Quantum Mechanics M.T. 2000

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Problem Set No. 1

There are likely to be more problems in this set than many students could comfortably tackle for one tutorial, and besides some of the material will not be new to those who took Physics 2 in Prelims. Those who have done Physics 2 should concentrate first on problems which go beyond that, i.e numbers 7, 9, 10, 11 and 12, and then do any of the others that they haven't tried before. Those who did not take Physics 2 should try to do as many of the set as they can, omitting numbers 6, 10 (iv) and 11 (iii)-(v) if out of time.

1. Quanta of radiation

   A beam of ultraviolet light of wavelength $\lambda = 124$ nm and intensity 1.6×10^{-12} W is suddenly turned on and falls on a metal surface, ejecting electrons through the photo-electric effect. The beam has a cross-sectional area of 10^{-4}m^2 and the work function of the metal is 5 eV. Estimate the time delay before the first photoelectron appears by the following approaches:
   (a) A crude estimate using classical physics is to calculate the time needed for the work function energy to be accumulated over the area of one atom (radius $\approx 10^{-10}$ m).
   (b) Lord Rayleigh showed that this is a bit pessimistic and that a better estimate of the effective area (or *scattering cross section*) that the atom presents is λ^2. Use this to revise your estimate of the time delay.
   (c) In the quantum approach, emission can occur as soon as the first photon arrives (why?). To obtain a time delay that can be compared to these classical estimates, calculate the average time interval between arrival of successive photons.

2. Matter waves

   The single-slit diffraction pattern for a monochromatic wave of wavelength λ incident normally on a narrow slit of width a is described (in the "Fraunhofer region") by the intensity $I(\theta)$:

$$I(\theta) = I(0)\frac{\sin^2[(\pi a/\lambda)\sin\theta]}{[(\pi a/\lambda)\sin\theta]^2}$$

   where θ is the deflection angle perpendicular to the incident wavefront.

(i) What is the value of $I(\theta)$ as $\theta \to 0$?

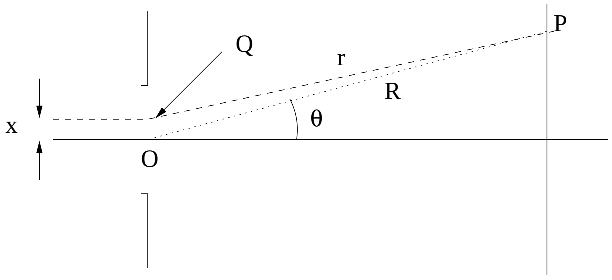
(ii) Sketch the form of $I(\theta)$ versus θ for the particular case $\lambda = a/2$. How does the sketch change as λ decreases? Show that the intensity peak centred on $\theta = 0$ falls to half its central intensity at

$$\theta = \sin^{-1}(0.443\lambda/a).$$

(iii) Reactors provide high fluxes of neutrons with energies $\sim 10^{-2}$ or $10^{-3} eV$. For neutrons with an energy of 4.18 x 10^{-3} eV, what is (a) their speed, (b) their wavelength?

(iv) In an experiment (C. G. Schull, Physical Review 179 (1969) 752-754) neutrons of energy 4.18 x 10^{-3} eV were incident on a slit of width 5.6 μm. It was found that the full width at half maximum of the central intensity peak (measured downstream from the slit) was 15.4 arc seconds. Is this consistent with the above diffraction formula $I(\theta)$?

(v) A simple derivation of $I(\theta)$ can be given as follows. Referring to the



diagram, the amplitude at the detection point P can be regarded as sum of the amplitudes for particles to arrive at P having started from any point Q in the aperture of the slit. The amplitude to arrive at P having started at Q is $\frac{e^{ik.PQ}}{PQ} = \frac{e^{ikr}}{r}$ where r = PQ. Let OP = R.

Show that $r \approx R - x\sin\theta$ where θ is the angle made by OP with the horizontal and $x = OQ$. Setting $r = R$ in the denominator of e^{ikr}/r, and retaining the x-dependence only in the phase factor, we can write

$$e^{ikr}/r \approx (e^{ikR}/R).e^{-ikx\sin\theta}.$$

"Summing" these amplitudes over x amounts to integrating over x from $\frac{-a}{2}$ to $\frac{a}{2}$. Show that this leads to the above formula for $I(\theta)$.

3. <u>The importance of h (or $\hbar$).</u>

   Planck's constant h is equal to 6.6×10^{-34} Js, to two significant figures. Retaining only two significant figures, evaluate $\hbar$ ($= h/2\pi$)in the following units: (i) eV . s (ii) eV/c . Å(iii) MeV/c . fm (1 Å= 10^{-10}m, 1 fm = 10^{-15} m). What physical systems are these units each likely to be useful for?

4. A system (e.g. a mechanical watch) has moving parts of size d and mass m, and the movement occurs on a timescale τ.
   (i) Construct a quantity having the dimensions of *action* from d, m and τ.
   (ii) For the particular case $d = 10^{-4}$m, $m = 10^{-4}$ kg and $\tau = 1$ s, evaluate your action expression and compare it to $\hbar$. What do you conclude?
   (iii) Now take d to be a typical atomic size, and m to be the electron mass. Find τ such that your action quantity is equal to $\hbar$ in this case. Defining an average velocity by $v = d/\tau$, calculate the corresponding kinetic energy of the electron, in eV.

5. Microscopes using waves with wavelength λ can resolve objects roughly as small as λ but no smaller. Determine the kinetic energy of electrons in an electron microscope needed to resolve (a) a DNA molecule (10^{-8}m) (b) a proton (10^{-15} m). [Use the de Broglie relation $\lambda = h/p$; in each case consider whether you should use the non-relativistic expression $T = p^2/2m$ for the kinetic energy T, or the relativistic one $T + mc^2 = [m^2c^4 + c^2p^2]^{1/2}$.

6. (i) Two particles of mass M interact gravitationally. When they are a certain distance l apart, their mutual gravitational potential energy is equal to the rest mass energy of one of them. Obtain an algebraic expression for l in terms of M, c and Newton's gravitational constant G_N. Taking M to be 100 proton masses, evaluate l in metres. How does this l compare with the size of a typical nucleus? [$G_N = 6.67 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$]
   (ii) For a particular value of M, $M = M_P$ say, the Compton wavelength of an M-particle is equal to l as calculated in (i). Find an expression for M_P in terms of h, c and G_N. Check that your expression has the correct dimension (i.e. mass). Evaluate M_P in kg, and also evaluate $M_P c^2$ in GeV (1 GeV = 10^9 eV).
   (iii) The length l_P is defined to be the l of (i) with $M = M_P$. Find an expression for l_P in terms of G_N, h and c. Check that your expression has the correct dimension (length). Evaluate l_P in metres. How does l_P compare with the size of a typical nucleus?
   (iv) Which branches of physics are likely to be relevant at distances of order l_P or energies of order $M_P c^2$?

7. Einstein - de Broglie - Schrödinger waves

A plane wave solution of the time-dependent Schrödinger equation in one dimension is given by

$$\psi(x,t) = Ae^{ikx-i\omega t}.$$

(i) By substituting this solution into the TDSE find the relation between ω and k for such waves.
(ii) The "phase velocity" v_p is defined to be ω/k and the "group velocity" v_g is defined to be $d\omega/dk$. Find expressions for v_p and v_g in terms of the "particle velocity" defined by $v = p/m$. Are the results what you expect, and why?

8. Probability interpretation of the wavefunction

In Max Born's original paper (Zeitschrift für Physik 37 863-67 (1926)) the sentences proposing the probability interpretation of the wavefunction read as follows (quoting from the English translation, printed in "Quantum Theory and Measurement", Edited by J A Wheeler and W H Zurek, Princeton University Press, 1983, pages 52-55):

"If one translates this result into terms of particles, only one interpretation is possible. $\Phi_{\eta\tau m}(\alpha,\beta,\gamma)$ [the wavefunction for the particular problem he is considering] gives the probability *for the electron, arriving from the z-direction, to be thrown out into the direction designated by the angles α,β,γ ... .

* Addition in proof: More careful considerations show that the probability is proportional to the square of the quantity $\Phi_{\eta\tau m}$. "

Give as many "considerations" as you can why a general wavefunction ψ does not have suitable properties to be interpreted as a probability density, but the square modulus $|\psi|^2$ does.

9. Continuous probability distributions

For a certain continuous variable x, the probability that it has a value lying between x and $x + dx$ is $\rho(x)dx$. The possible values of x range from a to b.
(i) What conditions must $\rho(x)$ satisfy?
(ii) Define the average value $\langle f(x)\rangle$ of a function of x, $f(x)$.
(iii) The *variance* of the distribution is σ^2, defined by $\sigma^2 = \langle (x - \langle x\rangle)^2\rangle$. Show that $\sigma^2 = \langle x^2\rangle - (\langle x\rangle)^2$. (A customary measure of the "spread" of a distribution

is σ, which by the above result is equal to $[\langle x^2\rangle - (\langle x\rangle)^2]^{1/2}$. Frequently this may be written as Δx.)

10. The needle on a broken gauge is free to swing and after being tapped comes to rest randomly at any point between the angles 0 and π; it bounces perfectly off pins fixed at $\theta = 0$ and $\theta = \pi$. The probability that the needle will come to rest between θ and $\theta + d\theta$ is $\rho(\theta)$.
(i) What is $\rho(\theta)$? Sketch $\rho(\theta)$ as a function of θ for $-\pi/2 \leq \theta \leq 3\pi/2$.
(ii) Calculate $\langle\theta\rangle, \langle\theta^2\rangle$ and σ.
(iii) Calculate $\langle \sin\theta\rangle, \langle\cos\theta\rangle$, and $\langle\cos^2\theta\rangle$.
(iv) Suppose we are now interested in the x-coordinate of the needle point (i.e. the projection of that point onto the x-axis). Let the needle have length l, and let $\rho(x)$ be the probability that this projected value lies between x and $x + dx$. (a) Derive a formula for $\rho(x)$ [hint: how is x related to θ? And dx to $d\theta$?]. Sketch $\rho(x)$ for $-l \leq x \leq l$. (b) Calculate $\langle x\rangle, \langle x^2\rangle$, and σ for this probability distribution $\rho(x)$. Explain how you could have obtained these results from (iii) above.

11. <u>Solutions of the TDSE separated in x and t</u>

The 1-D TDSE is

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t).$$

Consider solutions in which the x- and t- dependence is *separated*: $\psi(x,t) = \phi(x).T(t)$.
(i) Show that

$$\frac{1}{T(t)} i\hbar \frac{dT(t)}{dt} = \frac{1}{\phi(x)}\left(-\frac{\hbar^2}{2m}\frac{d^2\phi(x)}{dx^2}\right) + V(x)$$

and explain why each side of this equation must equal the same constant "A".
(ii) Solve the T-equation for $T(t)$ given that $T(0) = 1$. Show that if A is real, $\mid \psi(x,t) \mid^2$ is independent of t. What is the frequency ω of the wave in terms of A? Assuming the Einstein relation $E = \hbar\omega$, find E in terms of A, and obtain an expression for the average value of x. Such solutions are called *stationary state solutions*: why?
(iii) Suppose V depends on t as well as on x: $V(x,t)$. Will such a separation of the x and t variables be possible, in general? Can you invent a $V(x,t)$ for which it would be (mathematically if not physically) possible?
(iv) Returning to the case $V(x)$, suppose A is in fact complex, $A = E - i\Gamma/2$. Show that the total (integrated over x) probability decays exponentially with a half-life of $(\hbar \ln 2)/\Gamma$. Suggest a physical problem in which such a solution

might be useful.
(v) Consider the x-equation for such a complex A:

$$-\frac{\hbar^2}{2m}\frac{d^2\phi(x)}{dx^2} + V(x)\phi(x) = (E - i\Gamma/2)\phi(x).$$

If we take $V(x) = V_0(x) - i\Gamma/2$ where $V_0(x)$ is real, we return to a standard time-independent Schrödinger equation $-\frac{\hbar^2}{2m}\phi'' + V_0\phi = E\phi$. For such a complex $V(x)$, discuss whether the probability current conservation equation $\frac{\partial\rho}{\partial t} + \frac{\partial j}{\partial x} = 0$ holds or not.

12. Particle in a box: average values

(i) Find the allowed energy values E_n and the associated normalised eigenfunctions, $\phi_n(x)$, for a particle of mass m confined by "infinitely high" potential barriers to the region $0 \leq x \leq a$.

(ii) For a particle with energy $E_n = \hbar^2 n^2 \pi^2/2ma^2$, calculate the quantum-mechanical expectation value of x, denoted by $\langle x \rangle$.

(iii) Without working out any integrals, show that $\langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - a^2/4$. Hence find $\langle (x - \langle x \rangle)^2 \rangle$ using the result

$$\int_0^a x^2 \sin^2\left(\frac{n\pi x}{a}\right) dx = \frac{a^3}{6} - \frac{a^3}{4n^2\pi^2}.$$

(iv) The classical equivalent of this problem is that of a particle bouncing back and forth between two perfectly elastic walls, with uniform velocity between bounces. Calculate the classical averages values $\langle x \rangle_c$ and $\langle (x - \langle x \rangle_c)^2 \rangle_c$, and show that for high values of n the quantum and classical results tend to each other.

[An extra problem]

13. [The Klein-Gordon equation]

Note: this topic is definitely outside the Part A syllabus

Consider again the de Broglie wave in one dimension

$$\psi(x,t) = Ae^{ikx-i\omega t}$$

where as before the associated particle momentum is $p = \hbar k$ and the particle energy is $E = \hbar\omega$. Suppose now that E and p (and hence ω and k) are related by the *relativistic* relation $E^2 = c^2p^2 + m^2c^4$.

(i) Find a partial differential equation, analogous to the Schrödinger equation, such that when the plane wave is substituted into it as a solution (see Q 7(i)) this relation between E and p follows, rather than the non-relativistic one which followed from the Schrödinger equation. [The equation you want is called the Klein-Gordon equation and was found first by Schrödinger, but discarded by him.]

(ii) Find expressions for the phase velocity v_p and the group velocity v_g of these waves, and discuss your results (compare with Q 7(ii)).