Mathematical Methods (Second Year) MT 2002

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Tutorial Problems. Set No. 1

1 3-D Rotations

Which of these matrices represents a rotation?

$$(a)\quad \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (b)\quad \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & -\sqrt{\frac{3}{8}} \\ \frac{3}{4} & \frac{1}{4} & \sqrt{\frac{3}{8}} \\ \sqrt{\frac{3}{8}} & -\sqrt{\frac{3}{8}} & -\frac{1}{2} \end{pmatrix}.$$

Find the angle and axis of the rotation.

2 The three unit vectors $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ form a RH set of axes, and so do the three unit vectors $(\vec{e}_1{}', \vec{e}_2{}', \vec{e}_3{}')$. Explain carefully why $\sum_{k=1}^{3}(\vec{e}_k{}' \cdot \vec{e}_i)(\vec{e}_k{}' \cdot \vec{e}_j)$ is 1 if $i = j$ and is 0 if $i \neq j$.

3 Parity

Not examinable in Part A, but comes into 'Sub-atomic Physics' in Part B

The nucleus of ^{60}Co has a non-zero angular momentum ('spin') in its ground state, and at low temperatures the spin tends to align along the direction of an applied magnetic field $\mathbf{B}$. The nucleus decays by emitting an electron (and a neutrino). When a sample of ^{60}Co is cooled and aligned in this way, it is found that the electrons emerge preferentially in directions opposite to that of the applied field $\mathbf{B}$. Explain why this is evidence that the process involved ('β-decay') is not invariant under the parity transformation.

4 Foucault Pendulum

A pendulum fixed to earth at latitude λ in the northern hemisphere, whose bob is free to swing in any direction, executes small amplitude oscillations which can be regarded as occurring in a horizontal plane. The equations of motion for the components of its displacment are

$$\ddot{x} = -\frac{g}{\ell}x + 2\omega\dot{y}\sin\lambda$$

$$\ddot{y} = -\frac{g}{\ell}y - 2\omega\dot{x}\sin\lambda.$$

Explain the origin of these equations.

A neat way to solve them is to introduce $z = x + iy$. Show that $\ddot{z} + 2i\omega \sin\lambda\, \dot{z} + \omega_0^2 z = 0$ where $\omega_0^2 = g/\ell$. Try a solution of the form $z = Ae^{pt}$, and show that $p = -i\omega\sin\lambda \pm i\omega_1$, where $\omega_1 = [\omega_0^2 + \omega^2 \sin^2\lambda]^{\frac{1}{2}}$. Hence the general solution for z is $z = e^{-i\omega\sin\lambda t}(Ae^{i\omega_1 t} + Be^{-i\omega_1 t})$ where A and B are both complex. Describe the motion in the special case $A = B = \frac{1}{2}a$ where a is real (note that $\omega \ll \omega_0$).

Elgenvalues, eigenvectors and diagonalization of real symmetric matrices; eigenvector expansions

5 (a) Verify that the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

has eigenvalues -1, 1, 2 and find the associated normalized eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. Construct the matrix $R = (\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3)$ and show that it is orthogonal and that it diagonalizes A.

(b) A general 3-component column vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is written as a linear combination of $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{u}_3$:

$$\mathbf{v} = \alpha\mathbf{u}_1 + \beta\mathbf{u}_2 + \gamma\mathbf{u}_3. \qquad (1)$$

Find α, β, γ by (i) equating components on each side of equation (1), (ii) showing that $\alpha = \mathbf{u}_1^T\mathbf{v}, \beta = \mathbf{u}_2^T\mathbf{v}, \gamma = \mathbf{u}_3^T\mathbf{v}$ and evaluating these products.

6 (a) Show that the quadratic form $4x^2 + 2y^2 + 2z^2 - 2xy + 2yz - 2zx$ can be written as $\mathbf{r}^T V \mathbf{r}$ where V is a symmetric matrix and $\mathbf{r}^T = (x\ y\ z)$. Find the eigenvalues of V. Explain why, by rotating the axes, the quadratic form may be reduced to the simple expression $\lambda x'^2 + \mu y'^2 + \nu z'^2$; what are λ, μ, ν in this case?

(b) The components $\mathbf{j}$ of the current density in a conductor are proportional to the components $\mathbf{E}$ of the applied electric field in simple (isotropic) cases: $\mathbf{j} = \sigma\mathbf{E}$. In crystals, however, the relation may be more complicated, though still linear, namely of the form $j_i = \sum_{j=1}^{3} \sigma_{ij} E_j$, where σ_{ij} form the entries in a real symmetric 3×3 matrix, and i runs from 1 to 3.

In a particular case, the quantities σ_{ij} are given (in certain units) by $\begin{pmatrix} 4 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$.
Explain why by a rotation of the axes, the relation between $\mathbf{j}$ and $\mathbf{E}$ can be reduced to $j_1' = \tilde{\sigma}_1 E_1', j_2' = \tilde{\sigma}_2 E_2', j_3' = \tilde{\sigma}_3 E_3'$ and find $\tilde{\sigma}_1, \tilde{\sigma}_2$ and $\tilde{\sigma}_3$.

7 A particle of mass m moves in two-dimensions (x, y) subject to the potential $V(x, y) = x^2 + y^2 + 2\sqrt{2}xy$. Is the origin (0,0) a position of stable equilibrium for the particle? Explain your answer by reference to a sketch showing some contour lines of $V(x, y)$.

8 Three particles of masses m, M and m, lying in a straight line (with M in the middle), are connected by light springs of constant k, and execute small oscillations along the line of the springs.

If x_i is the displacement of the particle i from its equilibrium position, show that the equations of motion for the x_i are

$$\sum_{j=1}^{3}(T_{ij}\ddot{x}_j + V_{ij}x_j) = 0, \qquad \text{(for } i = 1, 2 \text{ and } 3\text{)},$$

where the matrices T and V are given by

$$T = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}$$

and

$$V = k\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

By considering solutions of the form $x_i = a_i e^{i\omega t}$ show that the a_i satisfy the equations

$$\begin{pmatrix} k-\omega^2 m & -k & 0 \\ -k & 2k-\omega^2 M & -k \\ 0 & -k & k-\omega^2 m \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0.$$

Hence deduce the allowed values of ω. Describe qualitatively the type of motion associated with each value of ω.

If the CO_2 molecule can be considered as linear, with each oxygen atom connected to the carbon atom by a spring with constant 1000 N m^{-1}, calculate the frequencies of the natural vibrations of this molecule.

[The relative atomic masses of carbon and oxygen are 12 and 16 respectively].

9 Find the eigenvalues and eigenvectors of the matrix

$$F = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}.$$

Hence, proving the validity of the method you use, find the values of the elements of the matrix F^n where n is a positive integer.

Eigenvalues, eigenvectors and diagonalization of Hermitian matrices; eigenvector expansions.

Diagonalization of Hermitian matrices is not examinable in Part A, but the concept is useful in quantum mechanics

10 (a) Find the eigenvalues of the matrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Normalize the two corresponding eigenvectors, $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$, so that $\mathbf{u}^{(1)\dagger}\mathbf{u}^{(1)} = \mathbf{u}^{(2)\dagger}\mathbf{u}^{(2)} = 1$. Check that $\mathbf{u}^{(1)\dagger}\mathbf{u}^{(2)} = 0$. Form the matrix

$$U = \begin{pmatrix} \mathbf{u}^{(1)} & \mathbf{u}^{(2)} \end{pmatrix}.$$

Check that $U^\dagger U = I$ and evaluate

$$U^\dagger \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} U.$$

(b) A general 2-component complex vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ where $v_1 = a + ib, v_2 = c + id$ is expanded as a linear combination of the eigenvectors $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ via

$$\mathbf{v} = \alpha \mathbf{u}^{(1)} + \beta \mathbf{u}^{(2)} \qquad (2)$$

where α and β are complex numbers. Determine α and β in terms of v_1 and v_2 by (i) equating corresponding components of (2), (ii) showing that $\alpha = \mathbf{u}^{(1)\dagger}\mathbf{v}, \beta = \mathbf{u}^{(2)\dagger}\mathbf{v}$ and evaluating these products.