Mathematical Methods (Second Year) MT 2002

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Tutorial Problems. Set No. 2

1 General solution of p.d.e

The function $V(x, y)$ satisfies the partial differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + 6\frac{\partial^2 V}{\partial x \partial y} = 0.$$

Show that by a rotation of axes this equation can be transformed to

$$\lambda_1 \frac{\partial^2 v}{\partial x'^2} + \lambda_2 \frac{\partial^2 v}{\partial y'^2} = 0$$

where $v(x', y') = V(x, y)$ and λ_1, λ_2 are to be determined. Hence find the general solution of the original equation.

2 Legendre polynomials

Position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are such that $r_2 >> r_1$, where $r_1 = |\mathbf{r}_1|$ and $r_2 = |\mathbf{r}_2|$. Show that

$$\frac{1}{| \mathbf{r}_2 - \mathbf{r}_1 |} = \frac{1}{r_2} \left\{ 1 + \left(\frac{r_1}{r_2}\right) P_1(\cos\theta_{12}) + \left(\frac{r_1}{r_2}\right)^2 P_2(\cos\theta_{12}) + ... \right\}$$

where θ_{12} is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$, and $P_1(\cos\theta) = \cos\theta, P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$.

An electric quadrupole is formed by charges Q at coordinates $(0, \pm a, 0)$ and charges $-Q$ at coordinates $(\pm a, 0, 0)$. Show that the potential V in the (x, y) plane at a distance r large compared with a is approximately

$$V = \frac{-3Qa^2 \cos 2\theta}{4\pi\varepsilon_0 r^3},$$

where θ is the angle between $\mathbf{r}$ and the x-axis.

Derive an expression for the couple exerted on the quadrupole by a positive point charge Q at a position $\mathbf{r}$ in the (x, y) plane, where $r >> a$.

Deduce the angles θ for which this couple is zero. If the charges of the quadrupole are rigidly connected and free to rotate about the z-axis, determine whether the equilibrium is stable or unstable in each case.

Orthogonal, normalized, eigenfunctions

3 The real functions $u_n(x)(n = 1 \text{ to } \infty)$ are an orthogonal, normalized, set on the interval (a, b). The function $f(x)$ is expressed as a linear combination of the $u_n(x)$'s via

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x).$$

Show

(i)

$$a_n = \int_a^b u_n(x) f(x) dx;$$

(ii)

$$\int_a^b [f(x)]^2 dx = \sum_{n=1}^{\infty} a_n^2.$$

[Hint for this part: the left hand side is, when written out in long-hand notation,

$$\int_b^a (a_1 u_1(x) + a_2 u_2(x) + ...)(a_1 u_1(x) + a_2 u_2(x) + ...)dx = \int_a^b \{a_1^2[u_1(x)]^2 + a_2^2[u_2(x)]^2 + ...$$

$$+2a_1 a_2 u_1(x) u_2(x) + ...\}dx.$$

Why do the $\int [u_n(x)]^2 dx$ terms each give 1? Why do the $\int u_n(x).u_m(x)dx$ terms with $n \neq m$ each give 0?]

4 The transverse motion of a string stretched between $x = 0$ and $x = L$ is described by the wave equation $\frac{\partial^2 y(x,t)}{c^2 \partial t^2} = \frac{\partial^2 y(x,t)}{\partial x^2}$. The general solution for $y(x, t)$ can be written as a linear superposition $y(x, t) = \sum_{n=1}^{\infty} A_n(t) \sin \frac{n\pi x}{L}$ where $\ddot{A}_n = -\omega_n^2 A_n$ with $\omega_n = \frac{n\pi c}{L}$.

(i) Show that the total energy of the string is

$$E = \frac{1}{2} \int_0^L [\rho \left(\frac{\partial y}{\partial t}\right)^2 + \rho c^2 \left(\frac{\partial y}{\partial x}\right)^2]dx$$

where ρ is the (constant) density.

(ii) Show that

$$E = \frac{L}{2} \sum_{n=1}^{\infty} \frac{1}{2} \rho \dot{A}_n^2 + \frac{1}{2} \rho \omega_n^2 A_n^2.$$

Interpret this result physically in terms of simple harmonic oscillators.

5 The real function $f(x)$ is said to be 'normalized on the interval (a, b)' if $\int_a^b [f(x)]^2 dx = 1$.

(i) Normalize e^{-ax^2} on the interval $(-\infty, \infty)$.

(ii) Normalize $\cos\left(\frac{n\pi x}{a}\right)$ on the interval $(0, a)$.

(iii) Find a and b ($b > 0$) such that $a + bx^2$ is orthogonal to 1 and normalized to 2/5, on the interval (-1, 1).

Fourier series

6 $f(x)$ is an even function of x in the interval $(-a, a)$ and $g(x)$ is an odd function of x in $(-a, a)$. Show that

(i) $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$;

(ii) $\int_{-a}^{a} g(x)dx = 0$;

(iii) $\int_{-a}^{a} f(x)g(x)dx = 0$.

7 (a) The function $y(x)$ is defined in the interval $-1 \leq x \leq 1$ by

$$\begin{aligned} y &= \frac{1}{2\varepsilon} \text{ for } -\varepsilon \leq x \leq \varepsilon \\ &= 0 \text{ for } -1 \leq x < -\varepsilon \text{ and for } \varepsilon < x \leq 1. \end{aligned}$$

Verify that the Fourier expansion of y in the interval $-1 \leq x \leq 1$ is given by

$$y(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{\sin n\pi\varepsilon}{n\pi\varepsilon}\right) \cos n\pi x.$$

Discuss the behaviour of the coefficents $b_n = \frac{\sin n\pi\varepsilon}{n\pi\varepsilon}$ as ε becomes very small. What does the function $y(x)$ look like for very small ε? Is $\int_{-1}^{1} y(x)dx$ still well-defined as $\varepsilon \to 0$?

(b) Expand the same function $y(x)$ as a Fourier series of complex exponentials $e^{in\pi x}$, and verify that your series can be re-written so as to agree with the one found in (a).

Use of Fourier series in boundary value problems

8 The potential $V(x, y)$ satisfies $\frac{\partial^2 V}{dx^2} + \frac{\partial^2 V}{\partial y^2} = 0$ and the boundary conditions

(i)$V = 0$ on $y = 0$ and $y = a$;

(ii) $V \to 0$ as $x \to \infty$;

(iii) $V = V_0$ on $x = 0$ for $0 < y < a$.

Show that the potential in the region $0 < y < a, x > 0$ is given by

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,\ldots} \frac{1}{n} \sin\left(\frac{n\pi y}{a}\right) e^{-n\pi x/a}.$$

9 The temperature T is a one-dimensional bar whose sides are perfectly insulated obeys the heat flow equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

where κ is a positive constant. The bar extends from $x = 0$ to $x = L$ and is perfectly insulated at $x = L$. At $t < 0$ the temperature is 0^0 C throughout the bar and at $t = 0$ the uninsulated end is placed in contact with a heat bath at 100^0C. Show that the temperature of the bar at subsequent times is given by

$$\frac{T}{100} = 1 - \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin\left(\frac{(2n+1)\pi x}{2L}\right) \exp\left\{-\kappa \left(\frac{(2n+1)\pi}{2L}\right)^2 t\right\}.$$

Estimate the time taken for the average temperature of the bar to reach 90^0 C if $L = 1$m and $\kappa = 0.001\text{m}^2\text{s}^{-1}$.

Use of other eigenfunction equations

10 Laplace's equation in spherical polar coordinates (r, θ, ϕ) for a potential $V(r, \theta)$ which is independent of ϕ is

$$\frac{\partial}{\partial r}(r^2 \frac{\partial V}{\partial r}) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}(\sin\theta \frac{\partial V}{\partial \theta}) = 0.$$

(i) Show that there is a solution of the form

$$V(r, \theta) = \sum_{n=0}^{\infty} P_n(\cos\theta)(\frac{a_n}{r^{n+1}} + b_n r^n)$$

where $P_n(x)$ satisfies

$$\frac{d}{dx}\left((1 - x^2)\frac{dP_n(x)}{dx}\right) + n(n+1)P_n(x) = 0.$$

(ii) By first calculating the potential along the z-axis, find the potential at arbitrary θ and $r > a$ for the following charge distributions:

a) A dipole lying along the z-axis and consisting of a charge +1 at $z = +a$ and a charge -1 at $z = -a$.

b) A quadrupole lying along the z-axis and consisting of a charge -1 at $z = +a$, a charge +2 at $z = 0$, and a charge -1 at $z = -a$.

You may assume $P_n(1) = 1$.