Revision Lectures for Mathematics I
(Physical Sciences Prelims)

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These notes list most, but not all, of the topics which are examinable in Maths I. Each section begins with a simple *"Test"* question: if you can't do it straight away, you probably need to work more on that section. The problems (from Prelims or Mods) listed as *"Examples"* will be done in the lectures (time permitting); those listed as *"See also"* are suggestions for you to practise on.

1 Calculus 1: differentiation etc.

1.1 Taylor and Maclaurin series

Boas §1.12

Test: (a) expand $\cos x$ about $x = \pi/3$ in powers of $h = (x - \pi/3)$ up to and including the h^3 term.
(b) evaluate the limit as $x \to 0$ of $((1/x) - 1/(e^x - 1))$.

Taylor series for $f(x)$ expanded about the point $x = a$ is

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^{(n)}(a) + \ldots$$

where h is small.

Special case is $a = 0$, which is Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots \frac{x^n}{n!}f^{(n)}(0) + \ldots$$

for x near zero.
You should **know** the series listed on p 24 of Boas.

Example: June 93/1/4

See also: 88/1/7

Note: don't always go blindly ahead with brute force differentiation. See Boas §1.13.

Example: June 93/1/4

Use of these series: Boas §1.15

(a) numerical evaluation
Example June 93/1/4

(a) evaluation of integrals
Example Sept 94/1/3

(c) evaluation of limits
See *Test* (b) above

1.2 Leibnitz's theorem

Boas §12.3

Test: Evaluate $\frac{d^n}{dx^n}\left((1-x^2)\frac{d^2y}{dx^2}\right)$ for $n \geq 1$.

Theorem is

$$\begin{aligned} \frac{d^n}{dx^n}(uv) &= u\frac{d^n v}{dx^n} + n\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}} + \frac{n(n-1)}{2!}\frac{d^2u}{dx^2}\frac{d^{n-2}v}{dx^{n-2}} + \ldots \\ &+ \ldots {}_nC_r\frac{d^r u}{dx^r}\frac{d^{n-r}v}{dx^{n-r}} + \ldots + \frac{d^n u}{dx^n}v \end{aligned}$$

Example: 90/1/4

See also: 87/1/9

1.3 Partial differentiation

Boas chptr 4

1.3.1 Basic

Test: x and y are related to r and θ by $x = r\cos\theta$, $y = r\sin\theta$; show that

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta, \left(\frac{\partial r}{\partial x}\right)_y = \cos\theta, \left(\frac{\partial \theta}{\partial r}\right)_x = \frac{\cot\theta}{r}$$

Definition: for $f(u, v, w, \ldots)$, $\frac{\partial f}{\partial u}$ means differentiate *w.r.t.* u holding all other variables fixed.

$$\frac{\partial f}{\partial u} = \ limit\ as\ \Delta u \to 0\ of\ \left\{\frac{f(u+\Delta u, v, w\ldots) - f(u, v, w, \ldots)}{\Delta u}\right\}$$

1.3.2 Total differential

Boas §4.3
Test: If x, y, and z are related by $f(x, y, z) = 0$, show that

$$\left(\frac{\partial x}{\partial y}\right)_z = 1/\left(\frac{\partial y}{\partial x}\right)_z .$$

Consider $f(x, y)$: total change in f when $x \to x + dx, y \to y + dy$ is

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

to first order in dx, dy. This is the "total differential" of f, df. Note that if y is held fixed, for example, so $dy = 0$, then $\left(\frac{\partial f}{\partial x}\right)_y = (df \div dx)_{dy=0}$ as expected.
Sometimes introduce $z = f(x, y)$ and think of it as a surface in 3-D; $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$.

Subscripts left off if sense is obvious.

Example: June 92/1/3
See also: June 94/1/2
June 93/1/2

1.3.3 Chain rule

Boas §4.5, 4.6, 4.7

Test: The function w can be regarded as a function of x and y, or of r and s, where $r = xy$ and $s = 1/y$. Show that

$$y\frac{\partial w}{\partial y} = r\frac{\partial w}{\partial r} - s\frac{\partial w}{\partial s}.$$

(a) Cases where $x, y, \ldots$ depend on only one variable (Boas §4.5, 4.6)
e.g. $f(x(t)), y(t))$

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ \frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \end{aligned}$$

(b) Cases where $x, y, \ldots$ depend on more than one variable (Boas §4.7)

e.g. $f(x(, s, t), y(s, t))$

$$
\begin{aligned}
df &= \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy \\
&= \left(\frac{\partial f}{\partial x}\right)\left[\frac{\partial x}{\partial s}ds + \frac{\partial x}{\partial t}dt\right] + \left(\frac{\partial f}{\partial y}\right)\left[\frac{\partial y}{\partial s}ds + \frac{\partial y}{\partial t}dt\right]. \\
\frac{\partial f}{\partial s} &= (df \div ds)_{dt=0} \\
&= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.
\end{aligned}
$$

Similarly for $\frac{\partial f}{\partial t}$.

Sometimes people prefer to use a different symbol for "the function" when it is regarded as a function of (x, y) from when it is regarded as a function of (s, t) (in this case); for example, if $f(x, y) = x^2 + y^2$, when this is written in terms of s and t it is most unlikely that it will come out as $s^2 + t^2$! Then the above results would read

$$\frac{\partial F}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \quad etc.$$

Example: 1990/1/2

See also: 92/1/4 not (c) (transformation of second order coefficients is no longer
94/1/2 not (ii) in the syllabus).

1.3.4 Exact differentials

Boas §6.8

Test: Write $(3x^2yz - 3y)dx + (x^3z - 3x)dy + (x^3y + 2z)dz$ as $d(something)$.

If we are in possession of a function $f(x, y, z)$ we can calculate its total differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

But we may meet an expression having the form of the RHS - i.e. $Pdx + Qdy + Rdz$, say - and wonder if it is the "d" of anything. If it is, it is an exact differential.

Let's write $\mathbf{F} = (P, Q, R)$, $d\mathbf{r} = (dx, dy, dz)$. Then $Pdx + Qdz + Rdz = \mathbf{F}.d\mathbf{r}$.If this is exact,there must be an "f" such that $df = \mathbf{F}.d\mathbf{r}$. But

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \nabla f.d\mathbf{r}$$

So we must have $\mathbf{F} = \nabla f$ if $\mathbf{F}.d\mathbf{r}$ is exact. Now using curl grad (well-behaved function) $= 0$, we have the condition $\nabla \times \mathbf{F} = \mathbf{0}$ if $\mathbf{F}.d\mathbf{r}$ is exact.

This condition is actually necessary and sufficient for $\mathbf{F}.d\mathbf{r}$ to be exact.

Suppose we integrate $\mathbf{F}.d\mathbf{r}$ along some path, $\int_A^B \mathbf{F}.d\mathbf{r}$, where $\mathbf{F} = \nabla f$. Then

$$\int_A^B \mathbf{F}.d\mathbf{r} = \int_A^B \nabla f.d\mathbf{r} = \int_A^B df = [f]_A^B = f_B - f_A$$

which only depends on the **end-points** of the path, not the route of the path itself. If we go round a **closed path**, and f is **single-valued**, we end up where we started from and $f_B = f_A$ so $\oint_{closed\,path} \nabla f.d\mathbf{r} = 0$.

Application: work done by conservative force, $\mathbf{F} = -\nabla V$.

Example: June 93/1/6(i)
See also: Sept 92/1/6
Sept 94/1/6

1.4 Taylor's theorem in two variables, etc.

Test: expand $\cos(x+2y)$ about the point $(\pi/3, \pi/3)$ keeping terms up to and including second order in $(x-\pi/3)$ and $(y-\pi/3)$.

1.4.1 Basic

Boas §4.2

$f(x,y)$ expanded about (a,b):

$$\begin{aligned} f(x,y) &= f(a,b) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \\ &+ \frac{1}{2}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2hk\,\frac{\partial^2 f}{\partial x \partial y} + k^2\,\frac{\partial^2 f}{\partial y^2}\right] + \ldots \end{aligned}$$

where all differential coefficients are evaluated at (a,b), and $h = (x-a), k = (y-b)$.

See: June 94/1/3

1.4.2 Use in max/min problems

Boas Problem 2 from §4.8

Test: Find the stationary point of $x^2 - y^2 + 2x - 4y + 10$, identify its type, and sketch the contour lines, including the lines which pass through the stationary point.

Definition: (i) $f(x,y)$ has a **max** at (a,b) if $\Delta f = f(a+h,\ b+k) - f(a,b) < 0$ for **arbitrary** small h,k;

(ii) $f(x,y)$ has a **min** at (a,b) if $\Delta f = f(a+h,\ b+k) - f(a,b) > 0$ for **arbitrary** small h,k.

Stationary point (max, min, or saddle) if $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ at (a,b).

So at a stationary point, we have

$$\begin{aligned}\Delta f &= \frac{1}{2}\left\{h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right\} \\ &= \frac{1}{2f_{xx}}\left\{(hf_{xx} + kf_{xy})^2 - k^2\Delta\right\} \\ &= \frac{1}{2f_{yy}}\left\{(hf_{xy} + kf_{yy})^2 - h^2\Delta\right\}\end{aligned}$$

where $\Delta = f_{xy}^2 - f_{xx}f_{yy}$.

So if $\Delta < 0$ the sign of Δf is independent of h and k, and we have a max or min.

From definition (i), f is a max if $\Delta < 0$ and $f_{xx} < 0$; from definition (ii), f is a min if $\Delta < 0$ and $f_{xx} > 0$.

Notice that

$$\begin{aligned}\Delta < 0 \text{ and } f_{xx} < 0 &\Rightarrow f_{yy} < 0 \\ \Delta < 0 \text{ and } f_{xx} > 0 &\Rightarrow f_{yy} > 0.\end{aligned}$$

If on the other hand $\Delta > 0$, Δf will increase away from (a,b) in some directions but decrease away from it in others, and we have a saddle point.

Example: June 92/1/5 (i)

See also: June 94/1/3
June 93/1/5

1.4.3 Lagrange multipliers

Boas §4.9

Test: Using the method of Lagrange multipliers, find the maximum volume of a lidless rectangular box when the total area of the sides and bottom of the box is constrained to the value A. (90/1/3 (ii)).

We want max/min of $f(x,y,\ldots)$ subject to $\phi(x,y,\ldots) = 0$. Sometimes we can solve $\phi = 0$ for one of the variables and substitute it back in f; then we have a standard max/min problem for the remaining unconstrained variables. Often this is clumsy or impossible. Lagrange method introduces parameter ("multiplier") λ - which can in fact be **determined** ! - and considers $F(x, y \ldots) = f + \lambda\phi$. Seek max/min of F, treating all variables as unconstrained.

Stationary conditions:

$$\begin{aligned} \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} &= 0 \\ \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} &= 0 \\ etc. \end{aligned}$$

Example: June 92/1/5 (ii).

If more than one constraint, need one parameter for each: $F = f + \lambda\phi + \mu\psi + \ldots$.

See also: June 94/1/4
91/1/4

2 Calculus 2: integration

2.1 Line integrals

Boas page 209

Test: Find the length of the curve $y = \cosh x$ between $x = -1$ and $x = 1$.

2-D:

$$\begin{aligned} s &= \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{dx^2 + dy^2} \\ &= \int_{x_1}^{x_2} \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} dx \end{aligned}$$

(Sept 94/1/7 (i))

3-D:

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

If x, y, z all functions of one parameter t, say, we have a curve in 3-D, and

$$s = \int ds = \int \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right)^{\frac{1}{2}} dt.$$

Example: June 93/1/6 (ii)

Also integrals of "$\mathbf{F}.d\mathbf{r}$" type: see Boas §6.8 and Section 1.3.4 above.

2.2 Double (iterated) integrals

Boas §5.2

Test: Invert the order of integration in

$$\int_{y=0}^{1}\int_{x=y^2}^{1}\frac{e^x}{\sqrt{x}}\,dxdy$$

and hence evaluate the integral.

$$\begin{aligned} & \iint_A f(x,y)dxdy \\ = & \int_{x_1}^{x_2} dx\left\{\int_{y_-(x)}^{y_+(x)} f(x,y)dy\right\} \\ \text{or} \quad = & \int_{y_1}^{y_2} dy\left\{\int_{x_-(y)}^{x_+(y)} f(x,y)dx\right\}. \end{aligned}$$

Example: 89/1/6(a)

See also: 90/1/9 (i)

2.3 Triple integrals

Boas §5.2

Test: Find the volume of a solid consisting of the region below the plane $z = y + 1$, bounded by the coordinate planes $x = 0$, $y = 0$, and $z = 0$, and the plane $2x+y = 2$.

$$\begin{aligned} & \iiint f(x,y,z)\ dxdydz \\ = & \int_{x_1}^{x_2} dx\left\{\int_{y_-(x)}^{y_+(x)} dy\left[\int_{z_-(x,y)}^{z_+(x,y)} f(x,y,z)dz\right]\right\}. \end{aligned}$$

But it is often easier to evaluate such integrals (both double and triple) by a change of variables.

2.4 Change of variables in multiple integrals: Jacobians

Boas §5.4

Test: Evaluate the integral $\iint dxdy(9 - x^2 - 4y^2)$ over the area in the $x - y$ plane inside the ellipse $x^2 + 4y^2 = 9$, by means of the transformation $x = 3r\cos\theta$, $y = \frac{3r}{2}\sin\theta$ (from 91/1/8).

In an ordinary 1-D integral $\int f(x)dx$, we can often evaluate it by introducing some function of x as a new variable, say $u = u(x)$. (The "method of substitution".) Then

$$dx = \left(\frac{dx}{du}\right) du$$

and the integral becomes $\int f$ (regarded as a fn of u). $\left(\frac{dx}{du}\right) du$ where $\frac{dx}{du}$ also has to be written as a function of u.

What if we want to do a similar thing with

$$\int\int f(x,y)dxdz, \text{ or } \int\int\int f(x,y,z)dxdydz, \ldots ?$$

Answer: 2-D case $x, y \to u(x,y), v(x,y)$

$$\int\int f(x,y)dxdy \to \int\int f(u,v)J(u,v)dudv$$

where the Jacobian J is

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Generalises to 3-D,4-D.....

Special cases: plane polars $J = r(r = \sqrt{x^2+y^2}) : dxdy \to rdrd\theta$
cylindrical polars $J = r : dxdydz \to rdrd\theta\ dz$
$(r = \sqrt{x^2+y^2}, \theta =$azimuthal angle$)$
spherical polars $J = r^2 \sin\theta : dxdydz \to r^2 \sin\theta drd\theta\ d\phi$
$(r = \sqrt{x^2+y^2+z^2}, \theta =$ polar angle, $\phi =$ azimuthal angle$)$

Example: June 92/1/4

See also: Sept 94/1/6

June 92/1/7 (i) ($\frac{x}{a} \to x', \frac{y}{b} \to y', \frac{z}{c} \to z'$, and use spherical polars)

91/1/8

3 Vectors and vector calculus

3.1 Vector geometry

Boas §3.4 - 3.5, Boas §6.1 - 6.3

Test: (i) Find in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ the equation of the line of intersection of the planes

$$\begin{aligned} 2x + 6y - 3z &= 10 \\ 5x + 2y - z &= 12. \end{aligned}$$

(ii) The edges OA, OB, OC of a tetrahedron are mutually perpedicular and are represented respectively by $a\mathbf{i}, b\mathbf{j}, c\mathbf{k}$ where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along these edges. The midpoints of AB, OC, AC, OB are P, Q, R, S respectively. Write down vectors representing $\vec{PQ}, \vec{RS}$ and determine the cosine of the angle between them (90/1/7 (i)).

Line is $\mathbf{r} = \mathbf{a} + \lambda \mathbf{s}$, through $\mathbf{a}$ parallel to $\mathbf{s}$.

Plane is $\mathbf{r}.\mathbf{n}$ = constant where $\mathbf{n}$ is normal to plane. If $\mathbf{r} = \mathbf{a}$ lies in plane, equation is $\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$. If write in form $\mathbf{r}.\hat{\mathbf{n}}$=p, where $\hat{\mathbf{n}}$ is unit vector, then p is perpedicular distance from origin to plane.

Examples: Sept 94/1/5
June 93/1/3

See also: June 94/1/6
Sept 93/1/1
Sept 92/1/1

3.2 Vectors depending on t: mechanics

Boas §6.4

Test: The force on a moving charged particle in a magnetic field $\mathbf{B}$ is $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$ where q is the particle's charge and $\mathbf{v}$ is its velocity. Assuming Newton's law of motion in the form $m\,\frac{d\mathbf{v}}{dt} = \mathbf{F}$ (where m is the mass), show that $\mathbf{v}^2$ and $\mathbf{v}.\mathbf{B}$ are both constant. For the particular case $\mathbf{B} = (0, 0, B)$, find the solution for $\mathbf{v}(t)$ given that $\mathbf{v}(0) = (0, u, 0)$). Find the solution for $\mathbf{r}(t)$ (where $\mathbf{v} = \frac{d\mathbf{r}}{dt}$) given, in addition, that $\mathbf{r}(0) = (a, 0, a)$. Describe the motion of the particle in space.

$$\text{Newton} \quad m\frac{d\mathbf{v}}{dt} = \mathbf{F} = -\nabla V \quad \text{for conservative force.}$$

$$\text{Dot with } \dot{\mathbf{r}} : \ \frac{1}{2}m\ \frac{d}{dt}\ (\dot{\mathbf{r}}^2) = -\nabla V.\dot{\mathbf{r}} \ = \ -\ \frac{dV}{dt} \quad \text{(see 1.3.4).}$$

$$\text{Therefore} \ \ \frac{d}{dt}(\frac{1}{2}m\dot{\mathbf{r}}^2 + V) = 0.$$

So $\frac{1}{2}m\dot{\mathbf{r}}^2 + V$ is a constant (energy).

If $\mathbf{F} = f(r)\mathbf{r}$ (central force) then

$$\begin{aligned} \mathbf{r} \times m\frac{d\dot{\mathbf{r}}}{dt} &= \mathbf{r} \times \mathbf{F} = \mathbf{0} \\ &= \frac{d}{dt}(\mathbf{r} \times \mathbf{p}), \mathbf{p} = m\dot{\mathbf{r}} \end{aligned}$$

So $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is constant (angular momentum).

See also: 89/1/3

3.3 Vector operators

Boas §6.6

Note:will treat vector operators in **rectangular (Cartesian) coordinates only**.

Test: (i) Evaluate $\bigtriangledown(\mathbf{a}.\mathbf{r})^2, \bigtriangledown(e^{i\mathbf{k}.\mathbf{r}})$ where $\mathbf{a}$ and $\mathbf{k}$ are constant vectors.

(ii) Find the equation of the line through (1,1,1) and normal to the surface $x^2 - y^2 z = 0$ at that point.

$$\begin{aligned} \text{grad (scalar)} &= \text{vector} \\ \bigtriangledown\phi &= \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \\ \bigtriangledown &= \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \end{aligned}$$

$$\text{Surface} \quad \phi = 0 : d\phi = 0 = \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz = \bigtriangledown\phi.d\mathbf{r}$$

But $d\mathbf{r}$ is in tangent plane, so $\bigtriangledown\phi$ is perpendicular to this plane, i.e. to the surface.

Or, $\bigtriangledown\phi$ is parallel to the normal to the surface.

If we consider moving along a line $\mathbf{r} = \mathbf{a} + s\hat{\mathbf{u}}$ where $\hat{\mathbf{u}}$ is a unit vector, then ϕ is just a function of the parameter s, and

$$\begin{aligned} \frac{d\phi}{ds} &= \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds} \\ &= \bigtriangledown\phi.\hat{\mathbf{u}} \end{aligned}$$

which is greatest when $\hat{\mathbf{u}}$ is parallel to $\bigtriangledown\phi$.

$$\bigtriangledown(uv) = u\bigtriangledown v + v\bigtriangledown u.$$

Example: 87/1/4 (a)

See also: June 94/1/5

3.3.1 Surface integrals

Boas §5.5

Test: Find the area of the part of the surface $x^2 + y^2 + 2z = 4$ with $z \geq 0$ (from 89/1/7).

If surface is "one-sided" with respect to the $x-y$ plane (i.e. all lines parallel to the z-axis cut the surface only once) then we can project the area element on the surface down onto the $x-y$ plane, to get

$$dA = \frac{dxdy}{\mid \hat{\mathbf{n}}.\mathbf{k} \mid}$$

where $\hat{\mathbf{n}}$ is the unit normal to the surface at (x, y, z). From previous sections,

$$\begin{aligned} \hat{\mathbf{n}} &= \nabla\phi / \mid \nabla\phi \mid \\ \text{So } dA &= \frac{dxdy}{\mid \frac{\partial\phi}{\partial z} \mid} \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 \right]^{\frac{1}{2}}. \end{aligned}$$

Sometimes surface is written as $z = f(x, y)$, so

$$\mid \frac{\partial\phi}{\partial z} \mid = 1, \quad \mid \nabla\phi \mid = \left(1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)^{\frac{1}{2}}.$$

Example: Sept 92/1/7 (i)

See also: Sept 94/1/7 (ii)
87/1/4 (b)

3.3.2 Curl, div, ∇^2

Boas §6.7

Test: (i) Evaluate (for $\mathbf{r} = (x, y, z), r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$) $\nabla \times \mathbf{r}$, $\nabla \times (\mathbf{k} \times \mathbf{r})$, $\nabla.(\mathbf{r}/r)$, $\nabla^2 (1/r)$ for $r \neq 0$.

(ii) Verify that

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla.\mathbf{F}) - \nabla^2\mathbf{F}$$

in rectangular coordinates.

curl (vector) = vector

$\nabla \times \mathbf{F} =$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

div (vector) = scalar

$$\nabla.\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Worth remembering (and being able to prove):-

$$\begin{aligned} \nabla \times (\nabla\phi) &= \text{curl grad } \phi = 0 \\ \nabla.(\nabla \times \mathbf{F}) &= \text{div curl } \mathbf{F} = 0 \\ \nabla \times (\nabla \times \mathbf{F}) &= \nabla(\nabla.\mathbf{F}) - \nabla^2\mathbf{F} \end{aligned}$$

Note: the last identity is true for rectangular coordinate systems, not in general;it states that the three corresponding components of each side are equal.

$$\nabla.(\phi\mathbf{F}) = \phi\nabla.\mathbf{F} + (\nabla\phi).\mathbf{F}$$

These and other identities have to be slogged out by brute force - but often you can present one component in such a way as to make it clear what the others are by symmetry.

Example: 91/1/9

See also: Sept 93/1/7 (ii)
90/1/8

3.3.3 Divergence theorem (Gauss)

Boas §6.10

Test: Verify the divergence theorem for the vector field $\mathbf{F} = (x, y, z)$ where the closed surface is the bounding surface of the hemisphere

$$x^2 + y^2 + z^2 = a^2, z \geq 0.$$

$$d\mathbf{S} = \hat{\mathbf{n}}dS \qquad , \hat{\mathbf{n}} \text{ outward unit normal.}$$

$$\iint_S \mathbf{F}.d\mathbf{S} = \iiint_V (\nabla.\mathbf{F})dV$$

S is **closed** surface bounding the volume V.
Example: 90/1/10

See also: June 94/1/7
Sept 93/1/7 (i)
June 93/1/7

3.3.4 Stokes' theorem

Boas §6.11

Test: Verify Stokes' theorem for the field $\mathbf{F} = (y, -x, 0)$ where C is the boundary of the circle $x^2 + y^2 + 2x = 0$ in the $z = 0$ plane and the surface is the interior of this circle.

Stokes:

$$\oint_C \mathbf{F}.d\mathbf{r} = \int\!\!\int_S ((\nabla \times \mathbf{F}).d\mathbf{S}$$

where C is a simple closed curve bounding an **open** surface S, and the sense of the line integral is such as to keep the surface to the "left" as one proceeds round the boundary.

Example: Sept 93/1/6

See also: 91/1/7

3.3.5 Green's theorem in the plane

Boas §6.9

Test: Apply Green's theorem with $P = 0$ and $Q = \frac{1}{2}x^2$ to the triangle with vertices (0,0), (0,3), (3, 0); hence find the centroid of the triangle.

Green:

$$\oint_C P dx + Q dy = \int\!\!\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

where P, Q are functions of x and y only, and C is the boundary of S.

This follows from Stokes' theorem if we take $\mathbf{F} = (P(x, y), Q(x, y), 0)$.

Application: take $P = -y, Q = x$

Then $\oint_C -y dx + x dy = 2 \int\!\!\int_S dx dy = 2$ (area of S).

Therefore area enclosed by $C = \frac{1}{2} \oint_C x dy - y dx$.

For example, ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Boundary parametrised by $x = a \cos\theta, y = b \sin\theta, 0 \leq \theta \leq 2\pi$.

$$\begin{aligned} \text{Area} &= \tfrac{1}{2} \oint ba(\cos^2\theta + \sin^2\theta) d\theta \\ &= \pi ab. \end{aligned}$$