Determinants,Matrices,and Linear Equations

for

Subject 17:Mathematics 2 (Prelims) HT 1998 (Prof.I.J.R.Aitchison)

Problem Set

Linear Relations

1. Define what is meant by a 'linear relation' between 'input(s)' x and 'output(s)' y.For a single input x and output y,explain why the relation $y = ax + b$ is not a linear relation (even though it is a linear equation).
   Suppose now that the inputs are three-dimensional vectors $\boldsymbol{x}$,and the outputs are denoted by $f(\boldsymbol{x})$.Discuss whether the following are linear relations or not:-
   (i) $f(\boldsymbol{x}) = \boldsymbol{a}.\boldsymbol{x}$,for a fixed constant vector $\boldsymbol{a}$
   (ii) $f(\boldsymbol{x}) =\mid \boldsymbol{x} \mid$
   (iii) $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$,where $\boldsymbol{A}$ is a 3×3 matrix and $\boldsymbol{b}$ is a fixed constant vector.

2. A linear device which produces output $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ from input $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ via the matrix $\boldsymbol{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ is connected downstream from a similar device which produces output $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ from input $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ via the matrix $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, as shown in the figure:-

$$\begin{array}{cccccccc} x_1 & \quad & a & b & \quad y_1 \quad & e & f & \quad z_1 \\ x_2 & \quad & c & d & \quad y_2 \quad & g & h & \quad z_2. \end{array}$$

Write down y_1 and y_2 in terms of x_1 and x_2,and also z_1 and z_2 in terms of y_1 and y_2.Hence find explicit expressions for z_1 and z_2 in terms of x_1 and x_2.Show that your expressions are the same as those given by the formula

$$\boldsymbol{z} = \boldsymbol{BAx}$$

where $\boldsymbol{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\boldsymbol{BA}$ is the matrix product of $\boldsymbol{B}$ and $\boldsymbol{A}$ in that order ($\boldsymbol{B}$ to the left of $\boldsymbol{A}$).

Exercise

3. Given $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,$\boldsymbol{B} = \begin{pmatrix} 3 & 4 & 2 \\ 6 & 1 & 2 \\ 1 & 3 & 4 \end{pmatrix}$,$\boldsymbol{C} = \begin{pmatrix} 1 & 1 & 3 \\ 3 & 2 & 2 \end{pmatrix}$,
$\boldsymbol{D} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$, $\boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\boldsymbol{c} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{d}^T = \begin{pmatrix} 4 & 2 & 1 \end{pmatrix}$,
(a) state which of the following exists,and where possible evaluate them:
(i) $\boldsymbol{A}^2 = \boldsymbol{AA}$ (ii)$\boldsymbol{A} + \boldsymbol{B}$ (iii) $\boldsymbol{D} - \boldsymbol{A}$ (iv) $\boldsymbol{AC}$
(v) $\boldsymbol{CB}$ (vi) $\boldsymbol{BC}$ (vii) $\boldsymbol{A}.\boldsymbol{a}$ (viii) $\boldsymbol{d}^T\boldsymbol{Bc}$;
(b)calculate $\boldsymbol{AD}$ and $\boldsymbol{DA}$ and observe that it is not in general true that matrix multiplication is commutative.

Suffix notation

4. [The element in the i^{th} row and j^{th} column of a general matrix $\boldsymbol{A}$ is denoted by A_{ij}].Consider two 3×3 matrices $\boldsymbol{A}, \boldsymbol{B}$.Show explicitly that

$$(\boldsymbol{AB})_{23} = \sum_{j=1}^{3} A_{2j}B_{j3}.$$

$\boldsymbol{x}$ is a 3-component column vector.Show explicitly that

$$(\boldsymbol{Ax})_2 = \sum_{j=1}^{3} A_{2j}x_j.$$

Show for <u>general</u> $N \times N$ matrices $\boldsymbol{A}, \boldsymbol{B}$ that

$$(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T.$$

[Hint:take the i-j element of the LHS.This is $(\boldsymbol{AB})^T_{ij} = (\boldsymbol{AB})_{ji}$. Expand this as $\sum_{k=1}^{N} A_{jk}B_{ki}$.Now you have to rewrite this in terms of the elements of $\boldsymbol{A}^T$ and $\boldsymbol{B}^T$,and make it work].

Rotations

5. The components of a vector $\boldsymbol{r}$ in one frame are $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and in another frame,rotated by θ about the z-axis of the first,are $\boldsymbol{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ where

$$x' = \cos\theta \; x + \sin\theta \; y$$

$$y' = -\sin\theta \; x + \cos\theta \; y$$

$$z' = z \; .$$

Show that this transformation can be written as

$$\boldsymbol{r}' = \boldsymbol{R}_z(\theta)\boldsymbol{r}$$

where the matrix $\boldsymbol{R}_z(\theta)$ is given by

$$\boldsymbol{R}_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

The length of the vector is left unchanged if $x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$.Show that this is the same as the condition $\boldsymbol{r}'^T\boldsymbol{r}' = \boldsymbol{r}^T\boldsymbol{r}$. Suppose $\boldsymbol{r}' = \boldsymbol{R}\boldsymbol{r}$ for some general $\boldsymbol{R}$.Show that if this 'unchanged length' condition holds,then

$$\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}.$$

Check that this condition is satisfied for $\boldsymbol{R}_z(\theta)$. [Real 3×3 matrices satisfying $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$ are called 'orthogonal matrices',and they represent general 3-D rotations.]

Simultaneous equations

6. Solve the following linear equations and state whether the solution represents a point,a line,or a plane,or whether there is no solution.If the solution is a line,give the vector equation of the line.
(a)

$$\begin{aligned} 2x + y + z &= 2 \\ x + y + z &= 1 \\ 3x + y + 3z &= 1 \end{aligned}$$

(b)

$$\begin{aligned} x + y - 2z &= 7 \\ 2x + y - 4z &= 11 \\ x - y - 2z &= 1. \end{aligned}$$

7. For what values of β do the equations

$$\begin{aligned} x + y + z &= 0 \\ x + \beta y \quad &= s \\ 2x + 2y + z &= 3 \end{aligned}$$

have a unique solution?For what values of β and of s are the equations inconsistent?For the special case $\beta = 1$ and $s = 3$,obtain the solution in the form of the vector equation of a line.Illustrate your answer with a sketch.

8. What condition has to be satisfied by $\det \boldsymbol{A}$ for the homogeneous equations $\boldsymbol{Ax} = \boldsymbol{O}$ to have a non-trivial solution for $\boldsymbol{x}$? Show that this condition is satisfied for the equations

$$\begin{aligned} 2x_1 \qquad + 3x_3 &= 0 \qquad (1) \\ 4x_1 + 2x_2 + 5x_3 &= 0 \qquad (2) \\ x_1 - x_2 + 2x_3 &= 0. \qquad (3) \end{aligned}$$

Show that equation (1) can be written as a linear combination of equations (2) and (3),and explain the connection between this fact and the condition on det$\boldsymbol{A}$.Obtain the solution of the equations in the form of the vector equation for a line.

9. A very important type of simultaneous equation is the following: $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$.This 'says' that the application of $\boldsymbol{A}$ to $\boldsymbol{x}$ produces simply a scalar multiple of $\boldsymbol{x}$.The equation can be rewritten as $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{O}$, from which we deduce that it only has non-trivial solutions for $\boldsymbol{x}$ if det$\mid \boldsymbol{A} - \lambda\boldsymbol{I} \mid= 0$.<u>This gives an equation for λ.</u> The values of λ allowed by this equation are called 'eigenvalues of $\boldsymbol{A}$'. Suppose λ_1 is an eigenvalue;then there will be an $\boldsymbol{x}$,call it $\boldsymbol{x}_1$,such that $\boldsymbol{A}\boldsymbol{x}_1 = \lambda_1\boldsymbol{x}_1$.$\boldsymbol{x}_1$ is called the eigenvector of $\boldsymbol{A}$ belonging to the eigenvalue λ_1.
Find the eigenvalues of the matrix $\begin{pmatrix} -1 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix}$ and an eigenvector corresponding to the eigenvalue 4.(Note that since the equation for an eigenvector is homogeneous–i.e. $(\boldsymbol{A} - \lambda_1\boldsymbol{I}) = \boldsymbol{O}$–the vector can be freely multiplied by any scalar without changing anything.Often one <u>chooses</u> eigenvectors to have length 1).

Determinants,minors,cofactors,inverses

10. A 2×2 determinant is defined by

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21}.$$

The rule for evaluating the determinant of a general matrix $\boldsymbol{A}$ is :'multiply each element of one row (or one column) of $\boldsymbol{A}$ by its cofactor,and add up the results to give det$\boldsymbol{A}$.'
For the special case of an arbitrary 3×3 matrix,use this rule to prove
(i) if each element in any row (or column) is zero,then det$\boldsymbol{A}$ is zero
(ii)if each element of any row (or column) is multiplied by a factor k,det$\boldsymbol{A}$ is multiplied by k
(iii) if any two rows (or any two columns) are interchanged,det$\boldsymbol{A}$ changes sign

(iv) $\det \boldsymbol{A}$ is unchanged if a multiple of one row (column) is added to another row (column)
(v) if any two rows (columns) are proportional,$\det \boldsymbol{A}=0$
(vi)$\det \boldsymbol{A}=\det \boldsymbol{A}^T$.

11. Evaluate the determinants
$\begin{vmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 9 & 8 & 3 \end{vmatrix}$ and $\begin{vmatrix} 9 & 9 & 4 \\ 3 & 2 & 1 \\ 5 & 5 & 3 \end{vmatrix}$
(a) by directly expanding about a row (or column)
(b) by first using the results of Q 10 to reduce some elements to zero.

12. The variable x is to be found from the condition
$\begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix} = 0.$
Show that $(1 + x + x^2)$ is a factor of the determinant (hint: try adding the three rows (or columns) together).Show that $x = 1, -1/2 \pm \sqrt{3}/2\ i$.

13. Given the matrices
$\boldsymbol{A} = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & 1 & 4 \\ 1 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$
evaluate$\boldsymbol{AB}$ and $\boldsymbol{BA}$,and verify explicitly that
$\det(\boldsymbol{AB})=\det(\boldsymbol{BA})=\det(\boldsymbol{A}).\det(\boldsymbol{B})$.

14. Consider a general 3×3 matrix

$$\boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

Write down the matrix of cofactors (i.e. the matrix such that its (1,1) element is the cofactor of A_{11},its (1,2) element is the cofactor of A_{12},etc).Show that if you multiply each element of <u>one</u> row of $\boldsymbol{A}$ by the corresponding cofactor of <u>another</u> row,and add up the products, you get zero (hint:you need to spot that what you get is a determinant with two rows the same).
Hence explain why $\boldsymbol{A}^{-1}$ is given by the transpose of the matrix of cofactors of $\boldsymbol{A}$, divided by $\det \boldsymbol{A}$.

15. Find the inverse of the matrix

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix}$$

using the 'matrix of cofactors' formula.

16. The three quantities $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ are related to the three quantities $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ by $\boldsymbol{Ax} = \boldsymbol{y}$ where $\boldsymbol{A}$ is the general 3×3 matrix in Q14,and $\det\boldsymbol{A} \neq 0$.Find x_2 by using $\boldsymbol{A}^{-1}$,and verify that it is the same as the result obtained from 'Cramer's rule'.

17. (See Boas pages 121,122).Find the inverse of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix}$$

(same as in Q15) by the 'row reduction method',as follows.Let the inverse be $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.Then we must have

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{I}.$$

We can do elementary row (or column) processes on <u>both sides</u> of this equation.The aim is to bring $\boldsymbol{A}$ to 'echelon' form.So consider $R2 \to R2 - 3\ R1$,followed by $R3 \to R3 - R1$, followed by $R3 \to R3 - (3/5)\ R2$.Verify that this leads to

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 1 \\ 0 & 0 & 7/5 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4/5 & -3/5 & 1 \end{pmatrix}.$$

Take 'third row times first column' on LHS to get
$(7/5)a_{31} = (4/5) \Rightarrow a_{31} = 4/7$.Find a_{32}, a_{33} similarly.
Now look at the 'second row times first column,second column or third column' results,and use the values of a_{31}, a_{32}, a_{33} already found to get a_{21}, a_{22}, a_{23}.Finally,get a_{11}, a_{12}, a_{13}.Compare your answer with the answer of Q15.