Introduction to Symmetries

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Lecture 8:

Local Symmetry, Gauge Theories, and Spontaneously Broken Symmetry.

I have to admit that this lecture is really just a mad dash for the finishing post! It's an extremely rapid sketch of some of the main ideas, included more for "completeness" than with any hope that it's understandable.....

We have seen that $\overline{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi$ is invariant under the global transformation $\psi \to \psi' = e^{-i\alpha}\psi$ where α is a constant and does not depend on x. What if the transformation *did* depend on x? In this case it is called a local transformation. But then we have $\psi \to \psi' = e^{-i\alpha(x)}\psi$, which is not a symmetry of our free Dirac Lagrangian since this transforms to

$$\overline{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi \to \overline{\psi'}\,(i\gamma^\mu\partial_\mu - m)\,\psi' = \overline{\psi}e^{i\alpha(x)}\,(i\gamma^\mu\partial_\mu - m)\,e^{-i\alpha(x)}\psi = \overline{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi + \overline{\psi}\,(\gamma^\mu\partial_\mu\alpha)\,\psi.$$

We end up with an extra piece $\overline{\psi}\,(\gamma^\mu\partial_\mu\alpha)\,\psi$! However, if we modify the Lagrangian to

$$\overline{\psi}\,(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\,\psi = \overline{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi - eA_\mu\overline{\psi}\partial^\mu\psi$$

and demand that A^μ transforms according to:

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\alpha\,(x)$$

when

$$\psi \to \psi' = e^{-i\alpha(x)},$$

then we will have restored local invariance! In other words, this is an invariance which only exists if the paticles are *not* free! It's a kind of "explanation" for interactions....

We can therefore say that the modified (interacting) theory is invariant under local U(1) transformations,

$$\psi \to \psi' = e^{-i\alpha(x)}\psi$$

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\alpha\,(x)\,.$$

The two lines above define what is called a **U(1) gauge transformation.** The combination of $i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu$ is just $\gamma^\mu\,(\hat{p}_\mu - eA_\mu)$. Incidently, replacing $\hat{p}_\mu$ with $\hat{p}_\mu - eA_\mu$ is the standard way to incorporate electromagnetism in Hamiltonian or Lagrangian dynamics too, and we did it in Lecture 3.

The Maxwell Lagrangian for is $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$, in the sense that if you plug this into the E-L equation for A_μ you get Maxwell's equations for the free e-m field (this is a good and not trivial exercise). Note that there is no mass term in this Lagrangian e.g. of the form $\frac{1}{2}m^2A^\mu A_\mu$ which is not invariant under transformations $A_\mu \to A'_\mu = A_\mu + \partial_\mu\alpha\,(x)$. Note also that the U(1) symmetry current $\overline{\psi}\gamma^\mu\psi$ is what is now coupled to A^μ to form the *interaction* i.e. note how the *symmetry* is related to the *dynamics* .

We can do the same thing for SU(2) symmetry, as was done by Yang and Mills, e.g. require symmetry under the local transformation

$$\Psi \to \Psi' = e^{-\frac{i}{2}\boldsymbol{\alpha}\,(x)\cdot\boldsymbol{\tau}}\,\Psi.$$

If we try this for a free Dirac Lagrangian for the isospinor Ψ, we will get extra terms involving $\partial_\mu\,\boldsymbol{\alpha}\,(x)$ so we will require *three* gauge fields to keep our Lagrangian invariant under local SU(2) transformations like the one above. In the physical case of the SU(2)xU(1) gauge theory of weak interactions, we get four gauge fields, the W^+, W^-, Z^0 and γ.

How do we construct such a locally SU(2)-invariant Lagrangian? The key is to generalise the "covariant derivative" $D_\mu = \partial_\mu + ieA_\mu$ which appeared in the U(1) case. Consider the transformation property of $(D_\mu\psi)$:

$$(D_\mu\psi)' = D'_\mu\psi' = \left(\partial_\mu + ieA'_\mu\right) e^{-ie\alpha(x)}\psi = e^{-ie\alpha(x)}\partial_\mu\psi - ie\left(\partial_\mu\alpha(x)\right)e^{-ie\alpha(x)} + ie\left(A_\mu + \partial_\mu\alpha(x)\right)e^{-ie\alpha(x)}\psi$$

$$\Rightarrow D'_\mu\psi' = e^{-ie\alpha(x)}\left(\partial_\mu + ieA_\mu\right)\psi = e^{-ie\alpha(x)}D_\mu\psi$$

$$\Rightarrow D'_\mu\psi' = e^{-ie\alpha(x)}D_\mu\psi,$$

so $D_\mu\psi$ transforms like ψ. This means that $\overline{\psi}D_\mu\psi$ is trivially invariant!

In the SU(2) case the covariant derivative is $D_\mu = \partial_\mu + ig\,\boldsymbol{\tau}\,.\mathbf{W}_\mu/2$ when acting on SU(2) doublets, where $\mathbf{W}_\mu$ are three gauge fields i.e.

$$\left(\left(\partial_\mu + g\,\boldsymbol{\tau}\,.\mathbf{W}_\mu/2\right)\psi\right)' = e^{-i\,\boldsymbol{\alpha}(x).\,\boldsymbol{\tau}/2}\left(\left(\partial_\mu + g\,\boldsymbol{\tau}\,.\mathbf{W}_\mu/2\right)\psi\right),$$

or, for an infinitesimal transformation, $\delta\left(D_\mu\psi\right) = -\frac{i}{2}\mathbf{a}\left(x\right).\boldsymbol{\tau}\left(D_\mu\psi\right)$. This will tell us the required transformation property of the three fields $W_{1\mu}, W_{2\mu}\ W_{3\mu}$. We find that

$$\delta\mathbf{W}_\mu \quad = \quad \underset{\text{the gauge part}}{\partial_\mu\mathbf{a}(x)} \quad + \quad \underset{\text{says that } \mathbf{W} \text{ is an SU(2) triplet } = \text{SO(3) vector!}}{g\mathbf{a}\left(x\right)\times\mathbf{W}_\mu}$$

So to get a Dirac-type Lagrangian we replace ∂_μ in the free, non-interacting Lagrangian, with the appropriate covariant derivative above:

$$\overline{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - \frac{g}{2}\overline{\psi}\gamma^\mu\,\boldsymbol{\tau}\,\psi.\mathbf{W}_\mu$$

The last term shows the currents, which we met in the last Lecture associated with global SU(2) transformations, now coupled to W's in the local version . (Actually, for the weak interaction case, we need to include the left-handed projector $(1-\gamma_5)$ after the γ^μ since the SU(2) there refers only to the left-handed components of the fields.)

The Maxwell bit "$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$" is harder to derive. The result for the analogous field strength tensor is

$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{W}_\nu - \partial_\nu\mathbf{W}_\mu - g\mathbf{W}_\mu\times\mathbf{W}_\nu.$$

The resulting Lagrangian $-1/4\,\boldsymbol{F}_{\mu\nu}.\boldsymbol{F}^{\mu\nu}$ contains terms which are cubic and quartic in the W fields, as well as the usual free-particle quadratic pieces; this shows that the SU(2) gauge quanta *must interact with each other*.

QCD is very similar, being a local SU(3) gauge theory. The coupling of quarks to gluons we have seen before:

$$\overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - \frac{g_s}{2}\overline{\psi}\,\boldsymbol{\lambda}\,\gamma^\mu\psi.\,\boldsymbol{A}_\mu$$

but now of course it is motivated by the local symmetry argument. Here the $\boldsymbol{A}$'s are eight gluon fields, $\frac{1}{2}\overline{\psi}\,\boldsymbol{\lambda}\,\gamma^\mu\psi$ are the eight symmetry currents of SU(3) colour, and

$$\psi = \begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix}.$$

In this case, the covariant derivative acting on the SU(3) triplet field is

$$D_\mu = \partial_\mu + \frac{ig_s}{2}\,\boldsymbol{\lambda}\,.\,\boldsymbol{A}_\mu.$$

The field strength tensor is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f^{abc}A^b_\mu A^c_\nu.$$

The last term shows the two octet gluon fields being coupled via the f symbol to make another octet (compare what we said about the way octets transformed, in Lecture 6!).

Spontaneous Symmetry Breaking.

We shall continue not to put hats on field quantities....so we will properly speaking be confined to classical fields, not quantised ones. This section will be even more sketchy than the last....and is intended just to "get a foot in the door".

Let's consider a complex scalar field ϕ , which is a simple type of *Higgs field*. However, at first we will be considering a global symmetry and no gauge field, in which case it is the *Goldstone* rather than the Higgs model. Written in terms of its real and imaginary parts, the field is

$$\phi = (\phi_1 - i\phi_2)/\sqrt{2}$$

and

$$\phi^\dagger = (\phi_1 + i\phi_2)/\sqrt{2}.$$

We take the Lagrangian to be

$$L = \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi)$$

where

$$V(\phi) = -\frac{1}{2}\mu^2 \mid \phi \mid^2 + \frac{1}{2}\lambda^2 \mid \phi \mid^4 .$$

This Lagrangian has a global $U(1)$ symmetry

$$\phi \to \phi' = e^{-i\alpha}\phi$$

which in terms of the ϕ_1 and ϕ_2 fields is

$$\phi_1' = \cos\alpha\ \phi_1 - \sin\alpha\ \phi_2$$

and

$$\phi_2' = \sin\alpha\ \phi_1 + \cos\alpha\ \phi_2,$$

which can be thought of as rotation in the $\phi_1 - \phi_2$ plane. So the symmetry here is "rotations in the $\phi_1 - \phi_2$ plane".

The whole point here lies in the *sign of the* $\mid \phi \mid^2$ *term.* The notation used obviously implies that it is being chosen to be *negative*. Notice carefully that this is *not* the sign that a conventional mass term would have. Lagrangians are basically " $T - V$ ", and if you look back at the examples we gave in Lecture 7 you'll see that in the K-G case (a single scalar field) the mass term in the "potential" was definitely positive. If it *is* positive, this will mean that for small values of the field, near the "origin in field space", the potential will look like a bowl, and one would expect the field to settle at the origin, in equilibrium. On the other hand, if the sign of the quadratic term is negative, then the origin will be an unstable point, and the field will tend to slip away from it .

Where will the field "roll" to? Draw the above $V(\phi)$ as a surface, the vertical axis representing V, the horizontal axes being ϕ_1 and ϕ_2 ! The surface bends downwards near the origin, as we have indicated above, but at large enough $\mid \phi \mid$ it must bend back up again because of the $\mid \phi \mid^4$ term which must eventually win at large ϕ. So there will be a minimum, a point where the field will "roll" to, but it wont be at the origin any more. In fact, it is simple to find that it is at

$$\mid \phi \mid^2 = \mu^2/2\lambda^2,$$

or

$$\phi_1^2 + \phi_2^2 = \mu^2/\lambda^2 = f^2 \ \text{ say}.$$

But this only specifies a circle of points as a function of the two variables ϕ_1 and ϕ_2! (As you can see on your drawing.) *Anywhere on this circle will do for the field to settle on, in equilibrium.* All points are equally good. At this stage, we still have the original symmetry, that of rotations in the $\phi_1 - \phi_2$ plane. However, our basic calculational tool is *perturbation theory*, which assumes that the fields are "small", and are not going to evolve into "large" ones. But this clearly will happen if we "start" at the origin in field space. Instead, what we presumably have to do is start at a point of stable (not unstable) equilibrium, and consider small oscillations about that. But now we have a problem: which particular point out of all the equally good equilibrium positions (on that circle) do we choose?

We must choose one, in order to formulate a well-controlled perturbation theory. But, in choosing any particular one, we spoil - or "spontaneously break" - the symmetry. This is the essential idea.

OK, so let's choose the particular point $\phi_1 = f, \phi_2 = 0$, and consider small field motions away from this equilibrium position. There are obviously many ways in which we could represent such departures from the chosen equilibrium position. One way is to use "Cartesian" field coordinates, and set

$$\phi_1 = f + \chi_1(x)$$

and

$$\phi_2 = \chi_2(x).$$

Then it is an excellent little exercise to verify that, inserting this into our Lagrangian L, it becomes

$$L = (\frac{1}{2}\partial_\mu\chi_1\ \partial^\mu\chi_1 - \frac{1}{2}\mu^2\chi_1^2) + \frac{1}{2}\partial_\mu\chi_2\ \partial^\mu\chi_2 + A\chi_1(\chi_1^2+\chi_2^2) + B(\chi_1^2+\chi_2^2)^2 + C$$

where A, B and C are constants (which you can determine). By construction, we expect to be able to set up a decent perturbation theory in terms of the χ_1 and χ_2 fields. So what kind of particles do they describe? To answer this we look at the *quadratic* part of L, since this tells us what we've got in the absence of interactions i.e. it tells us the free-particle content of the theory (of course, to do this properly we'd have to quantize the fields). The quadratic part involving χ_1 is, amazingly enough, a standard spin-0 Lagrangian with the *correct* sign of the mass term to represent a genuine mass μ i.e. the corresponding potential starts with a $+\frac{1}{2}\mu^2\ \chi_1^2$ and is therefore "upturned", the origin in χ_1 space being therefore a stable equilibrium point, and OK for doing small-field perturbation theory about. The quadratic part involving χ_2 on the other hand has no mass term, and therefore χ_2 represents a *massless* field!

This particle content should not really be a surprise if you look at your sketch of $V(\phi)$. In particular, imagine looking down at the "bottom of the wine-bottle". The equilibrium point we have chosen is on the ϕ_1 axis at $\phi_1 = f,\ \ \phi_2 = 0$. Moving away from here along the ϕ_1 axis via $\phi_1 = f + \chi_1$, the field will be "held" in a restoring potential, which corresponds precisely to a genuine mass term. On the other hand, in the ϕ_2 direction at this point, the field will be moving along the "valley" of the wine-bottle, and will experience no restoring force, which means that it's a massless field when quantized.

The appearance of a *massless field* is actually a very general result in all such cases of *spontaneous breaking of a global continuous symmetry*, and is known as Goldstone's theorem.

It is instructive also to consider another parametrisation of the fields away from the chosen equilibrium position, namely one in which we use "polar" field coordinates (radial and angle variables) rather than the "Cartesian" ones above. We set

$$\phi = \frac{1}{\sqrt{2}}(f + \rho(x))\ e^{i\theta(x)/f}.$$

The "radial" field ρ and the "angle" field θ here replace the "Cartesian" ϕ_1 and ϕ_2. There are still, of course, two (field) degrees of freedom. Once again, it is a very useful exercise to stick this into L and find that it becomes

$$L = (\frac{1}{2}\partial_\mu\rho\ \partial^\mu\rho\ \ -\frac{1}{2}\mu^2\rho^2) + (\frac{1}{2}\partial_\mu\theta\ \partial^\mu\theta) + f\rho\partial_\mu\theta\ \partial^\mu\theta\ + \text{other interaction terms.}$$

The "θ" mode is the one for motion around the equilibrium circle, and it is not subject to any restoring potential, so it is massless. The "ρ" mode (radial) is restored, and has mass μ.

OK. The last topic is making the global $U(1)$ symmetry of this little model into a local symmetry by introducing a $U(1)$ gauge field - we'll then have the *Abelian Higgs model.*

All we have to do is change all derivatives to covariant derivatives and add in the Maxwell term for the A_μ field. This produces

$$L = [(\partial_\mu + ieA_\mu)\phi]^\dagger\ [(\partial^\mu + ieA^\mu)\phi]\ \ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\ \ +\ \ \frac{1}{2}\mu^2\mid\phi\mid^2\ \ -\ \ \frac{1}{2}\lambda^2\mid\phi\mid^4.$$

As in the Goldstone model, we can't read off the particle content from this, as the origin is unstable. So we expand about a point on the equilibrium circle as before......This time we shall choose to do that in terms of the "polar" field variables, and write

$$\phi = \frac{1}{\sqrt{2}}(f + \rho(x))\ e^{i\theta(x)/f}$$

where $f = \mu/\lambda$. This choice seems particularly appropriate in the present case, because it is a *gauge theory*, and that means (see above) that the theory is invariant under gauge transformations, which in the case of the ϕ field precisely involve $x-$ dependent phase transformations i.e. transformations on the "angle" variable θ. In fact, the theory is invariant under the combined transformations

$$\phi \to \phi' = e^{-ie\chi(x)}\ \phi$$

and

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\chi,$$

where χ is arbitrary. Under the first of these transformations, then, the fields ρ and θ transform by

$$\rho \to \rho$$

and

$$\theta \to \theta - ef\chi.$$

So, starting with any non-zero θ, we can by choosing the arbitrary field χ to be θ/ef arrange for the transformed θ to vanish! That is, *we can choose a gauge in which ϕ is real.* But we must remember that the gauge field A_μ and the complex scalar field ϕ come as a "package", in the sense that the gauge transformation affects both simultaneously (see above). So when we said "starting witn a non-zero θ ", we should have said "starting from a non-zero θ and a certain A_μ". Then, after the gauge transformation in which θ is reduced to zero, A_μ is changed to

$$A'_\mu = A_\mu + \frac{1}{ef}\partial_\mu\theta$$

and ϕ is

$$\phi' = \frac{1}{\sqrt{2}}(f + \rho).$$

To repeat the point again: these fields ϕ' and A'_μ are gauge transforms of the original ones ϕ and A_μ which appeared in our original $L(\phi, A_\mu)$. But since the theory is *gauge invariant* we have a prefect right to use them in L instead of ϕ and A_μ.

So, writing out L in terms of these primed fields (another useful little exercise) we find

$$L(\phi', A'_\mu) = (\frac{1}{2}\partial_\mu\rho\ \partial^\mu\rho \ - \ \frac{1}{2}\mu^2\rho^2) \ \ - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} \ \ + \ \ \frac{1}{2}e^2 f^2 A'_\mu A'^\mu \ \ + \text{interactions.}$$

We should be able to read off the particle content here, as the field ρ is correctly oscillating in a "restoring" potential, as in the Goldstone model. Indeed, just as in that case, we see that the ρ degree of freedom has a genuine mass μ. But what has happened to the massless mode θ? It is said to have been "swallowed" by the A_μ field! This is a graphic way of describing the fact that it does seem to have vanished from the Lagrangian...but it is there in a way, being present in A'_μ via

$$A'_\mu = A_\mu + \frac{1}{ef}\partial_\mu\theta.$$

So what kind of a particle field is A'_μ? *It is the field for a massive spin-1 particle, of mass ef*! This is the famous Higgs mechanism, whereby the massless gauge field A_μ has become a massive spin-1 field A'_μ by "eating" the scalar field θ.

You may be bothered by the sign of the claimed mass term $+\frac{1}{2}e^2f^2A'_\mu A'^\mu$. It's actually OK, but it deserves a word of explanation. Consider a massive spin-1 field W_μ. This has four components (one with $\mu = 0$ and the three spatial ones). But as a spin-1 field it should only have three degrees of freedom, corresponding to the possible spin projections. So one of these four degrees of freedom must be redundant. It turns out that the Euler-Lagrange equation of motion for such a field (indeed the one for A'_μ above) implies that the condition

$$\partial^\mu W_\mu = 0$$

holds. For free fields we have

$$W_\mu = \epsilon_\mu e^{ipx}$$

where ϵ_μ is a polarisation vector. Inserting this into the condition above, we get

$$\epsilon . p = 0.$$

Now we can always sit in the rest frame of a massive particle, in which the four-momentum p has vanishing spatial components. That means that our condition is $\epsilon_0 = 0$, and so only the spatial components of the field remain in play - and there are three of them, as required for a spin-1 field. So, going back to our Lagrangian in terms of A'_μ, we see that the mass term has, after all, the *correct* sign for the physical (spatial) degrees of freedom, bearing in mind that

$$A'_\mu A'^\mu = A'_0 A'_0 - \boldsymbol{A} . \boldsymbol{A} .$$

Note particularly that the above form of the Lagrangian in terms of the fields ρ and A' is *only one possible way of writing it, corresponding to a particular choice of gauge.* It is a good exercise tp try writing it out in some other gauge - for example one in which the phase degree of freedom θ is not reduced to zero! Nevertheless, this gauge choice is undoubtedly a physically appealing one, and it certainly gives us the particle content right away. It is called the "unitary gauge".

The above has been done for classical fields, really. To interpret it for quantized fields, the main step is to identify the "equilibrium value of the classical field" with the "vacuum expectation value of the quantum field". That is, we think of the qft vacuum as being the ground state of the interacting quantum field system. Usually, of course, vacuum expectation values of quantum fields are all zero (because the usual vacuum is defined by conditions like $\hat{a} \mid 0\rangle = 0$, $\langle 0 \mid \hat{a}^{\dagger} = 0$). The novel thing about the *Higgs vacuum* is precisely that, in it, a quantum field (the Higgs field) has a non-zero expectation value. Can we prove that this does actually happen? Yes, in some special soluble cases, but not (yet) for the actual$SU(2)xU(1)$ case we want. The difficulty is that, almost by definition, proving such a property can't be done in perturbation theory - it's a non-perturbative problem, and as such very hard to crack. What this means is that we have to say, frankly, that this "mechanism" is basically a phenomenological one. The Higgs sector of the Standard Model is put in "by hand", and in particular the crucial "minus sign" in the mass term is not derived. However, it is fair to say that an argument in favour of it can be made within a *supersymmetric extension* of the Standard Model.

The Weinberg-Salam model uses just the same kind of procedure , spontaneously breaking the local $SU(2) \times U(1)$ symmetry of the weak interactions by means of a (somewhat more complicated) Higgs field. In that case, the Higgs field plays another role as well - its Yukawa-like couplings to the fermions generate fermion masses when it is expanded about its vacuum value. Thus for both the gauge and fermion fields, it is their coupling to a field with a vacuum expectation value that is responsible (in the Standard Model) for their masses.