Introduction to Symmetries

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Lecture 6:

The Group SU(3).

By analogy with SU(2), SU(3) is the group of all 3×3 unitary matrices $\mathbf{Q}$ with determinant $+1$ (we call them by the same symbol!). As with SU(2) an infinitesimal SU(3) matrix has the form,

$$\mathbf{Q}^{infl} = \mathbf{I} + i\boldsymbol{\chi}$$

where $\boldsymbol{\chi}$ is an infinitesimal matrix. Imposing the requirement that $\mathbf{Q}^{infl}$ be unitary we have

$$\mathbf{Q}^{infl\dagger}\mathbf{Q}^{infl} = (\mathbf{I} - i\boldsymbol{\chi}^\dagger)(\mathbf{I} + i\boldsymbol{\chi}) = \mathbf{I} - i\boldsymbol{\chi}^\dagger + i\boldsymbol{\chi} + \boldsymbol{\chi}^\dagger\boldsymbol{\chi} = \mathbf{I}$$

so to first order in $\boldsymbol{\chi}$ this means,

$$\boldsymbol{\chi}^\dagger = \boldsymbol{\chi}.$$

Again, as with SU(2), $\boldsymbol{\chi}$ must be traceless. This means that infinitesimal SU(3) transformations are associated with **3×3 traceless Hermitian** matrices. The condition that the matrices be Hermitian reduces the number of free parameters in the matrix to eight:

$$\mathbf{Q}^{infl} = \begin{pmatrix} a_{11} & a_{12}+ib_{12} & a_{13}+ib_{13} \\ a_{12}-ib_{12} & a_{22} & a_{23}+ib_{23} \\ a_{13}-ib_{13} & a_{23}-ib_{23} & a_{33} \end{pmatrix}$$

where the a's and the b's are all real parameters. When we add in the additonal constraint that $\mathbf{Q}^{infl}$ be traceless the number of parameters is then reduced to eight (the same working gives three parameters in the case of SU(2)). As we have **eight free parameters** we will therefore have **eight generators**.

We start with the fundamental "**3**" representation (of dimension three, obviously) in which an element of SU(3) is represented by itself, and the generators are 3×3 matrices. SU(3) matrices act on three- component complex vectors:

$$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = q_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + q_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + q_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

[For SU(3) flavour symmetry we would have $q_1 = u$, $q_2 = d$, $q_3 = s$ while for colour symmetry we have $q_1 = r$, $q_2 = g$, $q_3 = b$.] Let's denote a transformation of our q_1, q_2, q_3 "coordinates" by:

$$q \to q' = \mathbf{Q}q \text{ where } \mathbf{Q} \in \mathrm{SU}(3).$$

We slightly rewrite the infinitesimal version as :

$$\mathbf{Q}^{infl} = \mathbf{I} + i\mathbf{a}.\boldsymbol{\lambda}/2$$

where $\mathbf{a}$ now stands for the eight real infinitesimal parameters, $\mathbf{a} = (a_1, ...a_8)$, which are just the previous a's and b's reorganised, and $\boldsymbol{\lambda}$ stands for eight Hermitian 3×3 matrices (the generators of SU(3)), $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_8)$, like

$\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$:

$$\lambda_1 = \begin{pmatrix} \mathbf{0} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{0} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} \mathbf{0} & \boldsymbol{-i} & 0 \\ \boldsymbol{i} & \mathbf{0} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix}$$

These are the *Gell-Mann matrices. These eight Hermitian* 3×3 *matrices represent the generators of SU(3) in the fundamental representation.*

Note that the first three generators $\lambda_1, \lambda_2, \lambda_3$ are simply the Pauli spin matrices, i.e. the SU(2) generators in the **2** representation, which are shown in bold type on this occasion, and they are augmented by a row and a column of zeros. The three generators $\lambda_1, \lambda_2, \lambda_3$ obey the same commutation relations as the corresponding spin matrices, obviously, which is just the SU(2) algebra. This is known as the SU(2) *sub-algebra* of SU(3); SU(2) is a *subgroup* of SU(3). Also note that λ_3 and λ_8 are diagonal: this means that we have *two* additively conserved quantum numbers (namely their eigenvalues), which in the case of flavour would be I_3 and Y for instance. The number of additively conserved quantum numbers appropriate to a given symmetry group is called the "rank" of the group: SO(3) and SU(2) are both of rank one, while SU(3) is of rank two.

A finite SU(3) transformation matrix can be written as,

$$e^{\frac{i}{2}\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}}$$

with eight finite real parameters ($\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_8)$).

We shall not take the time to calculate the *algebra of SU(3)* as we did for SU(2). We can easily find out what it is by evaluating the commutation relations among all the λ's. We find

$$[\lambda_i/2, \lambda_j/2] = f_{ijk}\lambda_k/2$$

where the f_{ijk} are known as the structure constants, and are tabulated in books. Generally there will also be bigger matrices representing the generators in other,larger, representations but they all obey the same algebra:

$$[G_i, G_j] = f_{ijk}G_k.$$

So $G_i^{(3)} = \lambda_i/2$ where we are labelling the generators by the dimensionality of the representation (not quite what we did for SU(2), but it will serve our purpose). Note that: trace $(\lambda_i\lambda_j) = 2\delta_{ij}$; compare trace($\tau_i\tau_j$) $= 2\delta_{ij}$.

Other Irreducible Representations of SU(3)

We have met two ways of finding representations - which is basically what physicists want to know about, since particle states correspond precisely to the contents of different representations. One way (used for rotations and the Lorentz group) relied on "angular momentum know-how", which was really the operator approach to angular momentum in quantum mechanics, starting from the angular momentum commutation relations - i.e. the algebra of the generators. We could do the same for SU(3), analysing the algebra of the 8 generators, and introducing generalisations of the angular momentum raising and lowering operators which take us from one state in a multiplet to any other. But this is a more general approach than we need - after all, it seems that in practice we are interested in rather few types of SU(3) representation. To learn about these few, it is more practical to follow the "tensor" method of Lecture 5, which as we saw has the advantage that it produces explicit wavefunctions. So we shall "multiply low order representations together" to get higher represtations.

The first thing we need to do is understand that there is another three- dimensional representation, the $\mathbf{3}^*$, which physically is the representation associated with the antiquarks. Note (before we start) that the $\mathbf{3}^*$ cannot, physically, be equivalent to the $\mathbf{3}$ as the $\mathbf{2}^*$ was to the $\mathbf{2}$, since antiquarks have different SU(3) quantum numbers from quarks (compare with SU(2): the antiproton and the antineutron are have the same SU(2) quantum numbers as the neutron and the proton, respectively).

The 3-dimensional ("self", or "fundamental") representation of SU(3) has a basis provided by the three "coordinates" q_i transforming by $q_i' = Q_{ij}q_j$. Taking the complex conjugate of this expression we find

$$q_i'^* = Q_{ij}^* q_j^*.$$

Using the fact that $\mathbf{Q}$ is Hermitian we have,

$$\mathbf{Q}^\dagger \mathbf{Q} = \mathbf{I}$$
$$\Rightarrow \mathbf{Q}^{-1} = \mathbf{Q}^\dagger$$
$$\Rightarrow \left(\mathbf{Q}^{-1}\right)_{ij} = \left(\mathbf{Q}^\dagger\right)_{ij} = \mathbf{Q}_{ji}^*$$
$$\Rightarrow \mathbf{Q}_{ij}^* = \mathbf{Q}_{ji}^{-1}$$

So $q_i'^*$ i.e. a component of a $\mathbf{3}^*$, transforms as,

$$q_i'^* = q_j^* \mathbf{Q}_{ji}^{-1}.$$

Is the $\mathbf{3}^*$ representation *equivalent* to the $\mathbf{3}$ representation? Look at the infinitesimal transformations:

$$q' = (1 + i\mathbf{a}.\boldsymbol{\lambda}/2)\, q$$

$$q'^* = (1 - i\mathbf{a}.\boldsymbol{\lambda}^*/2)\, q^*.$$

The question is, can we find a matrix $\mathbf{S}$ such that $\mathbf{S}\boldsymbol{\lambda}^*\mathbf{S}^{-1} = -\boldsymbol{\lambda}$? If so, we easily deduce that $\mathbf{S}q^*$ (which is just a linear combination of the components of q^*) would transform in just the same way as q, and the two of them would not be physically different (and they'd be mathematically equivalent). The analogous equivalence *does* exist in SU(2) i.e. betwen the $\mathbf{2}$ and the $\mathbf{2}^*$. In that case, the job is done by choosing $\mathbf{S} = \sigma_2$. Then, $\sigma_2\boldsymbol{\sigma}^*\sigma_2 = -\boldsymbol{\sigma}$ (check it!), and so $\mathbf{2}$ is equivalent to $\mathbf{2}^*$. In the case of SU(3) though, it can't be done. A simple way to see this is to consider the particular matrix λ_8, for which $\lambda_8 = \lambda_8^*$. This means we need $\mathbf{S}\lambda_8\mathbf{S} = -\lambda_8$ i.e. λ_8 and $-\lambda_8$ would have to have the same eigenvalues which is impossible with

$$\lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix}$$

(and is possible with τ_3!). Hence the $\mathbf{3}^*$ representation is not equivalent to $\mathbf{3}$ representation and we now have *two different three-dimensional representations of SU(3)* - the triplet and the antitriplet, or quarks and antiquarks if you like.

In flavour SU(3), $\lambda_3/2$ would be interpreted as I_3 and the hypercharge $Y = B + S$ would be

$$Y = \frac{\lambda_8}{\sqrt{3}} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{-2}{3} \end{pmatrix}$$

. What about the complex conjugate representation? We had $q_i'^* = \mathbf{e}^{-\frac{i}{2}\boldsymbol{\alpha}.\boldsymbol{\lambda}^*} q^* = e^{-\frac{i}{2}\alpha_3\lambda_3 - \frac{i}{2}\alpha_8\lambda_8 - \cdots\cdots} q$. So in this complex conjugate representation, the signs of the terms associated with the diagonal matrices λ_3 and λ_8 have been reversed relative to the original representation. . So for q^*, the "I_3" is,

$$\begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and "Y" is,

$$\begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix},$$

which are the quantum numbers of the antiquarks.

OK, now we are ready to start the programme of building larger representations by multiplying smaller ones together (and remember the snag in this approach - the "reducible/irreducible" problem). Consider the object $q_1^* q_1 + q_2^* q_2 + q_3^* q_3 = q_i^* q_i$. The usual transformation with $\mathbf{Q}$ leaves this quantity invariant:

$$q_i'^* q_i' = q_j^* Q_{ji}^{-1} Q_{ik} q_k = q_j^* \delta_{jk} q_k = q_j^* q_j .$$

This is a single (invariant) object, transforming according to the singlet representation $\mathbf{1}$. In colour language we write "$\frac{1}{\sqrt{3}}\left(\overline{r}r + \overline{b}b + \overline{g}g\right)$ for the normalised singlet wavefunction. The π^+ for instance is $\frac{1}{\sqrt{3}}(\bar{d}_r u_r + \bar{d}_b u_b + \bar{d}_g u_g)$.

An alternative way of writing "$q_1^* q_1 + q_2^* q_2 + q_3^* q_3$" is "$q^\dagger q$" , where we are thinking of the q as a column vector and the $q^\dagger$ as a row vector with entries the complex conjugates of those in q. We used this notation in SU(2). Less formally, it's often written as $\overline{q}q$, which tallies with the complex conjugate representation being the one for antiparticles, but you have to note that (at least at this stage) the "bar" is not the same as the one for Dirac wavefunctions which we introduced at the end of Lecture 5. In this form it transforms to $q'^\dagger q' = (\mathbf{Q}q)^\dagger \mathbf{Q}q = q^\dagger \mathbf{Q}^\dagger \mathbf{Q} q = q^\dagger q$ at once, as we saw in the SU(2) case.

So, we have learned how to make an SU(3) singlet $\mathbf{1}$ out of a $\mathbf{3}^*$ and a $\mathbf{3}$. It is the direct analogue of the singlet $q^\dagger q$ in SU(2). What is the SU(3) analogue of the *vector*, or SU(2) triplet, coupling $q^\dagger \boldsymbol{\tau}\, q$? We shall make a guess - that $q^\dagger \boldsymbol{\lambda}\, q$ is something like an SU(3) "vector". Consider an infinitesimal transformation:

$$q \to q' = \left(1 + \frac{i}{2}\mathbf{a}.\boldsymbol{\lambda}\right) q$$

$$\Rightarrow q^\dagger \lambda_j q \to q'^\dagger \lambda_j q' = q^\dagger \left(1 - \frac{i}{2}\mathbf{a}.\boldsymbol{\lambda}\right) \lambda_j \left(1 + \frac{i}{2}\mathbf{a}.\boldsymbol{\lambda}\right) q$$

$$\Rightarrow q^\dagger \lambda_j q \to q'^\dagger \lambda_j q' = q^\dagger \lambda_j q - \frac{i}{2} a_i q^\dagger \lambda_i \lambda_j q + \frac{i}{2} a_i q^\dagger \lambda_j \lambda_i q$$

$$\Rightarrow q^\dagger \lambda_j q \to q'^\dagger \lambda_j q' = q^\dagger \lambda_j q - \frac{i}{2} a_i q^\dagger \left[\lambda_i, \lambda_j\right] q$$

$$\Rightarrow q^\dagger \lambda_j q \to q'^\dagger \lambda_j q' = q^\dagger \lambda_j q + a_i q^\dagger f_{ijk} \lambda_k q$$

or,

$$V_j \to V_j' = V_j + a_i f_{ijk} V_k .$$

This should look familiar! Remember, for an infinitesimal 3-vector rotation we had:

$$x_j \to x_j' = x_j + a_i \epsilon_{ijk} x_k$$

which is just the "suffix" way of writing

$$\mathbf{r} \to \mathbf{r}' = \mathbf{r} + \mathbf{a} \times \mathbf{r}.$$

So the eight components V_i transform into linear combinations of each other and so form the basis of a representation. In fact they form an irreducible representation called $\mathbf{8}$, the octet. Note that although we arrived at this "bigger" representation by "multiplying" (or coupling) two smaller ones together, once we have got this transformation law, it tells us how *any* octet of states transforms - it doesn't have to be literally seen as a $\mathbf{3}^* \otimes \mathbf{3}$ composite state.

Under a finite SU(3) transformation, the components of the "octet" $\mathbf{V}$ transform according to

$$\begin{pmatrix} V_1 \\ . \\ . \\ . \\ V_8 \end{pmatrix} \to \begin{pmatrix} V_1' \\ . \\ . \\ . \\ V_8' \end{pmatrix} = e^{i\,\boldsymbol{\alpha}\,.\,\mathbf{G}^{(8)}} \begin{pmatrix} V_1 \\ . \\ . \\ . \\ V_8 \end{pmatrix},$$

where the $\mathbf{G}^{(8)}$'s are the generators (eight Hermitian 8×8 matrices) of SU(3) in the 8-dimensional representation. For an infinitesimal transformation of an SU(3) octet,

$$V_j' = (1 + i\mathbf{a}.\mathbf{G}^{(8)})_{jk} V_k = V_j + i a_i (G_i^{(8)})_{jk} V_k = V_j + a_i f_{ijk} V_k ,$$

by the previous result. So we are saying that $\left(G_i^{(8)}\right)_{jk} = -f_{ijk}$ (this is exactly like $\left(L_i^{(1)}\right)_{jk} = -i\epsilon_{ijk}$!) i.e. we can always find a representation by making use of the structure constants. This representation in which the matrix elements of the generators are essentially equal to the structure constants of the group is always possible for a Lie group, and is called the **regular** or **adjoint** representation.

In QCD the gluons are an **8** transforming in this way under SU(3)$_{colour}$. They transform under SU(3) transformations by the "f" coefficients as above. And to reiterate the point made earlier: the fact that they transform as the octet coupling of a $\mathbf{3}^*$ and a **3** doesn't mean they actually are literally that kind of composite state. However, it does explain why people often talk about the gluons as being like "colour-anticolour" combinations (with the singlet subtracted out).

Now that we have understood the transformation of the gluons, we can easily guess what is is *SU(3)-invariant coupling* of gluons to quarks - we need this, of course, since the QCD interactions are precisely SU(3) (colour) - invariant. To make the required invariant coupling, we ned to couple the **8** of gluons, $\boldsymbol{A}$, to an **8** made from the quarks - and the latter is of course just $q^\dagger \boldsymbol{\lambda} q$. It seems a good bet that the invariant coupling will be just the "dot product" of these: $q^\dagger \boldsymbol{\lambda}_i q.\boldsymbol{A}_i$, and this is indeed the case.

So we have learned that just as

$$\left(\text{spin}-\frac{1}{2}\right)\times\left(\text{spin}-\frac{1}{2}\right)=(\text{spin}-1)+(\text{spin}-0)\ \text{ in SO(3)}$$

or

$$\mathbf{2}\otimes\mathbf{2}\rightarrow\mathbf{3}\oplus\mathbf{1}$$

in SU(2), in SU(3) we have

$$\mathbf{3}\otimes\mathbf{3^*}\rightarrow\mathbf{8}\oplus\mathbf{1}\,.$$

For flavour SU(3), this would be an SU(3) - octet of mesons, and an SU(3) - singlet meson. For SU(2) we saw that it was nice to write $\mathbf{V}=q^\dagger\boldsymbol{\tau}\,q$ as the entries in the the 2×2 Hermitian matrix $\mathbf{V}.\boldsymbol{\tau}$ which produced combinations with physically meaningful quantum numbers. Here we want to consider $\mathbf{V}.\boldsymbol{\lambda}$ where

$$\begin{array}{ccccccccc}
V_1 & = & q^\dagger\lambda_1 q & = & q_1^*q_2 & + & q_2^*q_1 & = & \overline{u}d+\overline{d}u\\
V_2 & = & q^\dagger\lambda_2 q & = & iq_2^*q_1 & - & iq_1^*q_2 & = & i\overline{d}u-i\overline{u}d\\
V_3 & = & q^\dagger\lambda_3 q & = & q_1^*q_1 & - & q_2^*q_2 & = & \overline{u}u-\overline{d}d
\end{array}$$

etc. Then $\mathbf{V}.\boldsymbol{\lambda}$ is

$$\left(\begin{array}{ccc} V_3+\frac{1}{3}V_8 & V_1-iV_2 & V_4-iV_5\\ V_1+iV_2 & etc & \\ & & \end{array}\right)=\frac{1}{2}\left(\begin{array}{ccc} q_1^*q_1-\frac{1}{3}\mathbf{q}^\dagger.\mathbf{q} & q_2^*q_1 & q_3^*q_1\\ q_1^*q_2 & q_2^*q_2-\frac{1}{3}\mathbf{q}^\dagger\mathbf{q} & q_3^*q_2\\ q_1^*q_3 & q_2^*q_3 & q_3^*q_3-\frac{1}{3}\mathbf{q}^\dagger\mathbf{q}\end{array}\right)$$

or

$$\frac{1}{2}\left(\begin{array}{ccc} \overline{u}u-\frac{1}{3}\left(\overline{u}u+\overline{d}d+\overline{s}s\right) & \overline{d}u\sim\pi^- & \overline{s}u\sim\mathbf{K}^+\\ \overline{u}d\sim\pi^+ & \overline{d}d-\frac{1}{3}\left(\overline{u}u+\overline{d}d+\overline{s}s\right) & \overline{s}d\sim\mathbf{K}^0\\ \overline{u}s\sim\mathbf{K}^- & \overline{d}s\sim\pi^0 & \overline{s}s-\frac{1}{3}\left(\overline{u}u+\overline{d}d+\overline{s}s\right)\end{array}\right)$$

or

$$q_iq_j^*-\frac{1}{3}\left(q^\dagger q\right)\delta_{ij}=T_{ij}$$

which is precisely a *traceless second rank tensor*! As such, we expect it to provide the basis of an irreducible representation, and it does. Note that since the (*) represntation is inequivalent to the unstarred one, it makes no sense to symmetrise/antisymmetrise with respect to labels of q^*'s and q's (we don't expect to do that for quarks and antiquarks - only quarks among themselves, and antiquarks among themselves).

Now consider "$\mathbf{3}\times\mathbf{3}$" i.e. p_iq_j. Lets start with the following *antisymmetric* combinations:

$$\epsilon_{ijk}p_jq_k;$$

how do they transform? The components are $(p_2q_3-p_3q_2,\ p_3q_1-p_1q_3,\ p_1q_2-p_2q_1)$. In terms of flavours this is $(ds-sd,\ su-us,\ ud-du)$. It is simple to check that these components have the same quantum numbers Q,Y,I_3 as $\left(\overline{u},\overline{d},\overline{s}\right)$. In fact we can show by brute force that $(ds-sd,\ su-us,\ ud-du)$ does transform as $\mathbf{3}^*$.

This leaves the *symmetric* combinations $p_iq_j+p_jq_i$ of which there are six, which cannot be reduced further. So

$$\mathbf{3}\otimes\mathbf{3}=\mathbf{6}\oplus\mathbf{3}^*.$$

What about $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ then? (These give baryons!) The answer is

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{3} \otimes (\mathbf{6} \oplus (\mathbf{3}^*) = (\mathbf{3} \otimes \mathbf{6}) \oplus (\mathbf{3} \otimes \mathbf{3}^*) = \mathbf{8} \oplus \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{1}$$

and the bits on the RHS number 27 as they should. We sketch the proof.

We begin by considering the combinations $(\epsilon_{dab} p_a q_b) s_c$, which are *antisymmetric* in p and q. The part in brackets transforms as a $\mathbf{3}^*$ as we already know. So the whole thing behaves as $\mathbf{3}^* \otimes \mathbf{3}$. The trace is found by setting $d = c$ and summing, and this has to be subtracted out to get an irreducible remainder. This trace is $\epsilon_{cab} p_a q_b s_c$, which has no free indices and is therefore an SU(3) invariant, i.e. an SU(3) singlet, $\mathbf{1}$.

[Note that it is in fact the *determinant*

$$\begin{vmatrix} p_r & p_b & p_g \\ q_r & q_b & q_g \\ s_r & s_b & s_g \end{vmatrix}$$

for the colour case. This is the colour part of the wavefunction for three quarks in a baryon, and it is totally antisymmetric in the quark labels, as is required by the Pauli principle in that case.]

The remaining traceless part of the $\mathbf{3}^* \otimes \mathbf{3}$ is

$$\epsilon_{dab} p_a q_b s_c - \frac{1}{3} \epsilon_{eab} p_a q_b s_e \delta_{dc}$$

which is an $\mathbf{8}$. Next, note that we can start with an independent combination of p and q by considering the *symmetric* combination $(p_a q_b + p_b q_a)$, which has 6 different components (note how these dimensionalities figure in the "answer" given above). Multiplying this onto s_c gives 18 components, but again this is not an irreducible basis. We can "reduce" it by antisymmetrising in q and s, giving $\epsilon_{dbc}(p_a q_b + p_b q_a) s_c$ which seems to have 9 components. But you can check that it is in fact traceless, so there's only 8 components there, and they form another $\mathbf{8}$. This leaves 10 things from the 18, and these are *totally symmetric* and form the basis of the $\mathbf{10}$. For example, in SU(3) flavour this would be

$$\begin{array}{rcccl} p_1 q_1 s_1 & = & uuu & = & \Delta^{++} \\ p_2 q_2 s_2 & = & ddd & = & \Delta^{--} \\ p_3 q_3 s_3 & = & sss & = & \Omega^{-} \\ p_1 q_1 s_2 + p_2 q_1 s_1 + p_1 q_2 s_1 & = & uud + udu + duu & = & \Delta^{+} \\ p_1 q_1 s_3 + p_3 q_1 s_1 + p_1 q_3 s_1 & = & uus + usu + suu & = & \Sigma^{+} \end{array}$$

etc......members of the $\frac{3}{2}+$ decuplet.

The general result to note from all of this is that we obtain irreducible representations by considering traceless tensors of definite symmetry character. This procedure gives us, in fact, the actual SU(3)-space wavefunctions (amplitudes). Remember that, as noted after the flavour octet, the symmetrisation/antisymmetrisation is done on the quark-type ($\mathbf{3}$) indices separately from the antiquark-type ($\mathbf{3}^*$) indices.