<u>Introduction to Symmetries</u>

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Lecture 3:

Spin$-\frac{1}{2}$ Particles and Rotations on Spinors .

So far we have only considered (without saying so explicitly) spin-0 particles, and orbital angular momentum $\widehat{\mathbf{L}}$. Now we need to consider spin$-\frac{1}{2}$ particles and <u>spin</u> $\widehat{\mathbf{S}}$.

Let's begin by "deriving" a wave equation for a non-relativistic spin$-\frac{1}{2}$ particle. Define:

$$\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z),$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that if $S_x = \frac{\hbar}{2}\sigma_x, S_y = \frac{\hbar}{2}\sigma_y$ and $S_z = \frac{\hbar}{2}\sigma_z$, then $[S_x, S_y] = i\hbar S_z$ and these S's provide <u>a matrix representation of the algebra of the generators of SO(3)</u> with dimension d=2 (2×2 matrices). If they were "orbital angular momentum" matrices, their dimension would be $d = 2l + 1$, so the "l" here is $\frac{1}{2}$! In the example at the end of Lecture 2, we were dealing with the case $d = 3, l = 1$. For later reference, we note that both orbital and spin angular momenta can be included if we call the rotation generators $\widehat{\mathbf{J}}$; then in the case of the $d = 2$ represenation the $\widehat{\mathbf{J}}$'s are represented by $\mathbf{J}^{(\frac{1}{2})} = \mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$.

Consider the non-relativistic free particle equation,

$$\frac{\widehat{\mathbf{p}}^2}{2m}\psi = E\psi.$$

Note that,

$$\boldsymbol{\sigma}.\mathbf{A} = (\sigma_x A_x + \sigma_y A_y + \sigma_z A_z) = \begin{pmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{pmatrix}$$

$$\text{and} \ \ (\boldsymbol{\sigma}.\mathbf{A})^2 = \mathbf{A}^2\mathbf{I}$$

provided that the components A_x, A_y and A_z all commute with each other. So $(\boldsymbol{\sigma}.\mathbf{A})^2 = \mathbf{A}^2\mathbf{I}$ under these circumstances. So we can write $\frac{\widehat{\mathbf{p}}^2}{2m}\psi$ as $\frac{(\boldsymbol{\sigma}.\widehat{\mathbf{p}})^2}{2m}\psi$ if we want, interpreting ψ now as something with two components for the σ's to multiply on to. Now let's introduce an electromagnetic field by the usual "prescription" $\widehat{\mathbf{p}} \to \widehat{\mathbf{p}}' = \widehat{\mathbf{p}} - e\mathbf{A}$ where $\mathbf{A}$ is the magnetic vector potential and e is the charge on the particle. The wave equation is now

$$\left[(\boldsymbol{\sigma}.(\widehat{\mathbf{p}} - e\mathbf{A}))^2/2m\right]\psi = E\psi.$$

In this case the components of what is multiplying $\boldsymbol{\sigma}$ do **not** commute with each other - they are:

$$-i\hbar\frac{\partial}{\partial x} - eA_x, -i\hbar\frac{\partial}{\partial y} - eA_y, -i\hbar\frac{\partial}{\partial z} - eA_z,$$

and if $\mathbf{A}$ depends on x, y, z then the $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\frac{\partial}{\partial z}$ bits will not commute with the $\mathbf{A}$ bits.

So what is $(\boldsymbol{\sigma}.(\widehat{\mathbf{p}} - e\mathbf{A})^2)$ in this case? It is worth trying carefully to verify the following:

$$\frac{1}{2m}(\boldsymbol{\sigma}.(\widehat{\mathbf{p}} - e\mathbf{A}))^2 = \frac{1}{2m}(\widehat{\mathbf{p}} - e\mathbf{A})^2 - \frac{e\hbar}{2m}\boldsymbol{\sigma}.\mathbf{B}$$

where $\mathbf{B} = \nabla \times \mathbf{A}$ - the magnetic field. The first term is the same as that obtained when the replacement $\widehat{\mathbf{p}} \to \widehat{\mathbf{p}}' = \widehat{\mathbf{p}} - e\mathbf{A}$ is made in the spin-0 case; the second term $\frac{e\hbar}{2m}\boldsymbol{\sigma}.\mathbf{B}$ indicates that we have a spin$-\frac{1}{2}$ particle with $g = 2$ interacting with a magnetic field. Thus our wave equation is:

$$\left[\frac{1}{2m}\left(\widehat{\mathbf{p}} - e\mathbf{A}\right)^2 - \frac{e\hbar}{2m}\boldsymbol{\sigma}.\mathbf{B}\right]\psi = E\psi,$$

which is known as the Pauli equation.

Once again, we ask our standard question: *What wavefunction should people in the rotated coordinate system use?* Let's concentrate on the spin dependent part - the transformation of the spatial part of the wavefunction will go through just as in Lecture 2. So we are looking at

$$-\frac{e\hbar}{2m}\boldsymbol{\sigma}.\mathbf{B}\psi = E\psi \text{ where } \psi = \begin{pmatrix} a \\ b \end{pmatrix}.$$

We want to know what people in a rotated frame S' should use as their "spin" wavefunction i.e. what is $\psi' = \begin{pmatrix} a' \\ b' \end{pmatrix}$?

Note that our spin wavefunction has *two* components. Such two-component wavefunctions are called **spinors**. We shall see that the two components a' and b' are linear combinations of a and b - just as in the case of the *three component* wavefunction in our Example at the end of Lecture 2; that three-component wavefunction was, of course, a *vector*, with components x, y, z. The "exercise" referred to at the end of Lecture 2 will help us to guess what the corresponding transformation matrix should be in the spinor case.

We are going to work out a simple example explicitly. Take $\mathbf{B}$ to be along the z-axis in frame S ($\mathbf{B} = (0, 0, B)$) so the wavefunction ψ in this frame is a solution of:

$$\frac{-e\hbar}{2m}\sigma_z B\psi = E\psi$$

which we will rewrite as

$$\sigma_z\psi = \lambda\psi,$$

where $\lambda = \left(\frac{2mE}{-e\hbar B}\right)$. So the λ's are the eigenvalues of σ_z. In fact the eigenvalues of σ_z are just ± 1 as can be seen from:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is already "diagonal." So $\frac{2mE}{-e\hbar B} = \pm 1$ showing that the allowed energies are $E = \pm\frac{e\hbar B}{2m}$. The two eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $(\lambda = 1)$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $(\lambda = -1)$.

Now let's do a rotation of $-\alpha$ about the x-axis (not the z-axis!) . The components of $\mathbf{B}$ change from (B_x, B_y, B_z) to

$$\begin{array}{rcllcl} B'_x &=& B_x & & = & 0 \\ B'_y &=& B_y\cos\alpha &-& B_z\sin\alpha &= -B\sin\alpha \\ B'_z &=& B_y\sin\alpha &+& B_z\cos\alpha &= B\cos\alpha \end{array}$$

for $\mathbf{B} = (0, 0, B)$. In the primed frame the equation to be solved is therefore:

$$\boldsymbol{\sigma}.(0, -\sin\alpha, \cos\alpha)\,\psi' = \lambda\psi' \text{ where } \psi' = \begin{pmatrix} a' \\ b' \end{pmatrix}$$

with the same eigenvalue λ. This says

$$\begin{pmatrix} \cos\alpha & i\sin\alpha \\ -i\sin\alpha & \cos\alpha \end{pmatrix}\psi' = \lambda\psi'.$$

It is easy to check that the eigenvalues of the 2×2 matrix on the LHS are $\lambda = \pm 1$ i.e. we do have the same eigenvalues. What are the eigenvectors? For $\lambda = +1$ we have

$$\begin{pmatrix} \cos\alpha & i\sin\alpha \\ -i\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} a'_+ \\ b'_+ \end{pmatrix} = \begin{pmatrix} a'_+ \\ b'_+ \end{pmatrix}$$

and so

$$\begin{array}{rrcl} & a'_+\cos\alpha + ib'_+\sin\alpha &=& a'_+ \\ \Rightarrow & a'_+(1-\cos\alpha) &=& ib'_+\sin\frac{\alpha}{2} \\ \Rightarrow & a'_+.2\sin^2\frac{\alpha}{2} &=& ib'_+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \\ \Rightarrow & a'_+\sin\frac{\alpha}{2} &=& ib'_+\cos\frac{\alpha}{2} \end{array}$$

$$\Rightarrow \begin{pmatrix} a'_+ \\ b'_+ \end{pmatrix} = \begin{pmatrix} \cos\frac{\alpha}{2} \\ -i\sin\frac{\alpha}{2} \end{pmatrix}$$

is a possible normalised solution.

Now we need to understand how this " $\psi'(\lambda = 1)$ " is related to " $\psi(\lambda = 1)$ ", bearing in mind that the latter is $\psi_{\lambda=1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let's consider $e^{-i\alpha\frac{\sigma_x}{2}}$ by analogy with our other $\widehat{\mathbf{U}}$ operators, and with the Example at the end of Lecture 2 (we are replacing the 3×3 representation on the generators of SO(3) by the 2×2 one!). We have,

$$e^{-i\ \alpha\frac{\sigma_x}{2}} = 1 - i\alpha\frac{\sigma_x}{2} + \frac{1}{2!}\left(-i\alpha\frac{\sigma_x}{2}\right)^2 + \frac{1}{3!}\left(-i\alpha\frac{\sigma_x}{2}\right)^3 + \ldots$$

Now

$$\sigma_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \text{ and hence } \sigma_x^3 = \sigma_x.$$

So

$$\begin{array}{rcl} e^{-i\alpha\frac{\sigma_x}{2}} &=& 1 - \frac{i\alpha\sigma_x}{2} - \frac{1}{2!}\left(\frac{\alpha}{2}\right)^2 + i\left(\frac{\alpha}{2}\right)^3\sigma_x \ldots \\ &=& \cos\frac{\alpha}{2} - i\sigma_x\sin\frac{\alpha}{2} \\ &=& \begin{pmatrix} \cos\frac{\alpha}{2} & -i\sin\frac{\alpha}{2} \\ -i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}. \end{array}$$

Finally then, we arrive at

$$e^{-i\alpha\frac{\sigma_x}{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\alpha}{2} \\ -i\sin\frac{\alpha}{2} \end{pmatrix}$$

which is just $\begin{pmatrix} a'_+ \\ b'_+ \end{pmatrix}$!

The $\lambda = -1$ eigenvector for the primed frame is

$$\begin{pmatrix} -i\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} a'_- \\ b'_- \end{pmatrix},$$

which is $e^{-i\alpha\frac{\sigma_x}{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. **So the primed wavefunction is just $e^{-i\alpha\frac{\sigma_x}{2}}$ times the old one.** Compare this with the Example at the end of Lecture 2.

In the general case, $\psi' = e^{-i\alpha\frac{\mathbf{n}.\boldsymbol{\sigma}}{2}}\psi$ for a rotation about axis $\mathbf{n}$ by an angle α. Note how similar this is to

$$e^{-i\frac{a\widehat{p_x}}{\hbar}}, e^{-i\alpha\frac{\mathbf{n}.\widehat{\mathbf{L}}}{\hbar}} \text{ and } e^{-i\alpha\frac{\mathbf{n}.\mathbf{L}^{(1)}}{\hbar}}.$$

Writing $\mathbf{S}^{(\frac{1}{2})} = \frac{\hbar}{2}\boldsymbol{\sigma}$ our transformation is $e^{-i\alpha\frac{\mathbf{n}.\mathbf{S}^{(\frac{1}{2})}}{\hbar}}$ from which the **general prescription for spin $-j$ particles** follows: $\psi' = e^{-i\alpha\frac{\mathbf{n}.\mathbf{S}^{(j)}}{\hbar}}\psi$ where the $\mathbf{S}^{(j)}$ are the $(2j+1)\times(2j+1)$ spin matrices for spin$-j$ particles which obey $\left[S_x^{(j)}, S_y^{(j)}\right] = i\hbar S_z^{(j)}$. Note especially that this transformation law for the spinor wavefunction is of exactly the same form as the law we found in Lecture 2 for the ordinary Schrodinger wavefunction, except that the orbital angular momentum operators $\widehat{\mathbf{L}}$ have been replaced by the spin matrices $\mathbf{S}^{(j)}$.