Introduction to Symmetries

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Lecture 7:

Symmetry in Lagrangian Field Theory.

First, a crash course in Lagrangian field theory!

The basic idea is to start from an action principle which seems to work with everything (optics, mechanics, quantum mechanics, Q.F.T., strings). In mechanics the action S is defined as,

$$S = \int_{t_1}^{t_2} L\left(q(t), \dot{q}(t)\right) dt.$$

$q(t)$ denotes the position of particle as a function of time, t, $\dot{q}(t)$ is its velocity. We think of the $q(t)$'s as trajectories. The actual path taken (in classical mechanics) is the one for which the integral S is least; this is known as Hamilton's principle of least action. We shall now find the equation for the least action path, the classical trajectory. Taking the path of least action (which we are trying to find) to be $q(t)$, consider a small variation away from $q(t)$, $q(t) + \delta q(t)$, which has the same end points ($\delta q\left(t_1\right) = \delta q\left(t_2\right) = 0$). If $q(t)$ is the path for which S is least, then the resulting change in S has to be zero:

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q(t) + \frac{\partial L}{\partial \dot{q}} \delta \dot{q}(t) \right) dt = 0$$

where $\delta \dot{q}(t) = \frac{d\delta q(t)}{dt}$. Integrate the last term by parts:

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} \delta \dot{q}\ dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} \frac{d\delta q(t)}{dt} dt = \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q dt.$$

The first term $\left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2}$ vanishes as $\delta q\left(t_1\right) = \delta q\left(t_2\right) = 0$ and so

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q dt.$$

When the action is extremized we will have $\delta S = 0$ as q is varied. This must hold for *arbitrary* changes $\delta q(t)$. The only way this can be true is for the quantity multiplying $\delta q(t)$ in the preceding integral to vanish:

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0.$$

This is the *Euler-Lagrange equation.* It is basically the "equation of motion" for $q(t)$ in this way of doing business; note that we are back to a formulation in terms of a differential equation having started off rather differently, with an integral!

So what is L? Usually $L = T - V$ i.e. the kinetic energy minus the potential energy e.g. for a particle in a potential $V(q)$,

$$L = \frac{1}{2} m \dot{q}^2 - V\left(q\right)$$

$$\Rightarrow \frac{\partial L}{\partial q} = \frac{\partial V\left(q\right)}{\partial q} \text{ and } \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = m\ddot{q}$$

$$\Rightarrow m\ddot{q} = \frac{\partial V(q)}{\partial q},$$

as given by Newton.

Now we want to apply the principle of least action to *fields*. This means that we have to deal with a "q " at every point in space :

$$q(t) \to \phi(x,t)$$

and correspondingly

$$\int L dt \to \int \left[\int (L dx) \right] dt.$$

The previous " L" has become a density, but we are not troubling to change the notation. L is now a function of $\phi(x,t)\,, \frac{\partial \phi(x,t)}{\partial t}$and $\frac{\partial \phi(x,t)}{\partial x}$ in 1-dimension. The action is now

$$S = \int\int L\; dxdt.$$

As before, $\phi(x,t)$ will be determined from the condition that S is stationary under small variations $\phi(x,t) \to \phi(x,t) + \delta\phi(x,t)$. Under such variations, S changes by

$$\delta S = \int\int \left\{ \frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\delta\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial L}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\delta\left(\frac{\partial \phi}{\partial t}\right) \right\} dxdt$$

By analogy with the first time we calculated δS, here we integrate the second and third terms by parts and as before we end up with four terms two of which vanish because $\delta\phi = 0$ at the end points (as before, we are dealing with fixed end points); so

$$\delta S = \int\int \left\{ \frac{\partial L}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\right) \right\} \delta\phi\; dxdt.$$

Requiring, as before, that this variation is zero for em arbitrary $\delta\phi$, we deduce that the term in curly brackets must vanish:

$$\Rightarrow \frac{\partial L}{\partial \phi} - \nabla.\left(\frac{\partial L}{\partial(\nabla\phi)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\right) = 0$$

which is *the Euler-Lagrange field equation* for $\phi(x,t)$.

Example:

$$L = \frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2$$

$$\text{E-L eqn} \Rightarrow \frac{\partial}{\partial x}\left(-\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial t}\left(c^2\frac{\partial\phi}{\partial t}\right) = 0$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2}$$

the classical wave equation!

Here are a few examples of important field Lagrangians:

- The Schroedinger Lagrangian:

$$L = i\psi^*\psi - \frac{1}{2m}\left(\nabla(\psi^*)\right)(\nabla\psi)$$

- The Dirac Lagrangian:

$$L = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi$$

- The Klein-Gordon Lagrangian:

$$L = \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2}\left(\nabla\phi\right)\left(\nabla\phi\right) - \frac{1}{2}m^2\phi^2$$

- The Maxwell Lagrangian:

$$L = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Ok, now we are ready to go on to symmetries in field theories.

Noether's Theorem - Global Symmetries.

Suppose we have several fields, scalar to start with, $\phi_r(x)$, where x is now the 4-D vector $(t, \mathbf{x})$ with the same mass, and with interactions perhaps, such that L is invariant under a transformation among the ϕ's of the form

$$\phi_r\left(x\right) \to \phi'_r\left(x'\right) = \phi'_r\left(x\right) - i\epsilon\lambda_{rs}\phi_s\left(x\right)$$

for some coefficients λ_{rs}, and for infinitesimal ϵ. Obviously this is like an infinitesimal SO(3), SU(2) or SU(3) transformation. If L is invariant under the transformation, then

$$\delta L = \frac{\partial L}{\partial \phi_r}\delta\phi_r + \frac{\partial L}{\partial\left(\nabla\phi_r\right)}.\delta\left(\nabla\phi_r\right) + \frac{\partial L}{\partial\left(\frac{\partial\phi_r}{\partial t}\right)}\delta\left(\frac{\partial\phi_r}{\partial t}\right) = 0.$$

From the Euler-Lagrange equations we've just derived,

$$\frac{\partial L}{\partial\phi_r} = \nabla.\left(\frac{\partial L}{\nabla\phi_r}\right) + \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\left(\frac{\partial\phi_r}{\partial t}\right)}\right).$$

Using this in the previous equation,

$$\nabla.\left(\frac{\partial L}{\partial\left(\nabla\phi_r\right)}\right)\delta\phi_r + \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\left(\frac{\partial\phi_r}{\partial t}\right)}\right)\delta\phi_r + \frac{\partial L}{\partial\left(\nabla\phi_r\right)}.\nabla\left(\delta\phi_r\right) + \frac{\partial L}{\partial\left(\frac{\partial\phi_r}{\partial t}\right)}\frac{\partial\left(\delta\phi_r\right)}{\partial t} = 0$$

$$\Rightarrow \nabla.\left(\frac{\partial L}{\partial\left(\nabla\phi_r\right)}\delta\phi_r\right) + \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\left(\frac{\partial\phi_r}{\partial t}\right)}\delta\phi_r\right) = 0$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \nabla.\mathbf{j} = 0,$$

a *continuity equation*, just like in e-m theory or ordinary QM. Here, $\rho = \frac{\partial L}{\partial\left(\frac{\partial\phi_r}{\partial t}\right)}. - i\lambda_{rs}\phi_s$ and $\mathbf{j} = \frac{\partial L}{\partial\left(\nabla\phi_r\right)}. - i\lambda_{rs}\phi_s$ (dropping the irrelevant constant infinitesimal parameter ϵ). Integrating the continuity equation we have:

$$\int\frac{\partial\rho}{\partial t}d^3x + \int\nabla.\mathbf{j}d^3x = 0$$

$$\Rightarrow \frac{d}{dt}\int\rho d^3x + \int\mathbf{j}.d\mathbf{S} = 0.$$

As we are integrating over an infinite volume we can usually say that the currents on the surface are negligible i.e. $\int\mathbf{j}.d\mathbf{S} = 0$ from which it follows that

$$\frac{d}{dt}\int\rho d^3x = \frac{dQ}{dt} = 0.$$

This tells us that we have a conserved quantity Q, a kind of "charge", and an associated conserved "current." This is *Noether's theorem*, that for every continuous symmetry there is a conserved "charge" and a corresponding conserved "current".

For our first application, consider the Dirac Lagrangian $L = \overline{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi = \overline{\psi}\left(i\gamma^0\partial_0 + i\boldsymbol{\gamma}.\nabla - m\right)\psi$. This is obviously invariant under $\psi \to \psi' = e^{-i\alpha}\psi$ i.e. a simple multiplication by a phase factor. The associated symmetry is called, rather grandly, "U(1)". This just means (as in "U(2)" etc) that the transformation is described by a 1×1 unitary matrix - and of course a 1×1 matrix is just a single number, and since it's unitary it must be just a phase factor. Furthermore, there is some additional (rather important) jargon. This symmetry is called a *global* U(1) symmetry, because the phase factor is a constant throughout all of space and time - it is the same everywhere and at all times. In the next Lecture we shall consider the *local* version, in which α is allowed to vary with x.

To get at the symmetry current in this case, we recall that Noether's Theorem needs only the infinitesimal form of the transformation, which is $\psi \to \psi' = (1 - i\epsilon)\psi$, which means $\delta\psi = -i\epsilon\psi$. So, applying the above results for ρ and $\mathbf{j}$, we get (cancelling the ϵ)

$$\begin{aligned} \rho &= \frac{\partial L}{\partial\left(\frac{\partial\psi}{\partial t}\right)}. - i\psi &= \overline{\psi}.i\gamma^0. - i\psi &= \overline{\psi}\gamma^0\psi \\ \mathbf{j} &= \frac{\partial L}{\partial(\nabla\psi)}. - i\psi &= \overline{\psi}.i\boldsymbol{\gamma}. - i\psi &= \overline{\psi}\boldsymbol{\gamma}\psi \end{aligned}$$

$$\Rightarrow j^\mu = \overline{\psi}\gamma^\mu\psi.$$

In this case "Q"$= \int \overline{\psi}\gamma^0\psi d^3x = \int \psi^\dagger\psi d^3x$, which is just the *integrated probability density* for this fermion species. Note how the Example of Lecture 5 guarantees that this quantity does transform as a 4-vector under Lorentz transformations!

Now let's consider a more complicated scenario where we have two Dirac fields u and d with equal mass and interactions (which we shall neglect) which do not distinguish between u and d. We expect this will have to do with isospin again, this time in the context of fields rather than simple particle states. The Lagrangian is

$$\bar{\psi}_u\left(i\gamma^\mu\partial_\mu - m\right)\psi_u + \bar{\psi}_d\left(i\gamma^\mu\partial_\mu - m\right)\psi_d = \overline{\mathbf{\Psi}}\left(i\gamma^\mu\partial_\mu - m\right)\mathbf{\Psi}$$

where we have to watch the very compressed notation rather carefully; here

$$\mathbf{\Psi} = \begin{pmatrix} \mathbf{\Psi}_u \\ \mathbf{\Psi}_d \end{pmatrix}$$

in SU(2) space, with

$$\mathbf{\Psi}_u = \begin{pmatrix} \phi_u \\ \chi_u \end{pmatrix}$$

in Dirac space, and similarly for $\mathbf{\Psi}_d$. So the $\mathbf{\Psi}$ above has 8 components, 4 for the Dirac spinor and two for the "$u - d$" - ness.

This Lagrangian is invariant under $\mathbf{\Psi} \to \mathbf{\Psi}' = \mathbf{U}\mathbf{\Psi}$, where $\mathbf{U}$ is acting in the $u - d$ space, and is such that $\mathbf{U}^\dagger\mathbf{U} = \mathbf{I}$. So $\mathbf{U}$ is a 2×2 unitary matrix, $\mathbf{U} \in \mathrm{U}(2)$. When we studied isospin in Lecture 5, we "got rid of" an overall phase factor in $\mathbf{U}$, and concentrated our attention on $\mathbf{U}$'s which had determinant equal to 1, and which belonged to SU(2); these we called $\mathbf{Q}$. From our present perspective, we can understand rather better now why we should at this stage eliminate the phase factor from the $\mathbf{U}$'s: the reason is that such a phase factor is exactly the global U(1) transformation we have already considered.....so there is no need to include it again. The only slight difference this time is that since both ψ_u and ψ_d are transformed by the same phase factor, the resulting conserved "charge density" is $\rho_u + \rho_d$, which when integrated gives the total probablity for both ups and downs.

So, we remove a phase factor by requiring $\det\mathbf{U} = 1$, and rename the transformation matrix $\mathbf{Q}$ ($\mathbf{Q} \in \mathrm{SU}(2)$). Such transformations are called "global SU(2)" transformations in this context, because (again) the elements of $\mathbf{Q}$ do not depend on the space-time point x.

As before, we need only consider infinitesimal transformations of the familiar for $\mathbf{Q}$ we have,

$$\mathbf{Q} = 1 - i\mathbf{a}.\boldsymbol{\tau}/2$$

under which

$$\mathbf{\Psi}' = (1 - i\mathbf{a}.\boldsymbol{\tau}/2)\,\mathbf{\Psi}.$$

Here the $\mathbf{a}$ are the usual three infinitesimal parameters. Let's look at Noether's theorem for the case where only a_1 is non-zero. In this case we will have (remember the infinitesimal parameter is factored away) $\rho = \frac{\partial L}{\partial\left(\frac{\partial\mathbf{\Psi}}{\partial t}\right)}. - \frac{i}{2}\tau_1\mathbf{\Psi} = \frac{1}{2}\bar{\mathbf{\Psi}}\gamma^0\tau_1\mathbf{\Psi}$ and $\mathbf{j} = \frac{\partial L}{\partial(\nabla\psi)}. - \frac{i}{2}\tau_1\mathbf{\Psi} = \frac{1}{2}\bar{\mathbf{\Psi}}\boldsymbol{\gamma}\tau_1\mathbf{\Psi}$. We call this 4-vector current V_1^μ :

$$V_1^\mu = \left(\frac{1}{2}\bar{\mathbf{\Psi}}\gamma^0\tau_1\mathbf{\Psi}, \frac{1}{2}\bar{\mathbf{\Psi}}\boldsymbol{\gamma}\tau_1\mathbf{\Psi}\right) = \frac{1}{2}\bar{\mathbf{\Psi}}\gamma^\mu\tau_1\mathbf{\Psi}.$$

Similarly, for the general case where a_2 and a_3 are also non-zero, we have

$$V_2^\mu = \frac{1}{2}\bar{\Psi}\gamma^\mu \tau_2 \Psi$$

$$V_3^\mu = \frac{1}{2}\bar{\Psi}\gamma^\mu \tau_3 \Psi.$$

We can write all three currents in one package as $\mathbf{V}^\mu = \frac{1}{2}\bar{\Psi}\gamma^\mu\, \boldsymbol{\tau}\, \Psi$. Once again, the notation is very compressed. The "bold face, vector" aspect has to be understood as referring to an isospin triplet (compare $q^\dagger\, \boldsymbol{\tau}\, q$ in SU(2)), while the "μ" refers to the Lorentz 4-vector character. This package is really using everything we have learned up to now!

The three conserved charges are $I_i = \int \frac{1}{2}\Psi^\dagger \tau_i \Psi d^3x$, which are simply the three components of the isospin . Similarly for SU(3), the same working would lead to 8 currents

$$V_i^\mu = \frac{1}{2}\bar{\Psi}\gamma^\mu \lambda_i \Psi$$

where i=1,...,8, and where

$$\Psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \text{ for flavour, or } \begin{pmatrix} r \\ g \\ b \end{pmatrix} \text{ for colour.}$$

We have time for one more example.....

Chiral Symmetry:

Consider one species of free fermion, with $L = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi$. We are going to look at a *new* kind of global U(1) transformation, this one involving the Dirac matrix γ_5: $\psi \to \psi' = e^{i\alpha\gamma_5}\psi$. For infinitesimal transformations we have,

$$\psi \to \psi' = \left(1 + i\epsilon\gamma_5\right)\psi$$

$$\psi^\dagger \to \psi'^\dagger = \psi^\dagger\left(1 - i\epsilon\gamma_5\right).$$

The Lagrangian changes to

$$L \to L' = \psi^\dagger\left(1 - i\epsilon\gamma_5\right)\gamma^0\left(i\gamma^\mu\partial_\mu - m\right)\left(1 + i\epsilon\gamma_5\right)\psi$$

$$\Rightarrow L \to L' = \psi^\dagger\gamma^0\left(i\gamma^\mu\partial_\mu - m\right)\psi - i\epsilon\psi^\dagger\gamma_5\gamma^0\left(i\gamma^\mu\partial_\mu - m\right)\psi + i\epsilon\psi^\dagger\gamma^0\left(i\gamma^\mu\partial_\mu - m\right)\gamma_5\psi.$$

Now $\gamma^5\gamma^0\gamma^\mu\partial_\mu = \gamma^0\gamma^\mu\partial_\mu\gamma^5$ but $\gamma^5\gamma^0 = -\gamma^0\gamma^5$

$$\Rightarrow L \to L' = L - 2i\epsilon\overline{\psi}\gamma_5\psi m.$$

So actually this is *not* a symmetry of L. But we can still work out what the corresponding Noether current would be, which is $A^\mu = \overline{\psi}\gamma^\mu\gamma^5\psi$. The reason we call this one an "A" rather than a "V" is that the γ_5 changes the parity, so that this object is an *axial vector* rather than an ordinary one. Now, since L is not in fact invariant under this transformation, we dont expect A_μ to be divergenceless (i.e. conserved). And indeed we find

$$\partial_\mu A^\mu = 2im\overline{\psi}\gamma_5\psi.$$

Why did we go to all this trouble over something that wasn't a symmetry? Simply because it *is* a symmetry if $m = 0$! i.e. if the fermion is massless. So here is a case of a symmetry which exists if a certain term in the Lagrangian is absent (here, the mass), but is "broken" if that term is present. In our case, if our spin$-\frac{1}{2}$ particle is massless, then clearly the above result us that $\partial_\mu A^\mu = 0$ i.e. the current is conserved, and we have a good symmetry. So massless spin-$\frac{1}{2}$ particles have another symmetry called "chiral symmetry" which is to do with the handedness of the particles (remember $(1 - \gamma_5)$ projects out the left-handed Dirac component). This symmetry is called a global (independent of x) chiral U(1) symmetry. Recall the massless spinors ϕ and χ of Lecture 4 which had definite helicities!

There are also "Non-Abelian" global chiral symmetries. If the u and d fields are both effectively massless, then we have the following axial vector SU(2) currents:

$$A_i^\mu = \frac{1}{2}\bar{\Psi}\gamma^\mu\gamma_5\tau_i\Psi$$

where

$$\Psi = \begin{pmatrix} \Psi_u \\ \Psi_d \end{pmatrix}.$$

To the extent that the quark masses are "small", this might be expected to be "quite a good" symmetry. In fact, the situation is much more subtle: the chiral SU(2) flavour symmetry is "spontaneously" broken (as opposed to "explicitly broken" by the mass terms in the Lagrangian). This concept will be briefly explained in Lecture 8.

Remarkably enough, this SU(2) chiral-type symmetry arises somewhere else as well - as the symmetry associated with *weak* interactions! There, the symmetry currents we deal with are just the "left-handed" ones , of the type $\frac{1}{2}\bar{\Psi}\gamma^{\mu}(1-\gamma_5)\,\boldsymbol{\tau}\,\Psi$.

Enlarging Ψ to three components and replacing τ_i by λ_i, we get 8 axial vector currents which would be conserved (in the flavour case) if the u, d, s quarks were all massless. This is clearly a less good symmetry.....