Determinants,Matrices,and Linear Equations

for

Subject 17:Mathematics 2 (Prelims) HT 1998 (Prof.I.J.R.Aitchison)

Supplementary Notes

Linearity

Let f be some operation performed on the 'input' x (which might be more than one quantity),so as to produce an 'output' y: $y = f(x)$.And for input u,let the output be $v = f(u)$,say.Then f is a **linear** operation if

(i) $f(x+u) = f(x) + f(u) = y + v$

(ii) $f(sx) = sf(x)$ for any scalar multiple s.

General linear property for matrices acting on column vectors is
$\boldsymbol{A}(s\boldsymbol{x} + t\boldsymbol{u}) = s\boldsymbol{A}\boldsymbol{x} + t\boldsymbol{A}\boldsymbol{u}$.

Matrices

$$
\begin{array}{c}
A_{11}\ A_{12}\ A_{13}\ \ldots\ldots\ldots\ldots\ldots\ldots A_{1N} \\
A_{21}\ A_{22}\ A_{23}\ \ldots\ldots\ldots\ldots\ldots\ldots A_{2N} \\
\ldots \\
\ldots \\
\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots A_{ij}\ldots\ldots \\
\ldots \\
\ldots \\
\ldots \\
A_{N1}\ A_{N2}\ A_{N3}\ \ldots\ldots\ldots\ldots\ldots\ldots A_{NN}
\end{array}
$$

$(\boldsymbol{AB})_{ij} = \sum_{k=1}^{N} A_{ik}B_{kj}$

$(\boldsymbol{A}^T)_{ij} = (\boldsymbol{A})_{ji}$; $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$

$(\boldsymbol{Ax})_i = \sum_{j=1}^{N} A_{ij}x_j$

Simultaneous linear equations

$\boldsymbol{Ax} = \boldsymbol{y}$ or $\sum_{j=1}^{N} A_{ij}x_j = y_i$ for $i = 1, 2, ...N$

'Cramer's rule'
x_1=(1/det($\boldsymbol{A}$)).det(the matrix obtained from $\boldsymbol{A}$ by replacing the first column by $\boldsymbol{y}$)

x_2=(1/det($\boldsymbol{A}$)).det(the matrix obtained from $\boldsymbol{A}$ by replacing the second column by $\boldsymbol{y}$)

etc

Determinants

The 'minor' M_{ij} associated with the element A_{ij} of the matrix $\boldsymbol{A}$ is defined to be the determinant of the matrix obtained from $\boldsymbol{A}$ by deleting the i^{th} row and the j^{th} column (i.e. the whole row and the whole column that contain A_{ij}).The 'cofactor' of A_{ij} is $(-)^{i+j}M_{ij}$. To calculate det($\boldsymbol{A}$),take the elements of any one row (or column) and multiply each by its cofactor;the sum of these products is det($\boldsymbol{A}$).For properties of determinants see Q.10.

The inverse matrix:$\boldsymbol{A}^{-1}$

1. Form the matrix of cofactors (i.e. the (1,1) element of this matrix is the cofactor of A_{11},the (1,2) element is the cofactor of A_{12},etc).
2. Take the transpose of this matrix of cofactors and divide by det($\boldsymbol{A}$).The result (if det($\boldsymbol{A}$)$\neq$ 0) is $\boldsymbol{A}^{-1}$.

See Q.14

The condition det($\boldsymbol{A}$)=0

1. Necessary for $\boldsymbol{Ax} = 0$ to have a non-trivial solution for $\boldsymbol{x}$.

2. If det($\boldsymbol{A}$)=0,$\boldsymbol{A}^{-1}$ fails to exist.

3. If det($\boldsymbol{A}$)=0,two (or more) rows (or columns) of $\boldsymbol{A}$ are linearly dependent.

4. If $\boldsymbol{A}$ is 3×3,say
$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$
det($\boldsymbol{A}$)=0 implies that it must be possible to write one row as a linear combination of the other two-i.e. only two of $\boldsymbol{a}$,$\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly independent.(I exclude here the trivial cases where two or more of $\boldsymbol{a}$,$\boldsymbol{b}$,$\boldsymbol{c}$ are actually identical).The condition det($\boldsymbol{A}$)=0 can also be written,in this 3×3 case,as $\boldsymbol{a}.\boldsymbol{b}\times\boldsymbol{c} = 0$. This says $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}\times\boldsymbol{c}$.But $\boldsymbol{b}\times\boldsymbol{c}$ is perpendicular to both $\boldsymbol{b}$ and $\boldsymbol{c}$. So $\boldsymbol{a}$ must be parallel to the plane containing $\boldsymbol{b}$ and $\boldsymbol{c}$,or it must lie in that plane.(Can change $\boldsymbol{a}\rightarrow\boldsymbol{b}\rightarrow\boldsymbol{c}$ in these statements using the cyclic property of the triple scalar product,or the row interchange property of the det).

5. Consider three simultaneous equations for x, y, z :
$$a_1x + a_2y + a_3z = l \longrightarrow \boldsymbol{a}.\boldsymbol{r} = l$$
$$b_1x + b_2y + b_3z = m \longrightarrow \boldsymbol{b}.\boldsymbol{r} = m$$
$$c_1x + c_2y + c_3z = n \longrightarrow \boldsymbol{c}.\boldsymbol{r} = n.$$
$\boldsymbol{a}$,$\boldsymbol{b}$,$\boldsymbol{c}$ are the normals to these three planes.If det (matrix of coefficients)=0,then $\boldsymbol{a}.\boldsymbol{b}\times\boldsymbol{c}$=0.See 4. above.