Mathematical Methods (Second Year) MT 2002

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Supplementary Problem Set

1 (a) The inertial frame K' is moving with respect to the inertial frame K with velocity $\boldsymbol{v}$, but with the axes of K and K' remaining parallel. The Lorentz transformation relating space-time coordinates in the two frames is

$$\begin{aligned} x^{0\prime} &= \gamma(x^0 - \boldsymbol{\beta}.\boldsymbol{x}) \\ \boldsymbol{x}' &= \boldsymbol{x} + \frac{(\gamma-1)}{\beta^2}(\boldsymbol{\beta}.\boldsymbol{x})\boldsymbol{\beta} - \gamma\boldsymbol{\beta}x^0, \end{aligned}$$

where $\boldsymbol{\beta} = \boldsymbol{v}/c, \beta = \mid \boldsymbol{\beta} \mid$.

Explain how this is arrived at, by splitting $\boldsymbol{x}$ into a piece parallel to $\boldsymbol{\beta}$ and a piece perpendicular to $\boldsymbol{\beta}$, and using the simple "one-dimensional" transformation formula as appropriate [hint: write $\boldsymbol{x} = a\boldsymbol{\beta} + b\boldsymbol{x}_\perp$ where $\boldsymbol{x}_\perp.\boldsymbol{\beta} = 0$. Then $\boldsymbol{x}.\boldsymbol{\beta} = a\beta^2$, so $\boldsymbol{x} = (\frac{\boldsymbol{x}.\boldsymbol{\beta}}{\beta^2})\boldsymbol{\beta} + \boldsymbol{x}_\perp$]. This transformation is written as $x' = L(\boldsymbol{\beta})x$ for short, where x is the 4-vector $(x^0, \boldsymbol{x})$.

(b) Coordinates in a frame K' are related to those in a frame K by $x' = L(\beta\boldsymbol{i})x$ (i.e. $\boldsymbol{v}$ is along the x^1-axis). A third frame K'' is such that coordinates x'' are related to x by $x'' = L(\beta\boldsymbol{i} + \delta\beta\boldsymbol{j})x$ where $\delta\beta$ is small. We ask the question: how are x'' and x' related?

Since $x'' = L(\beta\boldsymbol{i} + \delta\beta\boldsymbol{j})x = L(\beta\boldsymbol{i} + \delta\beta\boldsymbol{j})L^{-1}(\beta\boldsymbol{i})x'$ and $L^{-1}(\beta\boldsymbol{i}) = L(-\beta\boldsymbol{i})$ we see that the matrix $L(\beta\boldsymbol{i} + \delta\beta\boldsymbol{j})L(-\beta\boldsymbol{i})$ describes the relation. Calculate the elements of this matrix by matrix multiplication, working only to first order in $\delta\beta$. Show that the resulting matrix is the product of a Lorentz transformation matrix $L(\gamma\delta\beta\boldsymbol{j})$ and a rotation matrix which corresponds to a rotation about the x^3-axis through an angle $(\gamma-1)\frac{\delta\beta}{\beta}$ where $\gamma = (1-\beta^2)^{-1/2}$.

Thus to go from K' to K'' is **not** just a simple velocity transformation, even in the case of infinitesimal $\delta\beta$, but it involves a rotation also. Describe one physical consequence this result.

2 Define the matrix $K_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The exponential of a matrix A is defined by

$$e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3} + ...$$

Verify that $e^{-\eta K_1} = \begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix}$.

Write down the 4 x 4 version of K_1, which is such that

$$e^{-\eta K_1} = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Give 4×4 matrices K_2, K_3 appropriate to Lorentz transformations along the x^2 and x^3 directions respectively.

Verify that $K_1K_2 - K_2K_1 = iL_3$

where $L_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

What transformation does $e^{-iL_3\theta}$ represent? Discuss the physical significance of these results.

3 For a spherically symmetric problem, the thermal diffusion equation for temperature $\theta(r,t)$ as a function of radial distance r and time t is

$$\frac{1}{\kappa}\frac{\partial\theta}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)$$

where κ is a constant. Show that $\theta = A$, $\theta = B/r$, and $\theta = \frac{1}{r}(A_\lambda\cos\lambda r + B_\lambda\sin\lambda r)e^{-\kappa\lambda^2 t}$ are all solutions, where $A, B, A_\lambda, B_\lambda$ and λ are constants.

A solid homogeneous sphere of constant radius a is heated to a uniform temperature θ_0 and then dropped at time $t = 0$ into a tank of cooling fluid which keeps the surface of the sphere at constant temperature θ_1 thereafter. What is the temperature distribution in the sphere as $t \to \infty$? Why are solutions of the form $\frac{\cos\lambda r}{r}$ not acceptable?

Show that the solution for $t > 0$ is

$$\theta = \theta_1 + \sum_{n=1}^{\infty}\frac{1}{r}B_n\sin\left(\frac{n\pi r}{a}\right)e^{-\kappa n^2\pi^2 t/a^2}$$

where $B_n = (-)^{n+1}2a(\theta_0 - \theta_1)/n\pi$.

Discuss the behaviour of this solution for $t = 0$ and $r \to 0$. What does your analysis imply for the 'value' of the series $1 - 1 + 1 - 1 + 1\ldots$?

4 The solution to Laplace's equation in spherical coordinates may be written

$$u(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}[Ar^l + Br^{-(l+1)}]P_l^m(\cos\theta)[C\sin m\phi + D\cos m\phi]$$

where A, B, C, D are constants which may depend on l and m. Determine the restrictions that apply to the solution if it is invariant under parity, i.e. unchanged if $\boldsymbol{r} \to -\boldsymbol{r}$.
[$P_l^m(\cos(\pi-\theta)) = (-1)^{(l-|m|)} P_l^m(\cos\theta)$ and $P_l(1) = 1$.]

Show that the gravitational potential due to a uniform disc of material of radius a and mass M is given by

$$\frac{2GM}{a}\left(1 - \frac{r}{a} \mid P_1(\cos\theta) \mid + \frac{1}{2}\left(\frac{r}{a}\right)^2 P_2(\cos\theta) - \frac{1}{8}\left(\frac{r}{a}\right)^4 P_4(\cos\theta) + ...\right)$$

for $r < a$, and

$$\frac{GM}{r}\left(1 - \frac{1}{4}\left(\frac{a}{r}\right)^2 P_2(\cos\theta) + \frac{1}{8}\left(\frac{a}{r}\right)^4 P_4(\cos\theta) - .....\right)$$

for $r > a$. Here the polar axis is normal to the plane of the disk centred at the origin.

5 (i) Find the Fourier cosine series representing the function : $f(x) = \sin x$ for $0 < x < \pi$. Sketch the function your series represents for $-2\pi < x < 2\pi$.
(ii) Find the Fourier sine series representing the function: $f(x) = \cos x$ for $0 < x < \pi$. Sketch the function your series represents for $-2\pi < x < 2\pi$.
(iii) Verify that your series in (ii) is obtained by differentiating the series in (i) term by term.
(iv) Discuss whether the series in (i) is obtainable by differentiating the series in (ii) term by term.

[Answer to (i): $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2x}{4n^2-1}$].
(v) Integrate the series (ii) term by term from 0 to x, and describe the function represented by the new series.

6 [c.f. 98VA2] An infinite homogeneous rod of uniform cross section, lying along the x axis, is lagged so that no heat is lost from the sides. The equation governing one-dimensional heat flow in the rod is

$$\kappa \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t} \qquad (1)$$

where $\theta(x,t)$ is the temperature in the rod at position x, time t. Consider solutions of the form

$$\theta(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k,t) e^{ikx} dk. \qquad (2)$$

Show that $g(k,t)$ must have the form

$$g(k,t) = a(k)exp(-\kappa k^2 t)$$

where $a(k)$ is (so far) an arbitrary function of k. [Hint: substitute (2) into (1)].

Insert this result for $g(k,t)$ into (2), put $t = 0$ and invert the resulting Fourier integral to prove that

$$a(k) = \frac{\perp}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$$

where $\theta(x,0) = f(x)$ (assuming $f(\pm\infty) = 0$).

Substituting these last two results into (2) we have

$$\theta(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} e^{-ikx'} f(x') dx' \right\} e^{-\kappa k^2 t} e^{ikx} dk.$$

(Note that we label the 'dummy' x-integration variable by x' to distinguish it from the 'external' variable x.)

Assuming the order of the x' and k integrals is immaterial, we can rewrite this as

$$\theta(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} e^{-ikx'} e^{ikx} e^{-\kappa k^2 t} dk \right\} f(x') dx'.$$

Now observe that

$$exp[-\kappa k^2 t - ikx' + ikx] = exp\left\{ -[\sqrt{\kappa t}\, k + i\frac{(x'-x)}{2\sqrt{\kappa t}}]^2 - \frac{(x'-x)^2}{4\kappa t} \right\}$$

and that $\int_{-\infty}^{\infty} dk e^{-(ak+b)^2} = \frac{\sqrt{\pi}}{a}$ for *real* a, and *complex* b.

Using these results show that

$$\theta(x,t) = \frac{1}{2\sqrt{\pi\kappa t}} \int_{-\infty}^{\infty} e^{-(x'-x')^2/4\kappa t} f(x') dx'.$$

At $t = 0$, the temperature distribution is given by $f(x) = 0$ for $-\infty < x < -\varepsilon$; $f(x) = \theta_0$ for $-\varepsilon \le x \le \varepsilon$; $f(x) = 0$ for $\varepsilon < x < \infty$. Without detailed calculation, sketch the form of $\theta(x,t)$ at a later time t, for which $\varepsilon << 2\sqrt{\kappa t}$.

7 Evaluate the 3-D F.T. of the function $\frac{e^{-r/a}}{r}$ where $r = \mid \boldsymbol{r} \mid$. [Hint: want $\int\int\int e^{i\boldsymbol{k}.\boldsymbol{r}} \frac{e^{-r/a}}{r} d^3\boldsymbol{r}$, omitting $\frac{1}{(2\pi)^{3/2}}$. Use polars $d^3\boldsymbol{r} \to r^2 dr \sin\theta d\theta d\phi$. Choose direction of $\boldsymbol{k}$, which is fixed when doing $\boldsymbol{r}$ integral, as the polar axis. So $\boldsymbol{k}.\boldsymbol{r} = kr\cos\theta$, where $k = \mid \boldsymbol{k} \mid$. No ϕ dependence, so $\int d\phi = 2\pi$. Introduce $u = \cos\theta$ and do the θ integral by converting it to a u integral. Do the remaining r integral (r goes from 0 to ∞). Answer: $\frac{4\pi}{\boldsymbol{k}^2 + a^{-2}}$].