Mathematical Methods (Second Year) MT 2002

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Class Problems. Set No. 1

1 Newton's Third Law and translation invariance

Two particles, of mass m_1 and m_2 respectively, interact via a mutual potential energy $V(r)$ which is a function only of the interparticle distance r, where $r = \mid \mathbf{r}_1 - \mathbf{r}_2 \mid$ and $\mathbf{r}_1 = (x_1, y_1, z_1,), \mathbf{r}_2 = (x_2, y_2, z_2)$ are the components of the position vectors of the two particles referred to a Cartesian coordinate system. The force on particle 1 is defined to have components

$$\mathbf{F}_1 = \left(-\frac{\partial V}{\partial x_1}, -\frac{\partial V}{\partial y_1}, -\frac{\partial V}{\partial z_1}\right)$$

and similarly for $\mathbf{F}_2$. Show that $\mathbf{F}_1 = -\mathbf{F}_2$ and deduce that $m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2$ is a constant vector (the total momentum). What property of V ensured this result?

2 2-D Rotations

(i) Multiply together the two matrices

$$A = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \qquad B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and interpret the result in terms of 2D rotations.

(ii) Evaluate the product $A^T A$ and interpret the result. Evaluate $A^T B$ and interpret the result.

3 A 3-D Rotation

The three unit vectors $\vec{e}_1, \vec{e}_2$ and $\vec{e}_3$ form an orthogonal set of axes, and so do $\vec{e}_1{}', \vec{e}_2{}'$ and $\vec{e}_3{}'$ where $\vec{e}_1{}' = \vec{e}_2, \vec{e}_2{}' = \vec{e}_3$ and $\vec{e}_3{}' = \vec{e}_1$.

Calculate the elements of the matrix R which has entries

$$R_{ij} = \vec{e}_i{}' \cdot \vec{e}_j.$$

Verify explicitly that R is an orthogonal matrix, and determine the axis and the angle of the rotation to which it corresponds.

4 What is a vector?

(x_1, x_2) are the components of a two-dimensional vector $\vec{r}$ when referred to Cartesian axes along the directions $\vec{i}, \vec{j}$. Derive the relations

$$\begin{aligned} x'_1 &= \cos\theta\ x_1 + \sin\theta\ x_2 & (1) \\ x'_2 &= -\sin\theta\ x_1 + \cos\theta\ x_2 & (2) \end{aligned}$$

for the components (x'_1, x'_2) of $\vec{r}$ referred to new axes $\vec{i}\,', \vec{j}\,'$ obtained by a rotation of the axes through an angle θ about the $\vec{k}$ direction.

(i) Which of the following pairs of quantities transform under rotations of the axes as the components of a two-dimensional vector:-

(a) $(x_2, -x_1)$; $(b)(x_2, x_1)$; $(c)((x_1)^2, (x_2)^2)$?

(ii) Show that $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ form the components of a two-dimensional vector. [Help: you need to show $\frac{\partial}{\partial x'_1} = \cos\theta\ \frac{\partial}{\partial x_1} + \sin\theta\ \frac{\partial}{\partial x_2}$ and similarly for $\frac{\partial}{\partial x'_2}$. This is a problem in calculus - transforming partial drivatives with respect to the pair x_1, x_2 to those with respect to the pair x'_1, x'_2, as in first year Calculus. From the 'chain rule', $\frac{\partial}{\partial x'_1} = \frac{\partial x_1}{\partial x'_1}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial x'_1}\frac{\partial}{\partial x_2}$; evaluate $\frac{\partial x_1}{\partial x'_1}$ and $\frac{\partial x_2}{\partial x'_1}$ (both at fixed x'_2) from the inverses of (1) and (2). Similarly for the other derivative.]

5 Parity

Not examinable in Part A, but will come into sub-atomic physics in Part B

(a) One of Maxwell's electromagnetic field equations is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Given that the equation is invariant under the parity transformation, and that $\mathbf{E}$ is a polar vector, deduce that $\mathbf{B}$ is an axial vector. Why should this result be expected, physically?

(b) The equation of motion for a particle of charge q in electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ is

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

where $\mathbf{p}$ is the momentum and $\mathbf{v}$ is the velocity of the particle. Discuss the invariance (or otherwise) of this equation under (i) rotations (ii) the parity transformation.

6 Relative and centre-of-mass coordinates

Two particles of masses m_1 and m_2 move non-relativistically under the influence of a mutual potential energy which is translationally invariant. Explain why it is possible to describe the motion in a reference frame such that $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/M$ is a constant vector, where $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of the particles and $M = m_1 + m_2$. Show that the equation of motion for $\vec{r} = \vec{r}_1 - \vec{r}_2$ is $\mu\ddot{\vec{r}} = \vec{F}$ where $\vec{F}$ is the force on particle 1 and $\mu = m_1 m_2/M$.

7 <u>Apparent gravity</u>

The apparent acceleration $\vec{g}^*$ due to gravity at a point $\vec{r}$ near the surface of the earth is $\vec{g}^* = \vec{g} - \vec{\omega} \times (\vec{\omega} \times \vec{r})$ where $\vec{g}$ is the acceleration due to gravity as measured in an inertial frame at $\vec{r}$, and $\vec{\omega}$ is the angular velocity of the earth about its axis. Take a set of axes, fixed on the rotating earth in the northern hemisphere, such that $\vec{i}$ is pointing south, $\vec{j}$ east, and $\vec{k}$ along the direction of $\vec{r}$ which is vertically outward from the earth's surface. For a point $\vec{r}$ at latitude λ, show that the components of $\vec{g}^*$ are $(\omega^2 r \sin\lambda\cos\lambda, 0, -g + \omega^2 r\cos^2\lambda)$ where $r = |\,\vec{r}\,|$. Using the values $\omega = 7.29 \times 10^{-5}\mathrm{s}^{-1}$ and $r = 6.37 \times 10^6\mathrm{m}$, calculate the difference between the magnitude of $\vec{g}^*$ at the North pole and at the equator. The true value for this difference is $5.2 \times 10^{-2}\mathrm{ms}^{-2}$: what is the discrepancy due to?

8 <u>Lorentz transformations</u>

The coordinate axes of two inertial frames K and K' are oriented so that K' is moving in the positive x direction of frame K with speed v, as viewed from K. The origins of the frames are coincident at $t = t' = 0$. The space-time coordinates of a point are (x^0, x^1, x^2, x^3) referred to K and $(x^{0\prime}, x^{1\prime}, x^{2\prime}, x^{3\prime})$ referred to K', where $x^0 = ct, x^{0\prime} = ct'$. These coordinates are related by the matrix equation

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix},$$

where $\tanh\eta = v/c$.

By considering the product of two such 4 x 4 matrices, derive the 'relativistic addition of velocities' formula (for two velocities in the same direction),

$$v = (v_1 + v_2)/(1 + v_1 v_2/c^2).$$

9 <u>4-vectors</u>

The components (x^0, x^1) of a space-time point transform under a simple Lorentz transformation as

$$x^{0\prime} = \cosh\eta\; x^0 - \sinh\eta\; x^1$$

$$x^{1\prime} = -\sinh\eta\; x^0 + \cosh\eta\; x^1$$

Write down the inverse of this transformation ie. express x^0 and x^1 as linear combinations of $x^{0\prime}$ and $x^{1\prime}$. Calculate $\frac{\partial}{\partial x^{0\prime}}$ in terms of $\frac{\partial}{\partial x^0}$ and $\frac{\partial}{\partial x^1}$, by using

$$\frac{\partial}{\partial x^{0\prime}} = \frac{\partial}{\partial x^0}\left(\frac{\partial x^0}{\partial x^{0\prime}}\right)_{x^{1\prime}} + \frac{\partial}{\partial x^1}\left(\frac{\partial x^1}{\partial x^{0\prime}}\right)_{x^{1\prime}}$$

(where the subscript means that $x^{1\prime}$ is held fixed) and evaluating the partial derivatives. Similarly, calculate $\frac{\partial}{\partial x^{1\prime}}$ in terms of $\frac{\partial}{\partial x^0}$ and $\frac{\partial}{\partial x^1}$. Hence show that $(\frac{\partial}{\partial x^0}, -\frac{\partial}{\partial x^1})$ transforms under the Lorentz transformation in the same way as (x^0, x^1).

$(\jmath^0, \jmath^1)$ are the "0,1" components of a 4-vector. Explain why $\frac{\partial \jmath^0}{\partial x^0} + \frac{\partial \jmath^1}{\partial x^1}$ is invariant under the Lorentz transformation.