Introduction to Symmetries

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March 27, 2001

Lecture 5:

The Group SU(2): "Isospin" and "Internal Symmetry".

In this and the next Lecture (on SU(3)) we move away from "space-time" symmetries (translations, rotations, LT's) and consider "internal" symmetries. The basic approach, however, will be very similar to that in the "Symmetry and Degeneracy" section of Lecture 2, at least to begin with. The first difference will be that whereas in Lecture 2 we were considering degeneracies such as the $2l+1$ - fold degeneracy of states with different $\widehat{L}_z$ eigenvalue for given l, here we shall be typically considering mass degeneracies between particle states differing in their *charge*, or other "internal" quantum numbers. The second difference is that the transformations we shall be considering are not real-space rotations, or whatever, but rather unitary transformations in the space of the degenerate states. This is always allowed in quantum mechanics, and of course the spatial rotations were, as we saw, unitary operations. Here, though, the thing is slightly more abstract.

Consider, for example, the (near) degeneracy between the proton and the neutron masses, or between the up and down quark masses. Let's pretend, in fact, that the up and down quarks are exactly degenerate in mass, and let's associate with them amplitudes ("flavour wavefunctions") ψ_u and ψ_d respectively. Then the u - d degeneracy means that we can equally well choose to describe the situation in terms of alternative amplitudes ψ'_u and ψ'_d where the primed amplitudes are linear combinations of the unprimed ones, as given by some matrix $\mathbf{Q}$:

$$\begin{pmatrix} \psi'_u \\ \psi'_d \end{pmatrix} = \mathbf{Q} \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$$

where $\mathbf{Q}$ is unitary so as to preserve the normalisation of the states. This is, of course, exactly the same kind of transformation as we saw for the spinors ϕ and χ , and the 2-component object $\begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$ is called an "isospinor".

The result at the end of Lecture 3 tells us the form of $\mathbf{Q}$, in fact. We obtained it there by a bit of guesswork, and by analogy with the transformation law we had worked out explicitly for a vector. We shall begin our internal symmetry discussion by deriving the form of $\mathbf{Q}$ another way, which will be useful in other cases.

The general unitary 2×2 matrix transforming a two-component doublet, as above, will be denoted by $\mathbf{U}$. Since $\mathbf{U}$ is unitary,

$$\mathbf{U}^\dagger\mathbf{U} = \mathbf{I}.$$

This implies

$$\det\left(\mathbf{U}^\dagger\mathbf{U}\right) = 1$$
$$\Rightarrow (\det\mathbf{U})^* (\det\mathbf{U}) = 1 \Rightarrow |\det\mathbf{U}| = 1$$
$$\Rightarrow \det\mathbf{U} = e^{i\alpha}.$$

Such matrices belong to the group U(2) (the group of unitary 2×2 matrices; recall the group axioms of Lecture 2). Suppose we transform ψ_u and ψ_d by $\psi'_u = e^{i\theta}\psi_u$ and $\psi'_d = e^{i\theta}\psi_d$ i.e an overall phase redefinition . Then for this transformation

$$\mathbf{U} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \text{ and } \det\mathbf{U} = e^{2i\theta}.$$

Such transformations, which change the ψ_u and ψ_d amplitudes by the <u>same</u> phase are physically irrelevant (in our little u-d world), and we can eliminate them by requiring $\det\mathbf{U} = 1$ (i.e. $\alpha = 0$). In this case $\mathbf{U}$ becomes a member of the group SU(2) (the S is for special i.e. $\det \mathbf{U} = 1$). Such matrices we denote by $\mathbf{Q}$. Note that this is another *group of matrices.*

As we shall see (again) we can establish all of the physical consequences by considering infinitesimal forms of the transformation (as in the case of infinitesimal rotations). An infinitesimal SU(2) trasformation differs infinitesimally from doing nothing, which is the identity matrix; so we write

$$\mathbf{Q}^{infl} = \mathbf{I} - i\boldsymbol{\xi}$$

where $\boldsymbol{\xi}$ is an infinitesimal 2×2 matrix i.e. its elements are all infinitesimal. The condition $\mathbf{Q}^\dagger \mathbf{Q} = \mathbf{I}$ then requires

$$\left(\mathbf{I} + i\boldsymbol{\xi}^\dagger\right)(\mathbf{I} - i\boldsymbol{\xi}) = \mathbf{I}$$

which implies $\boldsymbol{\xi}^\dagger = \boldsymbol{\xi}$ to first order i.e. $\boldsymbol{\xi}$ is Hermitian (as was the case for the $\widehat{\mathbf{J}}$'s in angular momentum). Also,

$$\det \mathbf{Q} = 1 \Rightarrow \begin{vmatrix} 1 - i\xi_{11} & -i\xi_{12} \\ -i\xi_{21} & 1 - i\xi_{22} \end{vmatrix} = 1$$

$$\Rightarrow \xi_{11} + \xi_{22} = 0$$

i.e. the $\det \mathbf{Q} = 1$ constraint means $\boldsymbol{\xi}$ is traceless. So we can write $\boldsymbol{\xi}$ in terms of <u>three real</u> parameters a, b and c:

$$\boldsymbol{\xi} = \begin{pmatrix} a & b - ic \\ b + ic & -a \end{pmatrix}$$

which we can write as

$$\boldsymbol{\xi} = \begin{pmatrix} \frac{a_3}{2} & \frac{a_1}{2} - \frac{ia_2}{2} \\ \frac{a_1}{2} + \frac{ia_2}{2} & -\frac{a_3}{2} \end{pmatrix} = \frac{1}{2}\mathbf{a}.\boldsymbol{\sigma}\,.$$

This is exactly as we found for the "space-time" spinors ϕ and χ. But in order to distinguish these "internal" transformations from the space-time ones, it is usual to use a different notation for the $\boldsymbol{\sigma}$ matrices; we call them the $\boldsymbol{\tau}$ matrices instead. So for internal rotations, $\mathbf{Q}^{infl} = \mathbf{I} - \frac{i}{2}\mathbf{a}.\boldsymbol{\tau}$ and $\mathbf{Q}^{finite} = e^{-i\frac{1}{2}\alpha\mathbf{n}.\tau}$. There are three parameters in play here, whether the transformation is infinitesimal (the three $\mathbf{a}$'s), or finite; in the latter case we have α and two parameters specifying the unit vector $\mathbf{n}$. And this way or writing $\mathbf{Q}^{finite}$ is in fact a way of writing the most general SU(2) matrix $\mathbf{Q}$.

So however you look at it, whether as a real space-time doublet or in internal space doublet, the transformation law is the same and is basically just an SU(2) matrix. This is why the u-d doublet is called an isospinor (from nuclear "isospin" originally). We shall see even more analogies between SU(2) and rotations (i.e. SO(3)) in a minute. Anyway, under finite SU(2) transformations, we have learned that

$$\begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}' = e^{-\frac{i}{2}\alpha\mathbf{n}.\tau}\begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}.$$

Note that

$$\frac{1}{2}\tau_3\begin{pmatrix} \psi_u \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \psi_u \\ 0 \end{pmatrix} \text{ and } \frac{1}{2}\tau_3\begin{pmatrix} 0 \\ \psi_d \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ \psi_d \end{pmatrix}.$$

So $\begin{pmatrix} \psi_u \\ 0 \end{pmatrix}$ is a state with third component of isospin equal to $\frac{1}{2}$, while $\begin{pmatrix} 0 \\ \psi_d \end{pmatrix}$ has component - $\frac{1}{2}$. And of course the manitude of the isospin is $I = \frac{1}{2}$.

So far we have only got as far as defining the group SU(2), writing the general element $\mathbf{Q}$ in a way that is after all quite familiar from earlier work, and seeing how the associated two-component column vectors transform. Now we want to start doing for SU(2) the sorts of things we did for SO(3) and the Lorentz group i.e we want to find the *algebra* and *matrix representations of the algebra.*

To find the algebra of SU(2), we follow the exact analogue of the "$\mathbf{U_R}\psi(\mathbf{r}) = \psi(\mathbf{R}^{-1}\mathbf{r})$" kind of thing we did for SO(3). Let's rename the two amplitudes ψ_u and ψ_d as $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$. The quantities q_1 and q_2 can be regarded as complex "coordinates" in this two-dimensional complex vector space. An SU(2) transformation in this "space" is:

$$q \to q' = \mathbf{Q}q$$

where $\mathbf{Q}$ is an SU(2) matrix. We can construct functions of these coordinates, $\psi(q_1, q_2)$, which we can think of as analogous to our wavefunction $\psi(\mathbf{r})$. Then for any such $\psi(q_1, q_2)$ the corresponding transformed ψ will be

$$\psi(q)' = \widehat{\mathbf{U}}_{\mathbf{Q}}\psi(q)$$

where for consistency between the primed and unprimed descriptions

$$\psi(q)' = \psi(\mathbf{Q}^{-1}q)$$

and this will give us the rule for how these ψ's transform. The right hand side is, for an infinitesimal transformation:

$$\begin{aligned}
\psi\left(\mathbf{Q}^{-1}q\right) &= \psi\left(\left(1 + \tfrac{i}{2}\mathbf{a}.\boldsymbol{\tau}\right)q\right) \\
&= \psi\left(\begin{pmatrix} 1 + i\frac{a_3}{2} & i\frac{a_1}{2} + \frac{a_2}{2} \\ i\frac{a_1}{2} - \frac{a_2}{2} & 1 - i\frac{a_3}{2} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right) \\
&= \psi\left(q_1 + i\frac{a_3q_1}{2} + i\frac{a_1q_2}{2}, i\frac{a_1q_1}{2} - \frac{a_2q_1}{2} + q_2 - i\frac{a_3q_2}{2}\right)
\end{aligned}$$

Expanding this as usual for a Taylor series of a function of two variables we have,

$$\begin{aligned}
\mathbf{U}_{\mathbf{Q}}\psi(q_1, q_2) = \psi(q_1, q_2) \quad & +i\frac{a_3q_1}{2}.\frac{\partial\psi}{\partial q_1} \quad +i\frac{a_1q_2}{2}.\frac{\partial\psi}{\partial q_1} \quad +i\frac{a_2q_2}{2}.\frac{\partial\psi}{\partial q_1} \\
& -i\frac{a_3q_2}{2}.\frac{\partial\psi}{\partial q_2} \quad +i\frac{a_1q_1}{2}.\frac{\partial\psi}{\partial q_2} \quad -\frac{a_2q_1}{2}.\frac{\partial\psi}{\partial q_2}
\end{aligned}$$

. We write this in the traditional way for all infinitesimal transformations:

$$\Rightarrow \mathbf{U}_{\mathbf{Q}}\psi(q_1, q_2) = \left(\mathbf{I} - i\mathbf{a}.\widehat{\mathbf{X}}\right)\psi(q_1, q_2)$$

where this time

$$\begin{aligned}
\widehat{X}_1 &= -\frac{q_2}{2}.\frac{\partial}{\partial q_1} - \frac{q_1}{2}.\frac{\partial}{\partial q_2} \\
\widehat{X}_2 &= \frac{iq_2}{2}.\frac{\partial}{\partial q_1} - \frac{iq_1}{2}.\frac{\partial}{\partial q_2} \\
\widehat{X}_3 &= -\frac{q_1}{2}.\frac{\partial}{\partial q_1} + \frac{q_2}{2}.\frac{\partial}{\partial q_2}.
\end{aligned}$$

$\widehat{X}_1, \widehat{X}_2$ and $\widehat{X}_3$ are the **generators** of SU(2) just as $\widehat{\mathbf{L}} = \mathbf{r} \times \widehat{\mathbf{p}}$ are for SO(3).

Now, what about the *algebra of these generators*? It is an interesting exercise to work out the commutation relations for the generators $\widehat{\mathbf{X}}$ as given above; one finds

$$\left[\widehat{X}_1, \widehat{X}_2\right] = i\widehat{X}_3$$

just as for the generators of SO(3)! We say that SU(2) and SO(3) "have the same algebra" (i.e. their generators obey the same commutation relations). Clearly, then, the groups SO(3) and SU(2) are very closely related, if their generators commute in exactly the same way. This implies, in fact, that the matrices representing their generators in any representation must be the same - since after all, a matrix representation of the generators is just a set of matrices with the same commutation relations as the generators. So the whole business of representations of SU(2) boils down, more or less, to taking over the results already learned in SO(3). This is of course an even better reason to regard SU(2) transformations as some kind of "internal space rotations".

However, rather than just take the stuff over blindly from SO(3), we shall investigate a way of getting SU(2) representations which is slightly different from the ways we used for SO(3), and which will be easily generalised to SU(3).

Representations of SU(2)

Remember that a "representation" of a group is a set of matrices that satisfy the same multiplication law as the elements of the group. SU(2) is a group of matrices so obviously one "representation" is the one in which $\mathbf{Q}$ is represented by itself! This is called the "self" or "fundamental" representation. (It is, again, perhaps slightly puzzling at first to contemplate other representations, namely bigger matrices which, in fact, have the same multiplication law as the $\mathbf{Q}$'s do. It's the same as in SO(3).) As indicated above, the representations of SU(2) must be closely related to those of SO(3). Remember that we can label representations of SO(3) with different spin quantum numbers $|j, m_j\rangle$. In the language of spin, clearly the two dimensional fundamental representation has "isospin" $\frac{1}{2}$, as we have already suggested. However there are three dimensional representations of SU(2) (with isospin 1) , four dimensional ones (with isospin $\frac{3}{2}$) etc. There are many ways of finding these representations explicitly - the (new to us) way we shall now discuss is called "The Tensor Method".

First consider an analogy from SO(3). Let's start with a vector $\mathbf{r}_1 = (x_1, y_1, z_1)$ which forms a basis for the three dimensional representation of SO(3) as we have seen. Now consider a second vector $\mathbf{r}_2$: what kind of <u>products</u> can we make from $\mathbf{r}_1$ and $\mathbf{r}_2$? Well, if we take any one component of $\mathbf{r}_1$, say $\mathbf{r}_{1i}$ $(i = x, y, z)$ and any component $\mathbf{r}_{2j}$ of $\mathbf{r}_2$ $(j = x, y, z)$ and then multiply them, we end up with nine things, $\mathbf{r}_{1x}\mathbf{r}_{2x}, \mathbf{r}_{1x}\mathbf{r}_{2y}, ..., \mathbf{r}_{1z}\mathbf{r}_{2z}$. Under an SO(3) transformation (a rotation), $\mathbf{r}_1 \to \mathbf{r}_1' = \mathbf{R}\mathbf{r}_1$ and $\mathbf{r}_2 \to \mathbf{r}_2' = \mathbf{R}\mathbf{r}_2$. Both $\mathbf{r}_1'$ and $\mathbf{r}_2'$ are just linear combinations of the original components $\mathbf{r}_1$ and $\mathbf{r}_2$. So under an SO(3) rotation,

$$\begin{array}{rcl} \mathbf{r}_{1i}\mathbf{r}_{2j} \quad \to \quad \mathbf{r}_{1i}'\mathbf{r}_{2j}' & = & \sum_p \mathbf{R}_{ip}\mathbf{r}_{1p} \sum_q \mathbf{R}_{jq}\mathbf{r}_{2q} \\ & = & \sum_p \sum_q \mathbf{R}_{ip}\mathbf{R}_{jq}\mathbf{r}_{1p}\mathbf{r}_{2q} \end{array}$$

which is some linear combination of the original nine things, $\mathbf{r}_{1p}\mathbf{r}_{2q}$:

$$\begin{pmatrix} \mathbf{r}_{1x}'\mathbf{r}_{2x}' \\ \mathbf{r}_{1x}'\mathbf{r}_{2y}' \\ \mathbf{r}_{1x}'\mathbf{r}_{2z}' \\ . \\ . \\ . \\ . \\ . \\ . \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ 9 & \times & 9 \\ & & \\ & \text{Matrix} & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} \mathbf{r}_{1x}\mathbf{r}_{2x} \\ \mathbf{r}_{1x}\mathbf{r}_{2y} \\ \mathbf{r}_{1x}\mathbf{r}_{2z} \\ . \\ . \\ . \\ . \\ . \\ . \end{pmatrix}$$

This 9×9 matrix does, in fact, represent this rotation using the 9 things $\{\ \mathbf{r}_{1i}\mathbf{r}_{2j}\ \}$ as a <u>basis</u>.

But there's a snag.....This representation is *reducible.* This is a crucial new concept. What it means, in this case, and in others by generalisation, is that we are not, in fact, looking at a genuinely nine-dimensional representation at all (which would correspond to a "j" of 4). If we were to consider some actual rotation, we would indeed be able to find all the entries in the above 9 ×9 matrix, showing that the nine base elements are transforming linearly among themselves as the members of a decent basis should. But all is not quite as it seems. In general, we are perfectly entitled to pick any linear combinations of basis elements, and use these linear combinations as the new basis elements in place of the old ones. And if we do this in the right way (see below) we shall find that when the transformation equation (for any rotation) is written out in terms of the new basis elements, it contains a great many zeros! In fact it turns out that *subsets* of the 9 new basis elements transform only among themselves, not mixing up with other basis elements at all (hence the zeros). But this must mean that they form the basis for representations of *lower dimensionality* than 9! That is why the apparent dimensionality of "9" is an illusion. The 9-dimensional representation we have got (in terms of 9×9 matrices) is indeed a representation, but it can be *reduced*, which means it contains within it representations of lower dimensionality, which can't be further reduced, and are *irreducible.* Physics wants to get down to the irreducible representations, which are the true "building blocks".

We actually know very well, from elementary vector algebra, that certain combinations of products of $\mathbf{r}_1$ and $\mathbf{r}_2$ transform only among themselves under SO(3) rotations. For example, $\mathbf{r}_1.\mathbf{r}_2$ is an invariant: $\mathbf{r}_1'.\mathbf{r}_2' = \mathbf{r}_1.\mathbf{r}_2$. This means that the particular combination $\mathbf{r}_{1x}\mathbf{r}_{2x} + \mathbf{r}_{1y}\mathbf{r}_{2y} + \mathbf{r}_{1z}\mathbf{r}_{2z}$ transforms all by itself, without involving any other bits of $\mathbf{r}_{1i}'\mathbf{r}_{2j}'$, and it transforms trivially, with the matrix $\mathbf{D}^{(0)}(\mathbf{R}) = 1$:

$$(\mathbf{r}_1'.\mathbf{r}_2') = (1).\,(\mathbf{r}_1.\mathbf{r}_2)\,.$$

The superscript "(0)" on the $\mathbf{D}$ matrix is consistent with our notation in Lecture 2: there we had three things, the components of a vector $\mathbf{r}$, transforming by a $\mathbf{D}^{(1)}$. The superscript is actually the "l" - value, in quantum-mechanical terms, such that $2l + 1$ gives the dimensionality of the representation. For the invariant, the dimensionality is just one (only one element), so the l is 0. So if we were to start our column vector of nine basis elements with the entry "$\mathbf{r}_1.\mathbf{r}_2$" instead of with "$\mathbf{r}_{1x}\mathbf{r}_{2x}$" we would find that the 9×9 matrix had a "1" in the top left hand corner, and that the rest of the first row and first column were all zeros. So the 9×9 matrix would be "partitioned" into a "1" and an 8×8 matrix bordered by zeros.

Is the 8×8 matrix irreducible? No, because we know of another subset of those products $\mathbf{r}_{1i}\mathbf{r}_{2j}$ which transform entirely among themselves - namely the *three* objects comprising the components of the cross product $\mathbf{r}_1 \times \mathbf{r}_2$:

$$\begin{pmatrix} \mathbf{r}_{1y}\mathbf{r}_{2z} & - & \mathbf{r}_{1z}\mathbf{r}_{2y} \\ \mathbf{r}_{1z}\mathbf{r}_{2x} & - & \mathbf{r}_{1x}\mathbf{r}_{2z} \\ \mathbf{r}_{1x}\mathbf{r}_{2y} & - & \mathbf{r}_{1y}\mathbf{r}_{2x} \end{pmatrix}.$$

These provide the basis for a three dimensional (or "vector") basis of SO(3). Just as before, this means that if we write these three quantities as the second, third and fourth entries in our column of 9 basis elements (after the invariant $\mathbf{r}_1.\mathbf{r}_2$), we shall find that rows 2, 3 and 4 of the 9×9 transformation matrix have entries in the columns 2, 3 and 4 and zeros elsewhere. Indeed, the non-zero bits form a 3×3 matrix, which is of course correct for the way in which a vector $\mathbf{r}_1 \times \mathbf{r}_2$ transforms. In fact this 3×3 matrix is just a $\mathbf{D}^{(1)}$ i.e. " $l = 1$" as appropriate to a three-dimensional representation.

So we have *reduced* the 9×9 matrix to a 1×1 matrix and a 3×3 matrix each bordered by zeros (i.e. disconnected from the rest), and this leaves a 5×5 matrix. At this point we must be very careful to make sure we do not include in the last 5 basis elements any which (in our reorganisation) we have already used up. The way to make sure there is no "overlap" with the three things in $\mathbf{r}_1 \times \mathbf{r}_2$ is to observe that the cross product is *antisymmetric* under interchange of the components of $\mathbf{r}_1$ and $\mathbf{r}_2$, so if we choose *symmetrical* combinations they can have nothing to do with the cross product bits. However the scalar product $\mathbf{r}_1.\mathbf{r}_2$ is symmetrical, so we have not to count this one twice. The symmetrical products are $\frac{1}{2}(\mathbf{r}_{1i}\mathbf{r}_{2j} + \mathbf{r}_{1j}\mathbf{r}_{2i})$ - the $\frac{1}{2}$ is put in for convenience - which amount to 6 objects in all. This already warns us that we have one too many. We must subtract from this set of 6 objects the sum of the three terms in which i is equal to j - because this is just $\mathbf{r}_1.\mathbf{r}_2$! This produces 5 elements which are indeed independent of the invariant $\mathbf{r}_1.\mathbf{r}_2$ and the vector $\mathbf{r}_1 \times \mathbf{r}_2$ and form the components of a *symmetrical traceless second rank tensor*:

$$\frac{1}{2}(\mathbf{r}_{1i}\mathbf{r}_{2j} + \mathbf{r}_{1j}\mathbf{r}_{2i}) - \frac{1}{3}\delta_{ij}\mathbf{r}_1.\mathbf{r}_2.$$

These five things do in fact form the basis for an irreducible 5-dimensional representation of the rotation group, and can't be reduced any further. So they transform by a matrix $\mathbf{D}^{(2)}$, the first time we have met this one (it has " $l = 2$ " appropriate to dimension $2l + 1 = 5$).

Some of the above words may be unfamiliar or a bit rusty. The "trace" of a matrix A_{ij} is defined to be the sum of its diagonal elements - i.e. the result of putting $i = j$ and summing over all their values. The symbol δ_{ij} is defined to be zero if $i \neq j$ and to be 1 if $i = j$. So when we take the trace of the matrix δ_{ij} with i and j running over three values we get just 3. You can now check that the above 5 things are indeed "traceless" in the sense that if you put $i = j$ and sum, you get zero.

It's a little curious that, when people are taught how to take "scalar products" and "vector products" of two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ they aren't also told what happens to the other 5 terms in all the 9 of "$\mathbf{r}_{1i}\mathbf{r}_{2j}$" ! Anyway, we have seen that they form the elements of something new - not a scalar, not a vector, not a spinor, but a *second rank tensor*.

There is a handy way of writing what we have been doing in terms of the $\mathbf{D}$'s. Our original *reducible* representation can usefully be denoted by $\mathbf{D}^{(1\otimes 1)}$ since we are "open producting" an $l = 1$ object ($\mathbf{r}_1$) with another $l = 1$ object ($\mathbf{r}_2$). This 9×9 matrix separates out (after judicious reorganisation of the basis) into a $\mathbf{D}^{(0)}$, a $\mathbf{D}^{(1)}$ and a $\mathbf{D}^{(2)}$:

$$\mathbf{D}^{(1\otimes 1)} = \mathbf{D}^{(0)} \oplus \mathbf{D}^{(1)} \oplus \mathbf{D}^{(2)}$$

the $\oplus$ denoting, as before, that the "addition" must be understood as being "on top of"- i.e. increasing the dimensionality .

It will not have escaped the student's notice that all the above is exactly the same as combining two quantum-mechanical angular momenta with $l = 1$. We can represent the process as (using bold face numbers to denote dimensionality):

$$\begin{array}{ccccccccc} \mathbf{3} & \otimes & \mathbf{3} & = & \mathbf{1} & \oplus & \mathbf{3} & \oplus & \mathbf{5} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ l=1 & & l=1 & & l=0 & & l=1 & & l=2 \end{array}$$

The generalisation to the "tensor product" of two higher-dimensional representations is similar:

$$\mathbf{D}^{(j_1\otimes j_2)} = \mathbf{D}^{(j_1+j_2)} \oplus \mathbf{D}^{(j_1+j_2-1)} \oplus .... \oplus \mathbf{D}^{|j_1-j_2|}.$$

What general rule can we learn from all this? We don't have time to develop the theory to explain why, but this is the rule: *the bases for irreducible representations of groups like the ones we are interested in are provided by the traceless tensors of definite symmetry* - recall that $\mathbf{r}_1.\mathbf{r}_2$ is symmetric, $\mathbf{r}_1 \times \mathbf{r}_2$ is antisymmetric, and the $l = 2$ tensor is symmetric.

All right...now we shall do much the same thing for SU(2). Let's combine two SU(2) doublets, just as we combined two SO(3) triplets. we label the components of the first doublet p_1, p_2 and those of the second doublet q_1, q_2. Consider then the four products $q_i p_j \quad (i = 1, 2 \; j = 1, 2)$. These separate into three symmetric combinations $q_i p_j + q_j p_i$ and one antisymmetric combination $q_i p_j - q_j p_i$ (i.e. $q_1 p_2 - q_2 p_1$). If we think of these as spins $\begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix}$ then what we are saying is that

$$\begin{array}{c} \uparrow \\ \uparrow\downarrow + \downarrow\uparrow \\ \downarrow \end{array}$$

are three things that form the basis of a 3-D representation (spin−1) and

$$\uparrow\downarrow - \downarrow\uparrow$$

is an antisymmetric thing that forms the basis of a 1-D representation (spin−0). Symbolically,

$$\begin{array}{ccccccc} \mathbf{2} & \otimes & \mathbf{2} & = & \mathbf{3} & \oplus & \mathbf{1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{spin} - \frac{1}{2} & & \text{spin} - \frac{1}{2} & & \text{spin} - 1 & & \text{spin} - 0 \end{array}$$

...as in quantum mechanics (again).
Example: Two Nucleons.

$$pp, \frac{np + pn}{\sqrt{2}}, nn \text{ have I} = 1$$

$$\frac{pn - np}{\sqrt{2}} \text{ is I} = 0 \text{ (deuteron).}$$

In the case of SU(2) it is interesting to introduce something that is not so far seen (for SO(3) or the Lorentz group), which is the complex conjugate representation (i.e. we go from considering $\mathbf{2}$ to $\mathbf{2^*}$). If $q' = \mathbf{Q}q$ then $q'^* = \mathbf{Q}^* q$ i.e it transforms by $\mathbf{Q}^*$ and not $\mathbf{Q}$. What is the physical interpretation of this "complex conjugate" representation?

The answer is that it describes antiparticles, this is because if $q' = e^{-\frac{i}{2}\alpha \mathbf{n}.\tau} q$ then $q'^* = e^{\frac{i}{2}\alpha \mathbf{n}.\tau^*} q^*$ which is saying, in particular, that the sign of the third component of $\boldsymbol{\tau}$ is reversed (since $\tau_3^* = \tau_3$). This is an additive quantum number and it is these which get reversed on the exchange particle↔antiparticle. So if we have, say, (dropping the irrelevant " ψ " now),

$$\begin{pmatrix} u \\ d \end{pmatrix} \begin{array}{l} I_3 = +\frac{1}{2} \\ I_3 = -\frac{1}{2} \end{array}$$

we will associate "u^*" with $I_3 = -\frac{1}{2}$ and "d^*" with $I_3 = +\frac{1}{2}$ and so we write

$$\begin{pmatrix} d^* \\ u^* \end{pmatrix} \text{ or } \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix},$$

to indicate the antiparticles.

However, there is a slightly tricky point here, which is that **if we want to use the usual spin−$\frac{1}{2}$ coupling rules,** we need to work with $\begin{pmatrix} -d^* \\ u^* \end{pmatrix}$ or alternatively $\begin{pmatrix} d^* \\ -u^* \end{pmatrix} \begin{array}{l} I_3 = +\frac{1}{2} \\ I_3 = -\frac{1}{2} \end{array}$.
For example, assuming this rule, the possible components of an antiquark doublet and a quark doublet would be

$$\begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{pmatrix} d^* \\ -u^* \end{pmatrix} \text{ and } \begin{pmatrix} u \\ d \end{pmatrix} \begin{array}{c} \uparrow \\ \downarrow \end{array}.$$

Then, using the usual quantum-mechanical rule for coupling two spin - $\frac{1}{2}$ particles, the singlet combination is proportional to

$$\begin{array}{ccccccc} \uparrow\downarrow & - & \downarrow\uparrow & & & & \\ & \text{i.e.} & & & & & \\ d^* d & - & (-u^*)\, u & = & d^* d & + & u^* d \end{array}$$

which is, in fact,

$$(u^* d^*) \begin{pmatrix} u \\ d \end{pmatrix}.$$

How would we get this if we hadn't been told about that minus sign needed when coupling the components of antiparticle isospinors? Well, if $q = \begin{pmatrix} u \\ d \end{pmatrix}$ then

$$(u^* d^*) \begin{pmatrix} u \\ d \end{pmatrix} = q^\dagger q$$

and under an SU(2) transformation

$$q^\dagger q \rightarrow q'^\dagger q' = q^\dagger \mathbf{Q}^\dagger \mathbf{Q} q = q^\dagger q$$

since $\mathbf{Q}^\dagger \mathbf{Q} = \mathbf{I}$, so indeed the combination $q^\dagger q$ is invariant, i.e. is a singlet.

What about the triplet? Applying the spin - $\frac{1}{2}$ coupling rules, this would be

$$\left.\begin{array}{ccc} \uparrow\uparrow & d^* u & I_3 = +1 \\ \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) & \frac{1}{\sqrt{2}}(d^* d - u^* u) & I_3 = 0 \\ \downarrow\downarrow & -u^* d & I_3 = -1 \end{array}\right\} I = 1.$$

We are here using freely the notation I for the magnitude of the total isospin (it would be J for genuine angular momentum) and I_3 for the third component.

There is another very useful way of writing this triplet in terms of the "$q^\dagger ....q$" notation. Consider the three quantities $q^\dagger \boldsymbol{\tau}\, q = \mathbf{V}$:

$$\begin{array}{lllllll} V_1 & = & q^\dagger \tau_1 q & = & (u^* d^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} & = & u^* d + d^* u \\ V_2 & = & q^\dagger \tau_2 q & = & (u^* d^*) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} & = & -iu^* d + id^* u \\ V_3 & = & q^\dagger \tau_3 q & = & (u^* d^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} & = & u^* u - d^* d \end{array}$$

Note that V_1 and V_2 do not have a definite value of I_3: the term $u^* d$ has $I_3 = -1$ and the term $d^* u$ has $I_3 = +1$. This is because they are the analogues of the Cartesian components x, y rather than of the combinations $x + iy, x - iy$ which have definite L_z, namely $L_z = +1$ and -1 respectively. So consider instead the analogous combinations:

$$\begin{array}{lllllll} V_1 + iV_2 & = & (u^* d + d^* u) + i(-iu^* d + d^* u) & = & 2u^* d & \rightarrow & u^*(\downarrow)\, d(\downarrow) \sim \pi^- \\ V_1 - iV_2 & = & (u^* d + d^* u) - i(-iu^* d + d^* u) & = & 2d^* u & \rightarrow & d^*(\uparrow)\, u(\uparrow) \sim \pi^+ \\ V_3 & = & u^* u - d^* d & = & & \rightarrow & \frac{u^*(\downarrow)u(\uparrow) - d^*(\uparrow)d(\downarrow)}{\sqrt{2}} \sim \pi^0 \end{array}$$

(note V_3 has been normalised). A common and compact way of writing these particular combinations is to regard them as the entries in the Hermitian 2×2 matrix $\mathbf{V}.\boldsymbol{\tau}$.

This exercise has constructed explicitly the "flavour wavefunctions" for the isospin singlet ($q^\dagger q$) and isospin triplet ($q^\dagger \boldsymbol{\tau}\, q$) couplings of a quark and an antiquark. In practice, it is just these kinds of wavefunctions that we need to know for many purposes.

Finally, we can usefully apply what we have just learned to the case of a "real" $J = \frac{1}{2}$ spinor ϕ. It must be the case that $\phi^\dagger \boldsymbol{\sigma}\, \phi$ transforms as an SO(3) vector.

This is easy to check. Under infinitesimal SO(3) transformations, $\phi \rightarrow (1 - \frac{i}{2}\mathbf{a}.\boldsymbol{\sigma})\, \phi$ so

$$\mathbf{V} = \phi^\dagger \boldsymbol{\sigma}\, \phi \rightarrow \mathbf{V}' = \phi^\dagger \left(1 + \frac{i}{2}\mathbf{a}.\boldsymbol{\sigma}\right) \boldsymbol{\sigma} \left(1 - \frac{i}{2}\mathbf{a}.\boldsymbol{\sigma}\right) \phi$$

Now

$$\boldsymbol{\sigma}^\dagger = \boldsymbol{\sigma}$$

and so

$$\mathbf{V} = \phi^\dagger \boldsymbol{\sigma}\, \phi \rightarrow \mathbf{V}' = \mathbf{V} + i\phi^\dagger \left(\frac{1}{2}\mathbf{a}.\boldsymbol{\sigma}\right) \boldsymbol{\sigma}\, \phi - i\phi^\dagger \boldsymbol{\sigma} \left(\frac{1}{2}\mathbf{a}.\boldsymbol{\sigma}\right) \phi.$$

Consider V_1:

$$\begin{aligned}
V_1' &= V_1 &+& \tfrac{i}{2}\phi^\dagger\left((\mathbf{a}.\boldsymbol{\sigma})\sigma_1 - \sigma_1(\mathbf{a}.\boldsymbol{\sigma})\right)\phi \\
&= V_1 &+& \tfrac{i}{2}\phi^\dagger\left((a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)\sigma_1 - \sigma_1(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)\right)\phi \\
&= V_1 &+& \tfrac{i}{2}\phi^\dagger\left(\epsilon_1 - ia_2\sigma_3 + i\sigma_2 a_3 - a_1 - ia_2\sigma_3 + i\sigma_2 a_3\right)\phi \\
&= V_1 &+& a_2\phi^\dagger\sigma_3\phi - a_3\phi^\dagger\sigma_2\phi \\
&= V_1 &+& a_2 V_3 - \epsilon_3 V_2
\end{aligned}$$

Compare this with "$\mathbf{V}' = \mathbf{V} + \mathbf{a} \times \mathbf{V}$" - it is correct for a vector $\mathbf{V}$! So $\phi^\dagger\,\boldsymbol{\sigma}\,\phi$ behaves as a vector under rotations!

Example Verify the 4-dimensional generalisation of this: that if ψ is a 4-component Dirac spinor, then the four quantities $(\psi^\dagger\psi, \psi^\dagger\,\boldsymbol{\alpha}\,\psi)$ transform under Lorentz trandformations as the components of a 4-vector ($\psi^\dagger\psi$ is the Dirac probability density ρ, and ($\psi^\dagger\,\boldsymbol{\alpha}\,\psi$) is the Dirac probability current density $\mathbf{j}$, and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ are the three Dirac matrices as in Lecture 4. The notation can be (and always is) streamlined by using Feynman's γ matrices, $\gamma^0 = \beta$, $\boldsymbol{\gamma} = \beta\,\boldsymbol{\alpha}$, and the definition $\bar{\psi} = \psi^\dagger\gamma^0$. Then this 4-vector current is $j^\mu = \bar{\psi}\gamma^\mu\psi$.