{G}}(\boldsymbol{p}, \omega) = \frac{1}{\boldsymbol{p}^2-\omega^2}$.

(ii) Show from the inverse of (B) that

$$\overline{G}(\boldsymbol{r}-\boldsymbol{r}', \omega) = \frac{1}{16\pi^2 i|\boldsymbol{r}-\boldsymbol{r}'|}\int_{-\infty}^{\infty} dp \left[e^{ip|\boldsymbol{r}-\boldsymbol{r}'|} - e^{-ip|\boldsymbol{r}-\boldsymbol{r}'|}\right]\left(\frac{1}{p-\omega} + \frac{1}{p+\omega}\right),$$

where $p = |\boldsymbol{p}|$.

(iii) Evaluate $\overline{G}(\boldsymbol{r}-\boldsymbol{r}', \omega)$ by contour integration, *assuming* that $\omega \to \omega + i\epsilon$ is the correct way to deal with the singularities. Show that this leads to

$$\overline{G}(\boldsymbol{r}-\boldsymbol{r}', \omega) = \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}e^{i\omega|\boldsymbol{r}-\boldsymbol{r}'|}.$$

Hence show that

$$G(\boldsymbol{r}-\boldsymbol{r}', t-t') = \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\delta(t' - t_{\text{ret}})$$

where the "retarded time" t_{ret} is given by

$$t_{\text{ret}} = t - |\boldsymbol{r}-\boldsymbol{r}'|.$$

(iv) Solve the equation

$$\Box\phi(\boldsymbol{r}, t) = \rho(\boldsymbol{r}, t)/\epsilon_0$$

and explain the answer physically; in particular, why should the choice $\omega \to \omega - i\epsilon$ be ruled out in part (iii)?

2 (i) The Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that $\sigma_2 \boldsymbol{\sigma}^* \sigma_2 = -\boldsymbol{\sigma}$ where * denotes complex conjugation.

(ii) The two-component spinor ϕ satisfies the wave equation (with $\hbar = c = 1$)

$$i\frac{\partial \phi}{\partial t} = -i\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}\phi + im\sigma_2\phi^*$$

or equivalently

$$(\partial_t + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\phi = m\sigma_2\phi^*. \qquad (1)$$

The complex conjugate equation is

$$(\partial_t + \boldsymbol{\sigma}^*.\boldsymbol{\nabla})\phi^* = -m\sigma_2\phi. \qquad (2)$$

By multiplying (2) from the left by σ_2 show that

$$(\partial_t - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\sigma_2\phi^* = -m\phi$$

and hence show that ϕ satisfies

$$(\partial_t^2 - \boldsymbol{\nabla}^2)\phi = -m^2\phi.$$

What is the significance of this equation?

(iii) Find the components $\begin{pmatrix} c \\ d \end{pmatrix}$ in terms of $\begin{pmatrix} a \\ b \end{pmatrix}$ such that

$$\phi = e^{-ip.x}\begin{pmatrix} a \\ b \end{pmatrix} + e^{ip.x}\begin{pmatrix} c \\ d \end{pmatrix}$$

is a solution of (1), where $p^\mu = (E, \boldsymbol{p})$ with $E = +\sqrt{\boldsymbol{p}^2 + m^2}$.

(iv) Discuss the covariance of equation (1) under (a) 3-D rotations (b) 4-D boosts (pure velocity transformations).

(v) Show that

$$\partial_t\rho + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

where $\rho = \phi^\dagger\phi$ and $\boldsymbol{j} = \phi^\dagger\boldsymbol{\sigma}\phi$.

(vi) What kind of particles might the wave equation (1) represent?

3 A rotation through an angle α about the x-axis is specified by a change in coordinates

$$x' = x, \quad y' = \cos\alpha\ y + \sin\alpha\ z, \quad z' = -\sin\alpha\ y + \cos\alpha\ z.$$

Under this transformation of coordinate axes, a Dirac spinor ω transforms by

$$\omega' = \exp(i\Sigma_x \alpha/2)\omega$$

where

$$\Sigma_x = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}.$$

Verify that

$$\omega^\dagger \alpha_y \omega$$

transforms as the y-component of a vector under this transformation, where α_y is the Dirac matrix $\begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}$.

(ii) Verify that $\overline{\psi}\psi$ (defined as $\psi^\dagger \beta \psi$) is invariant under the above transformation, and also under the 'boost' transformation

$$\omega' = \exp(\alpha_x \vartheta/2)$$

where α_x is the Dirac matrix $\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}$.

4 A positive energy spin-1/2 particle of mass m is incident from the left ($z < 0$) on a one-dimensional barrier of height V which exists in the region $z \geq 0$. What boundary condition should you impose on the wavefunction at $z = 0$? [Hint: consider ρ and $\boldsymbol{j}$.] Write down the appropriate Dirac equation and solve it, using your boundary condition. Show that if V is *large* enough the reflected current is larger than the incident current. Suggest a physical interpretation of this result.