Mathematical Methods (Second Year) MT 2002

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Class Problems. Set No. 3

Fourier transforms [Def: $\bar{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$]

1 Evaluate $\bar{f}(k)$ for

$$\begin{aligned} f(x) &= (a-x)/a^2 \quad \text{for} \quad 0 \le x \le a \\ f(x) &= (a+x)/a^2 \quad \text{for} \quad -a \le x \le 0 \\ f(x) &= 0 \quad \text{for} \quad x \le -a, \; x \ge a \end{aligned}$$

(you may find it helpful to sketch $f(x)$).

Sketch $\bar{f}(k)$ as a function of k, marking in particular (a) the value of $\bar{f}(k)$ at $k = 0$ (b) the position of the first zeros on either side of $k = 0$.

Explain the reason for the value of $\bar{f}(0)$ in geometrical terms.

2 Evaluate $\bar{f}(\omega)$ when

$$\begin{aligned} f(t) &= 0 \quad \text{for} \quad t < 0 \\ &= e^{-\alpha t} \quad \text{for} \quad t \ge 0 \end{aligned}$$

where α is a real positive constant. [Def: $\bar{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} f(t) dt$].

The current $I(t)$ in a circuit obeys the equation $L\frac{dI}{dt} + RI = f(t)$ with f as above. If $I(\omega)$ is the FT of $I(t)$ show that $\bar{I}(\omega) = [\sqrt{2\pi}(\alpha - i\omega)(R - Li\omega)]^{-1}$. Hence find $I(t)$. [Hint: use partial fractions and the first part of the question].

Diffraction, convolution

3 The amplitude distribution $A(\theta)$ in Fraunhofer diffraction is related to the FT of the light source distribution $f(x)$ by

$$A(\theta) = \bar{f}(2\pi \sin\theta/\lambda)$$

where λ is the wavelength of the light. Use this to calculate
(i) the 'single finite slit' diffraction pattern, given

$$\begin{aligned} f(x) &= 1 \quad \text{for} \quad |x| \le b/2 \\ &= 0 \quad \text{otherwise;} \end{aligned}$$

(ii) the 'two narrow slits' diffraction pattern, given

$$f(x) = \delta(x + a/2) + \delta(x - a/2);$$

(iii) the 'two finite slits' diffraction pattern, given

$$\begin{aligned} f(x) &= 1 \quad \text{for} \quad \tfrac{a-b}{2} \leq x \leq \tfrac{a+b}{2} \\ &= 1 \quad \text{for} \quad \tfrac{(-a-b)}{2} \leq x \leq \tfrac{-a+b}{2} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

where $b < a$.

How is the result (iii) related to those for (i) and (ii)? Can you explain this in terms of the convolution theorem?

4 Show that if $\bar{f}(k)$ is the FT of $f(x)$ then $e^{ika}\bar{f}(k)$ is the FT of $f(x + a)$. Use this result to answer part (iii) of question 3 directly, from the answers to parts (i) and (ii).

Applications of FT's.

5 The time dependence of a wavefunction (quantum amplitude) representing a decaying state in quantum mechanics is given by

$$\begin{aligned} \psi(t) &= A \exp(-iE_0 t/\hbar) \exp(-\Gamma t/2\hbar) \quad \text{for} \quad t \geq 0 \\ &= 0 \quad \text{for} \quad t < 0 \end{aligned}$$

where A is a constant, and E_0 and Γ are positive quantities. This wavefunction does not represent a *stationary state* - why not? So it cannot represent a state of *definite energy* - why not? Instead, therefore, it must represent a quantum state for which, if the energy were to be measured, different possible results would be obtained: in fact, a state for which a *spectrum* of possible energies would be obtained. To find the probability for obtaining any particular energy E, we need to express the given wavefunction $\psi(t)$ as a *superposition* of stationary state wavefunctions (states of definite energy) via

$$\psi(t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi(E) e^{-iEt/\hbar} dE$$

where the factor $1/\sqrt{2\pi\hbar}$ is conventional. $\Psi(E)$ is then the amplitude (wavefunction - but this time in 'E-space!) associated with the particular energy 'E'. Check that $\Psi(E)$ is given by

$$\Psi(E) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi(t) e^{iEt/\hbar} dt.$$

Find $\Psi(E)$ for the given $\psi(t)$.

$|\ \Psi(E)\ |^2\ dE$ gives the *probability* that the energy, if measured, will be found to lie in the range $(E, E + dE)$. Sketch $|\ \Psi(E)\ |^2$ for this $\psi(t)$. [Note: it has a characteristic *resonance* shape].

The lifetime of the state is determined by the exponential fall-off in $|\ \psi(t)\ |^2$, and is thus $\tau = \hbar/\Gamma$. What is the relation between τ and the *full width at half maximum* of the energy spectrum probability distribution $|\ \Psi(E)\ |^2$?

A certain state has a lifetime of 10^{-23}s. What is the corresponding full width at half maximum of the corresponding $|\ \Psi(E)\ |^2$, evaluated in MeV?

The energy spectrum of decay products from the Z^0 particle has a resonance shape with a full width at half maximum of about 2.5 GeV. What is the lifetime of the Z^0?

6 For all y, and for $x \geq 0$, the function $V(x, y)$ satisfies

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

On the line $x = 0, V(0, y) = f(y) = f(-y)$, and $V \to 0$ as $x \to \infty$. Show that $V(x, y)$ can be expressed as

$$V(x, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}(k) \cos ky \exp(-kx) dk$$

where

$$\bar{f}(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(y') \cos ky' dy'.$$

[Hint: Express $V(x, y)$ as a Fourier cosine integral by writing

$$V(x, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{V}(x, k) \cos ky\, dk$$

justifying why the cosine and not the full Fourier integral. Show that

$$\frac{d^2 \bar{V}}{dx^2} = k^2 \bar{V}.$$

So the problem is now reduced to an ordinary, rather than a partial, differential equation. Select the right solution of this equation to satisfy $V \to 0$ as $x \to \infty$. Determine the unknown function of k via the Fourier cosine transform of $V(0, y)$.]

Find $\bar{f}(k)$ if $f(y) = V_0$ for $|\ y\ | \leq a$ and $f(y) = 0$ for $|\ y\ | > a$, where $V_0 a = Q$, a constant. Show that in the limit $a \to 0$, keeping Q fixed,

$$V(x, y) \to g(x, y) = \frac{2Q}{\pi} \left(\frac{x}{x^2 + y^2} \right).$$

By considering your integral expression for $V(x, y)$, find what condition x must satisfy, in the case $a \neq 0$, for $g(x, y)$ to be a good approximation to $V(x, y)$.