Solutions to problems in "Gauge Theories in Particle Physics", Third Edition, volume 2: "QCD and the Electroweak Theory"

Chapter 12

12.1 The elements of the set (which we want to show form a group) are all 2×2 matrices $\mathbf{V}$ which are unitary (i.e. $\mathbf{V}^\dagger\mathbf{V} = \mathbf{V}\mathbf{V}^\dagger = \mathbf{I}_2$) and have determinant equal to $+1$. We need to show that conditions (i)-(iv) listed in section M.1 of Appendix M are satisfied, where the law of combination is ordinary matrix multiplication.
(i) Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be any two members of the set. Their product is $\mathbf{V}_1\mathbf{V}_2$. Clearly (by the rules of matrix multiplication) this is a 2×2 matrix. To see if it is unitary, we form $(\mathbf{V}_1\mathbf{V}_2)(\mathbf{V}_1\mathbf{V}_2)^\dagger$. Using the general matrix result $(AB)^\dagger = B^\dagger A^\dagger$ this becomes

$$\begin{aligned}(\mathbf{V}_1\mathbf{V}_2)(\mathbf{V}_1\mathbf{V}_2)^\dagger &= \mathbf{V}_1\mathbf{V}_2(\mathbf{V}_2)^\dagger(\mathbf{V}_1)^\dagger \\ &= \mathbf{V}_1\mathbf{I}_2(\mathbf{V}_1)^\dagger = \mathbf{I}_2\end{aligned}$$

using first the unitarity of $\mathbf{V}_2$ and then that of $\mathbf{V}_1$. So the product of any two such matrices is a unitary 2×2 matrix. Similarly, $\det(\mathbf{V}_1\mathbf{V}_2) = (\det\mathbf{V}_1)(\det\mathbf{V}_2) = 1$ using the '$\det \mathbf{V}=1$' condition for each matrix. So the product of any two such matrices is a unitary 2×2 matrix with determinant $+1$, and therefore the product also belongs to the same set, verifying property (i).
(ii) Matrix multiplication is associative.
(iii) The identity element is $\mathbf{I}_2$.
(iv) The inverse of any element $\mathbf{V}$ is $\mathbf{V}^\dagger$, by the unitarity property.

12.2 $\mathbf{V}_{\text{infl}}$ is given by equation (12.17) which we may write as

$$\mathbf{V}_{\text{infl}} = \begin{pmatrix} 1 + \mathrm{i}\xi_{11} & \mathrm{i}\xi_{12} \\ \mathrm{i}\xi_{21} & 1 + \mathrm{i}\xi_{22} \end{pmatrix}.$$

Hence

$$\begin{aligned}\det\mathbf{V}_{\text{infl}} &= (1 + \mathrm{i}\xi_{11})(1 + \mathrm{i}\xi_{22}) + \xi_{12}\xi_{21} \\ &\approx 1 + \mathrm{i}(\xi_{11} + \xi_{22}),\end{aligned}$$

to first order in ξ, so that the condition $\det\mathbf{V}_{\mathrm{infl}} = 1$ reduces to $\xi_{11} + \xi_{22} = 0$, which is just $\mathrm{Tr}\xi = 0$.

12.3 The commutation relations can be checked by straightforward matrix multiplication with the 2×2 τ matrices.

12.4 We have

$$\begin{aligned}
[T_i, T_j] &= \left[\frac{1}{2}\tau_{(1)i} + \frac{1}{2}\tau_{(2)i} + \dots \frac{1}{2}\tau_{(A)i}, \frac{1}{2}\tau_{(1)j} + \frac{1}{2}\tau_{(2)j} + \dots \frac{1}{2}\tau_{(A)j}\right] \\
&= \left[\frac{1}{2}\tau_{(1)i}, \frac{1}{2}\tau_{(1)j}\right] + \left[\frac{1}{2}\tau_{(2)i}, \frac{1}{2}\tau_{(2)j}\right] + \dots \left[\frac{1}{2}\tau_{(A)i}, \frac{1}{2}\tau_{(A)j}\right] \\
&= \mathrm{i}\epsilon_{ijk}\left(\frac{1}{2}\tau_{(1)k} + \frac{1}{2}\tau_{(2)k} + \dots \frac{1}{2}\tau_{(A)k}\right) \\
&= \mathrm{i}\epsilon_{ijk}T_k, \qquad (1)
\end{aligned}$$

where in the second line we used the fact that the τ's associated with different nucleons commute, and in the third line we used equation (12.28) for each of the independent τ's.

12.5 The (j, k) element (j=row label, k=column label) of $T_1^{(1)}$ is $-\mathrm{i}\epsilon_{1jk}$. The only non-zero entries are for $j = 2, k = 3$, giving $(T_1^{(1)})_{23} = -\mathrm{i}$, and for $j = 3, k = 2$, giving $(T_1^{(1)})_{32} = \mathrm{i}$. Hence

$$T_1^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}.$$

Similarly,

$$T_2^{(1)} = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix},$$

and

$$T_3^{(1)} = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is then a matter of straightforward matrix multiplication to check (12.47) for the case $i, j, k = 1, 2, 3$.

12.6 We expand equation (12.61):

$$\begin{aligned} v_i' &= q^\dagger(\mathbf{1}_2 - \mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{\tau}/2)\tau_i(\mathbf{1}_2 + \mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{\tau}/2)q \\ &\approx q^\dagger\tau_i q - \mathrm{i}q^\dagger(\boldsymbol{\epsilon}\cdot\boldsymbol{\tau}/2)\tau_i q + \mathrm{i}q^\dagger\tau_i(\boldsymbol{\epsilon}\cdot\boldsymbol{\tau}/2)q \\ &= v_i - \mathrm{i}\frac{1}{2}q^\dagger\epsilon_j\tau_j\tau_i q + \mathrm{i}\frac{1}{2}q^\dagger\tau_i\epsilon_j\tau_j q \\ &= v_i + \mathrm{i}\frac{1}{2}\epsilon_j q^\dagger(\tau_i\tau_j - \tau_j\tau_i)q, \end{aligned} \tag{2}$$

as required, where in the second line we have dropped the term quadratic in the ϵ's, and used $\mathbf{1}_2\tau_i = \tau_i\mathbf{1}_2 = \mathbf{1}_2\tau_i\mathbf{1}_2 = \tau_i$, and a summation on $j = 1, 2, 3$ is understood.

12.7 A matrix H is Hermitian if the condition

$$H_{ji} = H_{ij}^*$$

is satisfied. Putting $i = j$, we see that the diagonal elements must be real, while the off-diagonal element $H_{ji}(i \neq j)$ is equal to the complex conjugate of the element H_{ij}. So the general 3×3 Hermitian matrix takes the form

$$\begin{pmatrix} a & d & e \\ d^* & b & f \\ e^* & f^* & c \end{pmatrix}$$

where a, b, c are real (three parameters in all), and d, e, f are complex (six parameters in all). The traceless condition is $a + b + c = 0$, so only two of a, b, c are independent, leaving 8 real parameters.

12.8 The infinitesimal form of (12.81) is

$$w_a' = w_a + \mathrm{i}\eta_b\left(G_b^{(8)}\right)_{ac} w_c,$$

where a sum from 1 to 8 on the repeated indices b and c is understood, and a runs from 1 to 8. From equation (12.84) we have

$$\left(G_b^{(8)}\right)_{ac} = -\mathrm{i}f_{bac},$$

and so

$$w_a' = w_a + \eta_b f_{bac} w_c = w_a - f_{abc}\eta_b w_c,$$

as in (12.80), where in the last step we used the antisymmetry of the f's (see section M.4.5).
The next part is a special case of the general result concerning the regular or adjoint representation given in section M.5. We begin with the Jacobi identity for the generators of SU(3) (in any representation):

$$[G_a, [G_b, G_c]] + [G_b, [G_c, G_a]] + [G_c, [G_a, G_b]] = 0.$$

Using equation (M.58) for the SU(3) algebra we obtain

$$[G_a, \mathrm{i} f_{bcd} G_d] + [G_b, \mathrm{i} f_{cad} G_d] + [G_c, \mathrm{i} f_{abd} G_d] = 0.$$

Using (M.58) again this becomes

$$(\mathrm{i} f_{bcd}\, \mathrm{i} f_{ade} + \mathrm{i} f_{cad}\, \mathrm{i} f_{bde} + \mathrm{i} f_{abd}\, \mathrm{i} f_{cde}) G_e = 0.$$

Since the G's are independent (no linear combination of them can equal zero) we can infer that the bracketed quantity multiplying G_e vanishes:

$$\mathrm{i} f_{bcd}\, \mathrm{i} f_{ade} + \mathrm{i} f_{cad}\, \mathrm{i} f_{bde} + \mathrm{i} f_{abd}\, \mathrm{i} f_{cde} = 0.$$

Using equation (12.84), the first term can be written as

$$\left(G_b^{(8)}\right)_{cd} \left(G_a^{(8)}\right)_{de}$$

which is just the (ce) matrix element of the product of matrices $G_b^{(8)} G_a^{(8)}$ (note that the repeated index d is summed over, as usual). Using the antisymmetry of the f's, the second term becomes

$$-\left(G_a^{(8)}\right)_{cd} \left(G_b^{(8)}\right)_{de}$$

which is the (ce) matrix element of $-G_a^{(8)} G_b^{(8)}$. The third term is $\mathrm{i} f_{abd} \left(G_d^{(8)}\right)_{ce}$. Putting it all together we have verified the (ce) matrix element of (12.83) (replacing the dummy - summed over - index c by d).

12.9 Dropping the term of second order in the ϵ's, equation (12.99) is

$$\hat{q} - \mathrm{i}\boldsymbol{\epsilon} \cdot \boldsymbol{\tau}/2\ \hat{q} = \hat{q} + \mathrm{i}\boldsymbol{\epsilon} \cdot \hat{\boldsymbol{T}}^{(\frac{1}{2})}\ \hat{q} - \mathrm{i}\hat{q}\ \boldsymbol{\epsilon} \cdot \hat{\boldsymbol{T}}^{(\frac{1}{2})}.$$

This becomes

$$\epsilon_i(-\tau_i/2)\hat{q} = \epsilon_i[\hat{T}_i^{(\frac{1}{2})}, \hat{q}]$$

from which (12.100) follows, equating the coefficients of ϵ_i on both sides.

12.10 (We will give a 'brute force' solution to this problem, and use a slightly more sophisticated method for the next one, 12.11.)

In (12.100), each of $\hat{\boldsymbol{T}}^{(\frac{1}{2})}$ and $\boldsymbol{\tau}$ has three components; we shall check (12.100) for one of them:

$$[\hat{T}_1^{(\frac{1}{2})}, \hat{q}] = -(\tau_1/2)\hat{q}.$$

We need to be more explicit, and include both the space-time arguments and the spinor indices of the $\hat{q}$ fields, in order to be able to use the fermionic commutation relations of the form (7.44). First of all, from (12.101) and (12.86) we have

$$\begin{aligned}\hat{T}_1^{(\frac{1}{2})} &= \int (\hat{u}^\dagger(x)\hat{d}^\dagger(x)) \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \hat{u}(x) \\ \hat{d}(x) \end{pmatrix} \mathrm{d}^3\boldsymbol{x} \\ &= \frac{1}{2}\int (\hat{u}_\alpha^\dagger(x)\hat{d}_\alpha(x) + \hat{d}_\alpha^\dagger(x)\hat{u}_\alpha(x))\, \mathrm{d}^3\boldsymbol{x},\end{aligned}$$

where α is a (summed over) Dirac spinor index, running from 1 to 4. So we need to verify the result

$$\begin{aligned}\left[\frac{1}{2}\int (\hat{u}_\alpha^\dagger(x)\hat{d}_\alpha(x) + \hat{d}_\alpha^\dagger(x)\hat{u}_\alpha(x))\, \mathrm{d}^3\boldsymbol{x}, \begin{pmatrix} \hat{u}_\beta(y) \\ \hat{d}_\beta(y) \end{pmatrix}\right] &= \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \hat{u}_\beta(y) \\ \hat{d}_\beta(y) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2}\hat{d}_\beta(y) \\ -\frac{1}{2}\hat{u}_\beta(y) \end{pmatrix},\end{aligned}$$

where β is an (un-summed) Dirac spinor index. We will check the top (isospin) component on both sides, i.e.

$$\left[\frac{1}{2}\int (\hat{u}_\alpha^\dagger(x)\hat{d}_\alpha(x) + \hat{d}_\alpha^\dagger(x)\hat{u}_\alpha(x))\, \mathrm{d}^3\boldsymbol{x}, \hat{u}_\beta(y)\right] = -\frac{1}{2}\hat{d}_\beta(y). \qquad \text{(A)}$$

There are four terms in the commutator:

$$\hat{u}_\alpha^\dagger(x)\hat{d}_\alpha(x)\hat{u}_\beta(y) + \hat{d}_\alpha^\dagger(x)\hat{u}_\alpha(x)\hat{u}_\beta(y) - \hat{u}_\beta(y)\hat{u}_\alpha^\dagger(x)\hat{d}_\alpha(x) - \hat{u}_\beta(y)\hat{d}_\alpha^\dagger(x)\hat{u}_\alpha(x). \quad \text{(B)}$$

$\hat{u}$ fields anticommute with other $\hat{u}$ fields, and with $\hat{d}$ fields; the only non-zero anti-commutator is (c.f. (7.44))

$$\{\hat{u}_\alpha^\dagger(\boldsymbol{x}, t), \hat{u}_\beta(\boldsymbol{y}, t)\} = \delta(\boldsymbol{x} - \boldsymbol{y})\delta_{\alpha\beta}.$$

Notice that the fields in this anti-commutator are at equal times t. On the other hand, the fields in the commutator (A) are apparently at different times x_0 and y_0. However, equation (12.102) assures us that the operators $\hat{\boldsymbol{T}}^{(\frac{1}{2})}$ are in fact independent of time, and so we are free to choose x_0 in (A), and thus in (B) also, equal to y_0. With this understanding, in the second term in (B) the factor $\hat{u}_\beta(y)$ can be anti-commuted first past $\hat{u}_\alpha(x)$ and then past $\hat{d}^\dagger_\alpha(x)$, giving $+\hat{u}_\beta(y)\hat{d}^\dagger_\alpha(x)\hat{u}_\alpha(x)$, which cancels with the fourth term of (B). The first term in (B) can be written as

$$\hat{u}^\dagger_\alpha(x)\hat{d}_\alpha(x)\hat{u}_\beta(y) = -\hat{u}^\dagger_\alpha(\boldsymbol{x}, y_0)\hat{u}_\beta(\boldsymbol{y}, y_0)\hat{d}_\alpha(\boldsymbol{x}, y_0)$$

which can be combined with the third term of (B) to give

$$-(\hat{u}^\dagger_\alpha(\boldsymbol{x}, y_0)\hat{u}_\beta(\boldsymbol{y}, y_0)+\hat{u}_\beta(\boldsymbol{y}, y_0)\hat{u}^\dagger_\alpha(\boldsymbol{x}, y_0))\hat{d}^\dagger_\alpha(\boldsymbol{x}, y_0) = -\delta^3(\boldsymbol{x}-\boldsymbol{y})\delta_{\alpha\beta}\hat{d}_\alpha(\boldsymbol{x}, y_0),$$

which is therefore the value of the commutator (B). Inserting this into the left hand side of (A) gives

$$-\frac{1}{2}\int \delta^3(\boldsymbol{x}-\boldsymbol{y})\delta_{\alpha\beta}\hat{d}_\alpha(\boldsymbol{x}, y_0)\,\mathrm{d}^3\boldsymbol{x} = -\frac{1}{2}\hat{d}_\beta(\boldsymbol{y}, y_0)$$

which is the same as the right hand side, as was to be shown.

12.11 We introduce the notation $\hat{q}_{\alpha a}$ for the 8-component field $\hat{q}$ of (12.86): 'a' runs over the two (isospin) values, $a = 1$ being the 'up' component, and $a = 2$ being the 'down' component, while 'α' runs over the four Dirac components in $\hat{u}$ and $\hat{d}$. Thus $\hat{q}_{11}$ means the first spinor component of $\hat{u}$, and $\hat{q}_{32}$ means the third spinor component of $\hat{d}$, for example. With this notation, the only non-vanishing anti-commutator is

$$\{\hat{q}^\dagger_{\alpha a}(\boldsymbol{x}, t), \hat{q}_{\beta b}(\boldsymbol{y}, t)\} = \delta_{\alpha\beta}\delta_{ab}\delta^3(\boldsymbol{x}-\boldsymbol{y}).$$

We have, then,

$$\hat{T}^{(\frac{1}{2})}_i = \int \hat{q}^\dagger_{\alpha a}(x)(\tau_i/2)_{ab}\hat{q}_{\alpha b}(x)\,\mathrm{d}^3\boldsymbol{x},$$

and

$$\hat{T}^{(\frac{1}{2})}_j = \int \hat{q}^\dagger_{\beta c}(y)(\tau_j/2)_{cd}\hat{q}_{\beta d}(y)\,\mathrm{d}^3\boldsymbol{y}.$$

The commutator on the left hand side of (12.103) is

$$\frac{1}{4}\int \mathrm{d}^3\boldsymbol{x}\int \mathrm{d}^3\boldsymbol{y}\,(\tau_i)_{ab}(\tau_j)_{cd}(\hat{q}^\dagger_{\alpha a}(x)\hat{q}_{\alpha b}(x)\hat{q}^\dagger_{\beta c}(y)\hat{q}_{\beta d}(y)-\hat{q}^\dagger_{\beta c}(y)\hat{q}_{\beta d}(y)\hat{q}^\dagger_{\alpha a}(x)\hat{q}_{\alpha b}(x)). \quad \text{(C)}$$

Note that quantities such as $(\tau_i)_{ab}$ are single numbers (here, the (a,b) element of the matrix τ_i), so they can be freely moved to any convenient position in the expression. Also, since the $\hat{T}^{(\frac{1}{2})}$'s are time-independent, we are free to choose $x_0 = y_0$, and we shall assume this has been done, without indicating it explicitly. Consider the two products of four field operators in (C). The first one may be written as

$$\begin{aligned}\hat{q}^\dagger_{\alpha a}(x)\hat{q}_{\alpha b}(x)\hat{q}^\dagger_{\beta c}(y)\hat{q}_{\beta d}(y) &= \hat{q}^\dagger_{\alpha a}(x)[-\hat{q}^\dagger_{\beta c}(y)\hat{q}_{\alpha b}(x)+\delta_{\alpha\beta}\delta_{bc}\delta^3(\boldsymbol{x}-\boldsymbol{y})]\hat{q}_{\beta d}(y)\\ &= -\hat{q}^\dagger_{\beta c}(y)\hat{q}^\dagger_{\alpha a}(x)\hat{q}_{\beta d}(y)\hat{q}_{\alpha b}(x)+\hat{q}^\dagger_{\alpha a}(x)\hat{q}_{\alpha d}(y)\delta_{bc}\delta^3(\boldsymbol{x}-\boldsymbol{y}),\end{aligned}$$

where in the second line we have performed some anti-commutations in the first term. Similarly, the second product of four field operators in (C) is

$$\hat{q}_{\beta c}(y)\hat{q}^\dagger_{\alpha a}(x)\hat{q}_{\beta d}(y)\hat{q}_{\alpha b}(x)-\hat{q}^\dagger_{\alpha c}(y)\hat{q}_{\alpha b}(x)\delta_{ad}\delta^3(\boldsymbol{x}-\boldsymbol{y}).$$

The first term here cancels with the first term of the preceding expression, so that (C) becomes

$$\frac{1}{4}\int\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{y}\,(\tau_i)_{ab}(\tau_j)_{cd}[\hat{q}^\dagger_{\alpha a}(x)\hat{q}_{\alpha d}(y)\delta_{bc}-\hat{q}^\dagger_{\alpha c}(y)\hat{q}_{\alpha b}(x)\delta_{ad}]\,\delta^3(\boldsymbol{x}-\boldsymbol{y})$$

where $x_0 = y_0$ is understood. Now note that

$$(\tau_i)_{ab}(\tau_j)_{cd}\delta_{bc} = (\tau_i)_{ab}(\tau_j)_{bd} = (\tau_i\tau_j)_{ad}$$

while

$$(\tau_i)_{ab}(\tau_j)_{cd}\delta_{ad} = (\tau_j\tau_i)_{cb}.$$

Thus the expression (C) is

$$\frac{1}{4}\int\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{y}\,[\hat{q}^\dagger_{\alpha a}(x)(\tau_i\tau_j)_{ad}\hat{q}_{\alpha d}(y)-\hat{q}^\dagger_{\alpha c}(y)(\tau_j\tau_i)_{cb}\hat{q}_{\alpha b}(x)]\,\delta^3(\boldsymbol{x}-\boldsymbol{y}).$$

In the second term, the indices c and b are both summed over, and can be replaced by a and d, respectively, as in the first term. The expression

$$\hat{q}^\dagger_{\alpha a}(x)(\tau_i\tau_j)_{ad}\hat{q}_{\alpha d}(x)$$

is just

$$\hat{q}^{\dagger}(x)(\tau_i\tau_j)\hat{q}(x)$$

with the explicit isospin and (Dirac) spinor indices inserted, and similarly for the $\tau_j\tau_i$ term, so finally (C) is

$$\int\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{y}\,\hat{q}^{\dagger}(x)[\tau_i/2,\tau_j/2]\hat{q}(y)\delta^3(\boldsymbol{x}-\boldsymbol{y}) = \mathrm{i}\epsilon_{ijk}\int \mathrm{d}^3\boldsymbol{x}\,\hat{q}^{\dagger}(x)(\tau_k/2)\hat{q}(x)$$

using (12.28), which is precisely the right hand side of (12.103), as required.

Chapter 13

13.1 For the infinitesimal case, (13.34) is

$$\begin{aligned}\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}'^{\mu} &= \frac{\mathrm{i}}{g}(\partial^{\mu}(1+\frac{\mathrm{i}}{2}g\boldsymbol{\tau}\cdot\boldsymbol{\epsilon}(x)))(1-\frac{\mathrm{i}}{2}g\boldsymbol{\tau}\cdot\boldsymbol{\epsilon}(x))\\ &+ (1+\frac{\mathrm{i}}{2}g\boldsymbol{\tau}\cdot\boldsymbol{\epsilon}(x))\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}^{\mu}(1-\frac{\mathrm{i}}{2}g\boldsymbol{\tau}\cdot\boldsymbol{\epsilon}(x)),\end{aligned}$$

and expanding this to first order in $\boldsymbol{\epsilon}$ gives

$$\begin{aligned}\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}'^{\mu} &\approx \frac{\mathrm{i}}{g}\frac{\mathrm{i}g}{2}\boldsymbol{\tau}\cdot\partial^{\mu}\boldsymbol{\epsilon}(x)+\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}^{\mu}\\ &+ (\frac{\mathrm{i}g}{2}\tau_i\epsilon_i(x))(\frac{1}{2}\tau_j W_j^{\mu}) + (\frac{1}{2}\tau_j W_j^{\mu})(\frac{-\mathrm{i}g}{2}\tau_i\epsilon_i(x))\\ &= -\frac{1}{2}\boldsymbol{\tau}\cdot\partial^{\mu}\boldsymbol{\epsilon}(x)+\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}^{\mu}+\mathrm{i}g\left(\frac{\tau_i}{2}\frac{\tau_j}{2}-\frac{\tau_j}{2}\frac{\tau_i}{2}\right)\epsilon_i(x)W_j^{\mu}\\ &= -\frac{1}{2}\boldsymbol{\tau}\cdot\partial^{\mu}\boldsymbol{\epsilon}(x)+\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}^{\mu}+\mathrm{i}g\,\mathrm{i}\epsilon_{ijk}\frac{\tau_k}{2}\epsilon_i(x)W_j^{\mu}.\end{aligned}$$

The last term may be re-written as $-g\epsilon_{kij}\frac{\tau_k}{2}\epsilon_i(x)W_j^{\mu}$, which is the triple scalar product $-g\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{\epsilon}(x)\times\boldsymbol{W}^{\mu}$. Equating the components of $\frac{1}{2}\boldsymbol{\tau}$ on both sides then leads to (13.26).

13.2 Multiplying out the left hand side of (13.79) we obtain

$$
\begin{aligned}
&(\partial^\mu + \mathrm{i}eA^\mu)(\partial^\nu\psi + \mathrm{i}eA^\nu\psi) - (\partial^\nu + \mathrm{i}eA^\nu)(\partial^\mu\psi + \mathrm{i}eA^\mu\psi)\\
&= \partial^\mu\partial^\nu\psi + \mathrm{i}e(\partial^\mu A^\nu)\psi + \mathrm{i}eA^\nu\partial^\mu\psi + \mathrm{i}eA^\mu\partial^\nu\psi\\
&-e^2A^\mu A^\nu\psi - \partial^\mu\partial^\nu\psi - \mathrm{i}e(\partial^\nu A^\mu)\psi - \mathrm{i}eA^\mu\partial^\nu\psi - \mathrm{i}eA^\nu\partial^\mu\psi + e^2A^\nu A^\mu\psi\\
&= \mathrm{i}e(\partial^\mu A^\nu - \partial^\nu A^\mu)\psi = \mathrm{i}eF^{\mu\nu}\psi
\end{aligned}
$$

as required.

13.3 The left hand side of (13.82) is

$$(\partial^\mu + \frac{\mathrm{i}g}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}^\mu)(\partial^\nu\psi^{(\frac{1}{2})} + \frac{\mathrm{i}g}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}^\nu\psi^{(\frac{1}{2})}) - (\partial^\nu + \frac{\mathrm{i}g}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}^\nu)(\partial^\mu\psi^{(\frac{1}{2})} + \frac{\mathrm{i}g}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}^\mu\psi^{(\frac{1}{2})}).$$

After expanding this out some terms cancel and we are left with

$$\frac{\mathrm{i}g}{2}\boldsymbol{\tau}\cdot(\partial^\mu\boldsymbol{W}^\nu - \partial^\nu\boldsymbol{W}^\mu)\psi^{(\frac{1}{2})} - g^2\left[\left(\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}^\mu\right)\left(\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}^\nu\right) - \left(\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}^\nu\right)\left(\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}^\mu\right)\right]\psi^{(\frac{1}{2})}.$$

The second term is

$$-g^2\left(\frac{\tau_i}{2}\frac{\tau_j}{2} - \frac{\tau_j}{2}\frac{\tau_i}{2}\right)W_i^\mu W_j^\nu\psi^{(\frac{1}{2})} = -g^2\mathrm{i}\epsilon_{ijk}\frac{\tau_k}{2}W_i^\mu W_j^\nu\psi^{(\frac{1}{2})} = \frac{\mathrm{i}g}{2}\boldsymbol{\tau}\cdot(-g\boldsymbol{W}^\mu\times\boldsymbol{W}^\nu)\psi^{(\frac{1}{2})}$$

from which the result follows.

13.4 Using (13.83) and (13.23) we have

$$
\begin{aligned}
\delta\boldsymbol{F}^{\mu\nu} &= \partial^\mu\delta\boldsymbol{W}^\nu - \partial^\nu\delta\boldsymbol{W}^\mu - g\delta\boldsymbol{W}^\mu\times\boldsymbol{W}^\nu - g\boldsymbol{W}^\mu\times\delta\boldsymbol{W}^\nu\\
&= \partial^\mu(-\partial^\nu\boldsymbol{\epsilon}(x) - g\boldsymbol{\epsilon}(x)\times\boldsymbol{W}^\nu) - \partial^\nu(-\partial^\mu\boldsymbol{\epsilon}(x) - g\boldsymbol{\epsilon}(x)\times\boldsymbol{W}^\mu)\\
&- g(-\partial^\mu\boldsymbol{\epsilon}(x) - g\boldsymbol{\epsilon}(x)\times\boldsymbol{W}^\mu)\times\boldsymbol{W}^\nu - g\boldsymbol{W}^\mu\times(-\partial^\nu\boldsymbol{\epsilon}(x) - g\boldsymbol{\epsilon}(x)\times\boldsymbol{W}^\nu)\\
&= -g\boldsymbol{\epsilon}(x)\times(\partial^\mu\boldsymbol{W}^\nu - \partial^\nu\boldsymbol{W}^\mu) + g^2(\boldsymbol{\epsilon}(x)\times\boldsymbol{W}^\mu)\times\boldsymbol{W}^\nu\\
&+ g^2\boldsymbol{W}^\mu\times(\boldsymbol{\epsilon}(x)\times\boldsymbol{W}^\nu).
\end{aligned}
$$

The last two terms are

$$-g^2\boldsymbol{W}^\nu\times(\boldsymbol{\epsilon}(x)\times\boldsymbol{W}^\mu) = -g^2[\boldsymbol{\epsilon}(x)(\boldsymbol{W}^\nu\cdot\boldsymbol{W}^\mu) - \boldsymbol{W}^\mu(\boldsymbol{W}^\nu\cdot\boldsymbol{\epsilon}(x))]$$

and

$$g^2\boldsymbol{W}^\mu\times(\boldsymbol{\epsilon}(x)\times\boldsymbol{W}^\nu) = g^2[\boldsymbol{\epsilon}(x)(\boldsymbol{W}^\mu\cdot\boldsymbol{W}^\nu) - \boldsymbol{W}^\nu(\boldsymbol{W}^\mu\cdot\boldsymbol{\epsilon}(x))],$$

so that they can be combined to give

$$g^2[\boldsymbol{W}^\mu(\boldsymbol{\epsilon}(x)\cdot\boldsymbol{W}^\nu)-\boldsymbol{W}^\nu(\boldsymbol{\epsilon}(x)\cdot\boldsymbol{W}^\mu)]$$

which can also be written as

$$-g\boldsymbol{\epsilon}(x)\times(-g\boldsymbol{W}^\mu\times\boldsymbol{W}^\nu).$$

Hence

$$\begin{aligned}\delta\boldsymbol{F}^{\mu\nu} &= -g\boldsymbol{\epsilon}(x)\times[\partial^\mu\boldsymbol{W}^\nu-\partial^\nu\boldsymbol{W}^\mu-g\boldsymbol{W}^\mu\times\boldsymbol{W}^\nu]\\ &= -g\boldsymbol{\epsilon}(x)\times\boldsymbol{F}^{\mu\nu}\end{aligned}$$

as required. (Note that the above manipulations are still valid when the fields are quantized, since no non-commuting components are involved.)

13.5 We have

$$\begin{aligned}\delta(\boldsymbol{F}_{\mu\nu}\cdot\boldsymbol{F}^{\mu\nu}) &= (\delta\boldsymbol{F}_{\mu\nu})\cdot\boldsymbol{F}^{\mu\nu}+\boldsymbol{F}_{\mu\nu}\cdot(\delta\boldsymbol{F}^{\mu\nu})\\ &= -g(\boldsymbol{\epsilon}(x)\times\boldsymbol{F}_{\mu\nu})\cdot\boldsymbol{F}^{\mu\nu}+\boldsymbol{F}_{\mu\nu}\cdot(-g\boldsymbol{\epsilon}(x)\times\boldsymbol{F}^{\mu\nu})\\ &= \boldsymbol{F}_{\mu\nu}\cdot(-g\boldsymbol{F}^{\mu\nu}\times\boldsymbol{\epsilon}(x))+\boldsymbol{F}_{\mu\nu}\cdot(-g\boldsymbol{\epsilon}(x)\times\boldsymbol{F}^{\mu\nu})\end{aligned}$$

which vanishes, as required.

13.6 Using (13.87) and (13.88), (13.89) is

$$(\partial^\mu+\mathrm{i}g_\mathrm{s}\boldsymbol{\lambda}/2\cdot\boldsymbol{A}'^\mu)(1+\mathrm{i}g_\mathrm{s}\boldsymbol{\lambda}\cdot\boldsymbol{\eta}(x)/2)\psi=(1+\mathrm{i}g_\mathrm{s}\boldsymbol{\lambda}\cdot\boldsymbol{\eta}(x)/2)(\partial^\mu+\mathrm{i}g_\mathrm{s}\boldsymbol{\lambda}/2\cdot\boldsymbol{A}^\mu)\psi$$

from which we have to extract $\boldsymbol{A}'^\mu$ in terms of $\boldsymbol{A}^\mu$. Let us write

$$\boldsymbol{A}'^\mu=\boldsymbol{A}^\mu+\delta\boldsymbol{A}^\mu.$$

Then expanding the previous equation, and neglecting terms of second order (remembering that $\delta\boldsymbol{A}^\mu$ is itself first order), we have

$$\begin{aligned}&\partial^\mu\psi+\frac{\mathrm{i}g_\mathrm{s}}{2}(\boldsymbol{\lambda}\cdot\partial^\mu\boldsymbol{\eta}(x))\psi+\frac{\mathrm{i}g_\mathrm{s}}{2}(\boldsymbol{\lambda}\cdot\boldsymbol{\eta}(x))\partial^\mu\psi\\ &+\frac{\mathrm{i}g_\mathrm{s}}{2}(\boldsymbol{\lambda}\cdot\boldsymbol{A}^\mu)\psi+(\frac{\mathrm{i}g_\mathrm{s}}{2})^2(\boldsymbol{\lambda}\cdot\boldsymbol{A}^\mu)(\boldsymbol{\lambda}\cdot\boldsymbol{\eta}(x))\psi+\frac{\mathrm{i}g_\mathrm{s}}{2}(\boldsymbol{\lambda}\cdot\delta\boldsymbol{A}^\mu)\psi\\ &=\partial^\mu\psi+\frac{\mathrm{i}g_\mathrm{s}}{2}(\boldsymbol{\lambda}\cdot\boldsymbol{A}^\mu)\psi+\frac{\mathrm{i}g_\mathrm{s}}{2}(\boldsymbol{\lambda}\cdot\boldsymbol{\eta}(x))\partial^\mu\psi+(\frac{\mathrm{i}g_\mathrm{s}}{2})^2(\boldsymbol{\lambda}\cdot\boldsymbol{\eta}(x))(\boldsymbol{\lambda}\cdot\boldsymbol{A}^\mu)\psi\end{aligned}$$

After some cancellations we are left with

$$(\frac{\lambda_a}{2}\delta A_a^\mu)\psi = -(\frac{\lambda_a}{2}\partial^\mu \eta_a(x))\psi + \mathrm{i}g_\mathrm{s}[(\frac{\lambda_b}{2}\eta_b(x))(\frac{\lambda_c}{2}A_c^\mu) - (\frac{\lambda_c}{2}A_c^\mu)(\frac{\lambda_b}{2}\eta_b(x)]\psi.$$

The last term here is

$$\mathrm{i}g_\mathrm{s}[\frac{\lambda_b}{2}, \frac{\lambda_c}{2}]\eta_b(x)A_c^\mu\psi = \mathrm{i}g_\mathrm{s}\mathrm{i}f_{bca}\frac{\lambda_a}{2}\eta_b(x)A_c^\mu\psi = -g_\mathrm{s}\frac{\lambda_a}{2}f_{abc}\eta_b(x)A_c^\mu\psi$$

and so we have

$$(\frac{\lambda_a}{2}\delta A_a^\mu)\psi = -(\frac{\lambda_a}{2}\partial^\mu \eta_a(x))\psi - g_\mathrm{s}(\frac{\lambda_a}{2}f_{abc}\eta_b(x)A_c^\mu)\psi$$

which is satisfied if (13.90) holds. (We have checked (13.90) for the special case of the covariant derivative (13.87) acting on an SU(3) triplet, but (13.90) is in fact the general infinitesimal gauge transformation for the octet of gauge fields A_a^μ.)

13.7 We need to evaluate

$$(\partial^\mu + \frac{\mathrm{i}g_\mathrm{s}}{2}\boldsymbol{\lambda}\cdot\boldsymbol{A}^\mu)(\partial^\nu\psi + \frac{\mathrm{i}g_\mathrm{s}}{2}\boldsymbol{\lambda}\cdot\boldsymbol{A}^\nu\psi) - (\partial^\nu + \frac{\mathrm{i}g_\mathrm{s}}{2}\boldsymbol{\lambda}\cdot\boldsymbol{A}^\nu)(\partial^\mu\psi + \frac{\mathrm{i}g_\mathrm{s}}{2}\boldsymbol{\lambda}\cdot\boldsymbol{A}^\mu\psi).$$

After cancellations we are left with

$$[D^\mu, D^\nu]\psi = \mathrm{i}g_\mathrm{s}\boldsymbol{\lambda}/2\cdot(\partial^\mu\boldsymbol{A}^\nu - \partial^\nu\boldsymbol{A}^\mu)\psi + (\mathrm{i}g_\mathrm{s})^2[(\frac{\boldsymbol{\lambda}}{2}\cdot\boldsymbol{A}^\mu)(\frac{\boldsymbol{\lambda}}{2}\cdot\boldsymbol{A}^\nu) - (\frac{\boldsymbol{\lambda}}{2}\cdot\boldsymbol{A}^\nu)(\frac{\boldsymbol{\lambda}}{2}\cdot\boldsymbol{A}^\mu)]\psi.$$

The second term is

$$(\mathrm{i}g_\mathrm{s})^2[\frac{\lambda_b}{2}, \frac{\lambda_c}{2}]A_b^\mu A_c^\nu\psi = (\mathrm{i}g_\mathrm{s})^2\mathrm{i}f_{abc}\frac{\lambda_a}{2}A_b^\mu A_c^\nu\psi.$$

Hence

$$[D^\mu, D^\nu]\psi = \mathrm{i}g_\mathrm{s}\frac{\lambda_a}{2}F_a^{\mu\nu}\psi$$

(c.f. (13.82)) where $F_a^{\mu\nu}$ is as given in (13.95).

13.8 The ingoing $\bar{\mathrm{X}}$ with four-momentum k_2 in figure 13.9 is equivalent to an outgoing X with four-momentum $-k_2$; we take the polarization vectors to be real. Then (13.118) is (*note that* $\bar{\mathrm{X}}^*$ *should be replaced by* X^*)

$$-\mathrm{i}(-\mathrm{i}e)\int \epsilon_{2\mu}(\mathrm{e}^{+\mathrm{i}k_2\cdot x})^*\{\partial^\nu(\epsilon_{3\nu}\mathrm{e}^{-\mathrm{i}k_3\cdot x}\epsilon_1^\mu\mathrm{e}^{-\mathrm{i}k_1\cdot x}) + \epsilon_3^\nu\mathrm{e}^{-\mathrm{i}k_3\cdot x}\partial_\nu\epsilon_1^\mu\mathrm{e}^{-\mathrm{i}k_1\cdot x}\}\,\mathrm{d}^4x$$

The first term becomes, after a partial integration,

$$-e\int \epsilon_{2\mu}(-\partial^\nu \mathrm{e}^{-\mathrm{i}k_2\cdot x})\{\epsilon_{3\nu}\epsilon_1^\mu \mathrm{e}^{-\mathrm{i}(k_1+k_3)\cdot x}\}\,\mathrm{d}^4x$$
$$-e\int(\epsilon_{2\mu}\epsilon_1^\mu)(\mathrm{i}k_2^\nu\epsilon_{3\nu})\mathrm{e}^{-\mathrm{i}(k_1+k_2+k_3)\cdot x}\,\mathrm{d}^4x,$$

while the second term is

$$-e\int(\epsilon_{2\mu}\epsilon_1^\mu)(-\epsilon_3^\nu\mathrm{i}k_{1\nu})\mathrm{e}^{-\mathrm{i}(k_1+k_2+k_3)\cdot x}\,\mathrm{d}^4x.$$

The integrations give a factor $(2\pi)^4\delta^4(k_1+k_2+k_3)$ expressing four-momentum conservation, and the remaining part of the amplitude is

$$\mathrm{i}e(\epsilon_1\cdot\epsilon_2)(k_1-k_2)\cdot\epsilon_3$$

as in (13.120).

13.9 Replacing ϵ_1 by k_1 in (13.127) means that $\not\epsilon_1 d(p_1)$ is replaced by $\not k_1 d(p_1)$. Now note that $\not k_1 = \not k_2 + \not p_2 - \not p_1$ by four-momentum conservation, and that $\not p_1 d(p_1) = md(p_1)$ so that

$$\not k_1 d(p_1) = (\not k_2 + \not p_2 - m)d(p_1) = (\not k_1 + \not p_1 - m)d(p_1).$$

Hence after this replacement (13.127) becomes

$$(-\mathrm{i}e)^2\frac{\mathrm{i}}{2}\bar{d}(p_2)\not\epsilon_2 d(p_1)$$

as in (13.128).

13.10 When ϵ_1 is replaced by k_1 in (13.129), the expression in the first square bracket becomes, with $\delta=1$,

$$[k_1\cdot\epsilon_2(k_1+k_2)_\mu + \epsilon_{2\mu}k_1\cdot(-2k_2+k_1) + k_{1\mu}\epsilon_2\cdot(k_2-2k_1)]$$
$$= [k_1\cdot\epsilon_2(k_1+k_2)_\mu - 2\epsilon_{2\mu}k_1\cdot k_2 - 2k_{1\mu}\epsilon_2\cdot k_1]$$
$$= [k_1\cdot\epsilon_2(k_2-k_1)_\mu - 2\epsilon_{2\mu}k_1\cdot k_2]$$

using $k_1^2=0$ and $\epsilon_2\cdot k_2=0$. (13.129) then becomes

$$\frac{-\mathrm{i}e^2}{q^2}[k_1\cdot\epsilon_2\bar{d}(p_2)(-\frac{1}{2})(\not k_2-\not k_1)d(p_1) - 2k_1\cdot k_2\bar{d}(p_2)(-\frac{1}{2})\not\epsilon_2 d(p_1)].$$

In the first term here, note that

$$\bar{d}(p_2)(\not{k}_2 - \not{k}_1)d(p_1) = \bar{d}(p_2)(\not{p}_1 - \not{p}_2)d(p_1) = 0$$

using the Dirac equation for $d(p_1)$ and $\bar{d}(p_2)$. In the second term, note that $q^2 = -2k_1 \cdot k_2$, so that the whole expression reduces to

$$\frac{\mathrm{i}e^2}{2}\bar{d}(p_2)\not{\epsilon}_2 d(p_1)$$

which cancels (13.128).

13.11 $P_{\mu\nu}$ is a symmetric second-rank tensor. We write the most general expression for such a tensor, depending on the available quantities:

$$P_{\mu\nu} = Ag_{\mu\nu} + Bk_\mu k_\nu + C(k_\mu t_\nu + k_\nu t_\mu) + Dt_\mu t_\nu,$$

and determine the coefficients by imposing the conditions

$$k^\mu P_{\mu\nu} = 0 \quad \text{and} \quad t^\mu P_{\mu\nu} = 0$$

which follow from (13.157) (second equation) and (13.158). We also note that

$$g^{\mu\nu}P_{\mu\nu} = \sum_{\lambda=1,2} \epsilon^\mu(k,\lambda)\epsilon_\mu(k,\lambda) = -2$$

using the first equation in (13.157). We therefore have

$$0 = k^\mu P_{\mu\nu} = [A + C(k\cdot t)]k_\nu + D(k\cdot t)t_\nu$$

using $k^2 = 0$. Since t_μ and k_μ are independent we can infer that the coefficients of each must vanish:

$$D = 0 \quad \text{(i)}$$

and

$$A + C(k\cdot t) = 0. \quad \text{(ii)}$$

Similarly,

$$0 = t^\mu P_{\mu\nu} = [B(k\cdot t) + Ct^2]k_\nu$$

using (i) and (ii), and hence

$$B(k \cdot t) + Ct^2 = 0.$$

Finally,

$$-2 = g^{\mu\nu} P_{\mu\nu} = 4A + 2C(k \cdot t).$$

Solving for A, B, C and D gives

$$A = -1, B = -t^2/(k \cdot t)^2, C = 1/(k \cdot t), D = 0$$

as in (13.159).

Chapter 14

14.1 (a) Let's write the transformation (14.14) as

$$\psi'_\alpha = W_{\alpha\rho}\psi_\rho$$

where α and ρ run over the three colour indices (which we will call just 1, 2 and 3) and where as usual the repeated index ρ is summed over. We have a similar equation for each of ψ_α, ψ_β and ψ_γ in (14.2), and so the result of applying an SU(3) transformation to (14.2) is

$$\epsilon_{\alpha\beta\gamma}\psi'_\alpha\psi'_\beta\psi'_\gamma = \epsilon_{\alpha\beta\gamma}W_{\alpha\rho}W_{\beta\sigma}W_{\gamma\tau}\psi_\rho\psi_\sigma\psi_\tau.$$

Consider the product $\epsilon_{\alpha\beta\gamma}W_{\alpha\rho}W_{\beta\sigma}W_{\gamma\tau}$ for fixed ρ, σ and τ. It is summed over all values of α, β and γ, but there are only 6 non-zero values of $\epsilon_{\alpha\beta\gamma}$, namely $\epsilon_{123} = +1 = \epsilon_{231} = \epsilon_{312}$, $\epsilon_{213} = -1 = \epsilon_{132} = \epsilon_{321}$. Hence

$$\begin{aligned}\epsilon_{\alpha\beta\gamma}W_{\alpha\rho}W_{\beta\sigma}W_{\gamma\tau} &= W_{1\rho}W_{2\sigma}W_{3\tau} - W_{1\rho}W_{3\sigma}W_{2\tau} \\ &+W_{2\rho}W_{3\sigma}W_{1\tau} - W_{2\rho}W_{1\sigma}W_{3\tau} + W_{3\rho}W_{1\sigma}W_{2\tau} - W_{3\rho}W_{2\sigma}W_{1\tau}.\end{aligned}$$

Now the right hand side of this equation can be written as the determinant

$$\begin{vmatrix} W_{1\rho} & W_{1\sigma} & W_{1\tau} \\ W_{2\rho} & W_{2\sigma} & W_{2\tau} \\ W_{3\rho} & W_{3\sigma} & W_{3\tau} \end{vmatrix}.$$

In this form it is clear that $\epsilon_{\alpha\beta\gamma}W_{\alpha\rho}W_{\beta\sigma}W_{\gamma\tau}$ has the following properties: (a) it is non-vanishing only if ρ,σ and τ are all different; (b) it is antisymmetric in ρ,σ and τ; (c) when $\rho=1,\sigma=2$ and $\tau=3$ it has the value $\det W$. It follows that it is equal to $\epsilon_{\rho\sigma\tau}\det W$. But by definition $\det W=1$, and hence

$$\epsilon_{\alpha\beta\gamma}\psi'_\alpha\psi'_\beta\psi'_\gamma=\epsilon_{\rho\sigma\tau}\psi_\rho\psi_\sigma\psi_\tau$$

which (remembering that summed-over indices can be re-labelled as we please) confirms that the antisymmetric combination (14.2) is invariant under an SU(3) transformation.
(b) We shall calculate $Q'_1=p'_2q'_3-p'_3q'_2$ where (using the explicit forms for the $\boldsymbol{\lambda}$ matrices)

$$p'_2=p_2+\frac{\mathrm{i}}{2}(\eta_1+\mathrm{i}\eta_2)p_1+\frac{\mathrm{i}}{2}(-\eta_3+\frac{1}{\sqrt{3}}\eta_8)p_2$$

and

$$p'_3=p_3+\frac{\mathrm{i}}{2}\frac{-2}{\sqrt{3}}\eta_8p_3,$$

and similarly for q'_2 and q'_3. Inserting these expressions into $p'_2q'_3-p'_3q'_2$ we find

$$p'_2q'_3-p'_3q'_2=p_2q_3-p_3q_2-\frac{\mathrm{i}}{2}(\eta_1+\mathrm{i}\eta_2)(p_3q_1-p_1q_3)-\frac{\mathrm{i}}{2}(\eta_3+\frac{1}{\sqrt{3}}\eta_8)(p_2q_3-p_3q_2).$$

Setting $Q_1=(p_2q_3-p_3q_2)$ and $Q_2=(p_3q_1-p_1q_3)$ we recognize this as the first component of the transformation law

$$Q'=(1-\mathrm{i}\boldsymbol{\eta}\cdot\boldsymbol{\lambda}^*/2)Q$$

as required.

14.2 As usual, the repeated indices μ,ν,ρ and σ are all summed over, each taking the values 0, 1, 2, 3. The ϵ symbol is only non-zero if all its indices are different, when (see below) it has the value +1 or -1. The first index (μ) may be chosen arbitrarily i.e. in 4 ways, then the next in 3 ways, the next in 2, leaving the fourth fixed, which gives 24 different non-zero ϵ symbols. Consider the 6 in which '0' is the first index; these contribute

$$\epsilon_{0123}\hat{F}^{01}\hat{F}^{23}+\epsilon_{0132}\hat{F}^{01}\hat{F}^{32}+\epsilon_{0213}\hat{F}^{02}\hat{F}^{13}$$
$$\epsilon_{0231}\hat{F}^{02}\hat{F}^{31}+\epsilon_{0312}\hat{F}^{03}\hat{F}^{12}+\epsilon_{0321}\hat{F}^{03}\hat{F}^{21}.$$

ϵ_{0123} is defined to be $+1$, and in general $\epsilon_{\mu\nu\rho\sigma}$ is equal to $+1$ for all choices of indices which can be reached from '0123' by an even number of pairwise interchanges; it has the value -1 for those choices which are reached from '0123' by an odd number of interchanges. Thus $\epsilon_{0231}=+1$, $\epsilon_{0321}=-1$, etc. We also have, for example,

$$\hat{F}^{01}=\partial^0\hat{A}^1-\partial^1\hat{A}^0=\frac{\partial\hat{A}^1}{\partial t}+\frac{\partial\hat{A}^0}{\partial x^1}=-\hat{E}^1$$

using (3.10), and

$$\hat{F}^{23}=-\hat{B}^1,$$

so that

$$\epsilon_{0123}\hat{F}^{01}\hat{F}^{23}=\hat{E}^1\hat{B}^1.$$

On the other hand, since both the ϵ symbol and $\hat{F}^{\mu\nu}$ are antisymmetric under interchange of any pair of indices, it follows easily that

$$\epsilon_{0132}\hat{F}^{01}\hat{F}^{32}=\epsilon_{0123}\hat{F}^{01}\hat{F}^{23}=\hat{E}^1\hat{B}^1$$

also, so that these two terms give $2\hat{E}^1\hat{B}^1$. Evaluating the other four terms in which $\mu=0$ (or arguing from symmetry) we find that these six terms contribute $2\hat{\boldsymbol{E}}\cdot\hat{\boldsymbol{B}}$.

Now consider the six terms in which $\mu=1$, for example the term

$$\epsilon_{1023}\hat{F}^{10}\hat{F}^{23}.$$

Using the antisymmetry properties this is equal to

$$-\epsilon_{0123}(-\hat{F}^{01})\hat{F}^{23}=\hat{E}^1\hat{B}^1,$$

and it is clear that this set of six terms also gives $2\hat{\boldsymbol{E}}\cdot\hat{\boldsymbol{B}}$. Similarly, the contributions from the terms in which the first index is '2' and '3' are also all equal to this quantity, giving $8\hat{\boldsymbol{E}}\cdot\hat{\boldsymbol{B}}$ altogether.

14.3 The amplitude for the exchange of a massless scalar particle between two fermions is

$$\bar{u}(p_2')u(p_2)\frac{1}{(p_1-p_1')^2}\bar{u}(p_1')u(p_1).$$

The unpolarized cross section will involve the usual sums and averages over the fermion spins, which will lead to the familiar trace expression

$$\mathrm{Tr}[u(p_1')\bar{u}(p_1')u(p_1)\bar{u}(p_1)] = \mathrm{Tr}[\not{p}_1' \not{p}_1] = 4p_1 \cdot p_1'$$

in the massless limit. Similarly the spin sums for the other fermion yield a factor $4p_2 \cdot p_2'$. On the other hand, $\hat{t} = (p_1 - p_1')^2 = -2p_1 \cdot p_1' = -2p_2 \cdot p_2'$, also in the massless limit. Hence the numerator factors from the spin sums cancel the $1/\hat{t}^2$ factor from the exchange propagator.

14.4 (i) The equality we have to show is in fact the (s,l) matrix element of the matrix product

$$\chi(r)\chi^\dagger(r) + \chi(b)\chi^\dagger(b) + \chi(g)\chi^\dagger(g) \qquad (*)$$

on the left hand side, and of the unit 3×3 matrix on the right hand side. Evaluating $(*)$ using the given colour wavefunctions we find

$$\begin{aligned}
\sum_{c_2}\chi(c_2)\chi^\dagger(c_2) &= \begin{pmatrix}1\\0\\0\end{pmatrix}(100) + \begin{pmatrix}0\\1\\0\end{pmatrix}(010) + \begin{pmatrix}0\\0\\1\end{pmatrix}(001) \\
&= \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} \\
&= \mathbf{I}
\end{aligned}$$

as required.

(ii) In (14.72), since the matrix indices have all been indicated explicitly, quantities such as $a_d^*(c_3)$ and $\chi_m(c_1)$ for example are just numbers and can be freely moved about in the expression (compare the remarks following (8.62)). Thus (14.72) can be re-written as

$$\frac{1}{4}[\sum_{c_3} a_d^*(c_3)a_c(c_3)]\,[\sum_{c_2}\chi_s(c_2)\chi_l^\dagger(c_2)]\,[\sum_{c_1}\chi_m(c_1)\chi_r^\dagger(c_1)](\lambda_c)_{rs}(\lambda_d)_{lm}$$

$$= \frac{1}{4}\delta_{cd}\delta_{sl}\delta_{rm}(\lambda_c)_{rs}(\lambda_d)_{lm}$$
$$= \frac{1}{4}(\lambda_c)_{rl}(\lambda_c)_{lr}$$
$$= \frac{1}{4}\mathrm{Tr}(\lambda_c\lambda_c)$$

(summed over c), as required.
(iii) It is straightforward to check that all the λ matrices $\lambda_c (c = 1, 2, \ldots 8)$ satisfy

$$\mathrm{Tr}(\lambda_c)^2 = 2 \quad \text{no sum on } c.$$

Summing the eight (equal) contributions then gives 16 as required.

14.5 First note that the effect of the discussion after (13.127) is to assure us that for gluons, just as for photons (see the discussion in section 8.6.3 following equation (8.166)), polarization sums can be performed using

$$\sum_\lambda \epsilon^\sigma(p,\lambda)\epsilon^{*\rho}(p,\lambda) = -g^{\sigma\rho}. \quad (**)$$

The quantity we have to evaluate is then

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins},\,\lambda}|\mathcal{M}|^2 \quad = \quad & \frac{e_{\mathrm{a}}^2 e^4 g_{\mathrm{s}}^2}{Q^4}\frac{1}{4}\sum_{\text{spins},\,\lambda}\bar{v}(k_2)\gamma^\mu u(k_1)[\bar{u}(p_1)\gamma_\rho\frac{(\not p_1 + \not p_3)}{2p_1\cdot p_3}\gamma_\mu v(p_2) \\
& -\bar{u}(p_1)\gamma_\mu\frac{(\not p_2 + \not p_3)}{2p_2\cdot p_3}\gamma_\rho v(p_2)]\epsilon^{*\rho}(\lambda)\epsilon^\sigma(\lambda) \\
& [-\bar{v}(p_2)\gamma_\sigma\frac{(\not p_2 + \not p_3)}{2p_2\cdot p_3}\gamma_\nu u(p_1) \\
& +\bar{v}(p_2)\gamma_\nu\frac{(\not p_1 + \not p_3)}{2p_1\cdot p_3}\gamma_\sigma u(p_1)]\bar{u}(k_1)\gamma^\nu v(k_2)
\end{aligned}$$

after doing the usual tricks with the dagger operation and γ^0 as in section 8.2.3. The quantity $\bar{u}(k_1)\gamma^\nu v(k_2)$ is a number (not a matrix) and can be moved through the other factors and placed next to $\bar{v}(k_2)\gamma^\mu u(k_1)$. The expression

$$\sum_{\text{spins of } \mathrm{e}^+ \text{ and } \mathrm{e}^-}\frac{1}{2}\bar{v}(k_2)\gamma^\mu u(k_1)\bar{u}(k_1)\gamma^\nu v(k_2)$$

then yields, in the massless limit,

$$\begin{aligned}&\frac{1}{2}\mathrm{Tr}[\not{k}_2\gamma^\mu \not{k}_1\gamma^\nu]\\&=2[k_1^\mu k_2^\nu+k_1^\nu k_2^\mu-k_1\cdot k_2 g^{\mu\nu}]\end{aligned}$$

which we call $L^{\mu\nu}(k_1,k_2)$ in (14.56). Apart from the overall factor $e_a^2e^4g_s^2/Q^4$, the remaining part of the expression comes from multiplying out the two [.....] brackets, which involve the quark wavefunctions and will produce the hadron tensor $H_{\mu\nu}$, and making use also of $(**)$. There are four terms arising from the product of the square brackets. If, for the moment, we write the square brackets in shorthand notation as [(13)-(23)][-(23)+(13)], we see that the four terms split into two of the form (13)(13) and (23)(23), and two of the form -(13)(23) and -(23)(13). Consider then the '(13)(13)' term, which is

$$\begin{aligned}&\sum_{\text{quark spins}}\frac{1}{2}[\bar{u}(p_1)\gamma_\rho\frac{(\not{p}_1+\not{p}_3)}{2p_1\cdot p_3}\gamma_\mu v(p_2)\bar{v}(p_2)\gamma_\nu\frac{(\not{p}_1+\not{p}_3)}{2p_1\cdot p_3}\gamma_\sigma u(p_1)](-g^{\rho\sigma})\\&=-\frac{1}{2}\mathrm{Tr}[\gamma^\rho\not{p}_1\gamma_\rho\frac{(\not{p}_1+\not{p}_3)}{2p_1\cdot p_3}\gamma_\mu\not{p}_2\gamma_\nu\frac{(\not{p}_1+\not{p}_3)}{2p_1\cdot p_3}].\end{aligned}$$

We use (J.3) from Appendix J of volume 1, and the cyclic property of the Trace, to reduce this to

$$\frac{1}{4(p_1\cdot p_3)^2}\mathrm{Tr}[(\not{p}_1+\not{p}_3)\not{p}_1(\not{p}_1+\not{p}_3)\gamma_\mu\not{p}_2\gamma_\nu].$$

Since $(\not{p}_1)^2=(\not{p}_3)^2=0$ in the massless limit, this becomes

$$\frac{1}{4(p_1\cdot p_3)^2}\mathrm{Tr}[\not{p}_3\not{p}_1\not{p}_3\gamma_\mu\not{p}_2\gamma_\nu].$$

Using J.6 and $(\not{p}_3)^2=0$ this reduces to

$$\frac{1}{(p_1\cdot p_3)}\frac{1}{2}\mathrm{Tr}[\not{p}_3\gamma_\mu\not{p}_2\gamma_\nu]=\frac{1}{(p_1\cdot p_3)}L_{\mu\nu}(p_2,p_3).$$

The '(23)(23)' term is similar, and is

$$\begin{aligned}&\sum_{\text{quark spins}}\frac{1}{2}[\bar{u}(p_1)\gamma_\mu\frac{(\not{p}_2+\not{p}_3)}{2p_2\cdot p_3}\gamma_\rho v(p_2)\bar{v}(p_2)\gamma_\sigma\frac{(\not{p}_2+\not{p}_3)}{2p_2\cdot p_3}\gamma_\nu u(p_1)](-g^{\rho\sigma})\\&=-\frac{1}{2}\mathrm{Tr}[\not{p}_1\gamma_\mu\frac{(\not{p}_2+\not{p}_3)}{2p_2\cdot p_3}\gamma_\rho\not{p}_2\gamma_\rho\frac{(\not{p}_2+\not{p}_3)}{2p_2\cdot p_3}\gamma_\nu],\end{aligned}$$

which reduces, following steps similar to those carried out for the first term, to

$$\frac{1}{(p_2 \cdot p_3)} L_{\mu\nu}(p_1, p_3).$$

We turn now to the '-(13)(23)' term which is

$$\sum_{\text{quark spins}} -\frac{1}{2}[\bar{u}(p_1)\gamma_\rho \frac{(\not{p}_1 + \not{p}_3)}{2p_1 \cdot p_3}\gamma_\mu v(p_2)\bar{v}(p_2)\gamma_\sigma \frac{(\not{p}_2 + \not{p}_3)}{2p_2 \cdot p_3}\gamma_\nu u(p_1)](-g^{\rho\sigma})$$

$$= \frac{1}{2}\mathrm{Tr}[\not{p}_1\gamma_\rho \frac{(\not{p}_1 + \not{p}_3)}{2p_1 \cdot p_3}\gamma_\mu \not{p}_2\gamma^\rho \frac{(\not{p}_2 + \not{p}_3)}{2p_2 \cdot p_3}\gamma_\nu].$$

We use (J.5) to get rid of two γ-matrices via

$$\gamma_\rho(\not{p}_1 + \not{p}_3)\gamma_\mu \not{p}_2\gamma^\rho = -2\not{p}_2\gamma_\mu(\not{p}_1 + \not{p}_3)$$

but this still leaves a Trace involving 6 γ-matrices. For this we need a result which is not given in Appendix J of volume 1, but is a generalization of the theorem given there for a Trace involving four γ-matrices, namely (J. 31). The right hand side of (J.31) can be understood in a way that leads to the required generalization. Let's write the left hand side as $\mathrm{Tr}(\not{a}_1\not{a}_2\not{a}_3\not{a}_4)$. Then the right hand side is $4[(a_1 \cdot a_2)(a_3 \cdot a_4) - (a_1 \cdot a_3)(a_2 \cdot a_4) + (a_1 \cdot a_4)(a_2 \cdot a_3)]$. Consider the way the four indices 1, 2, 3 and 4 appear here: in the first term they are '1234', in the second '1324' and in the third '1423'. The second is an odd permutation of '1234', while the third is an even permutation (two pairwise interchanges are required to return it to '1234'.) The generalization to the Trace of the product of six such 'slash' factors is

$$\begin{aligned}\mathrm{Tr}(\not{a}_1\not{a}_2\not{a}_3\not{a}_4\not{a}_5\not{a}_6) \quad = \quad & 4[(12)(34)(56) - (12)(35)(46) + (12)(36)(45) \\ & -(13)(24)(56) - (13)(26)(45) + (13)(25)(46) \\ & \ldots + (16)(23)(45) - (16)(24)(35) + (16)(25)(34)]\end{aligned}$$

where '(12)(34)(56)', for example, means $(a_1 \cdot a_2)(a_3 \cdot a_4)(a_5 \cdot a_6)$. The signs here are '+' if the ordering of the indices is an even permutation of 123456 (for example (14)(26)(35)) and '-' if it is an odd permutation (such as (14)(25)(36)). Applying this result to our expression for the '-(13)(23)' term (omitting the factor $1/[(p_1 \cdot p_3)(p_2 \cdot p_3)]$) gives

$$-\frac{1}{4}\mathrm{Tr}[\not{p}_1\not{p}_2\gamma_\mu(\not{p}_1 + \not{p}_3)(\not{p}_2 + \not{p}_3)\gamma_\nu] =$$

$$-[p_1{\cdot}p_2(p_1{+}p_3)_\mu(p_2{+}p_3)_\nu{-}p_1{\cdot}p_2(p_2{+}p_3)_\mu(p_1{+}p_3)_\nu{+}p_1{\cdot}p_2(p_1{+}p_3){\cdot}(p_2{+}p_3)g_{\mu\nu}$$
$$-p_{1\mu}p_2{\cdot}(p_1{+}p_3)(p_2{+}p_3)_\nu{-}p_{1\mu}p_{2\nu}(p_1{+}p_3){\cdot}(p_2{+}p_3){+}p_{1\mu}p_2{\cdot}(p_2{+}p_3)(p_1{+}p_3)_\nu$$
$$+p_1{\cdot}(p_1{+}p_3)p_{2\mu}(p_2{+}p_3)_\nu{-}p_1{\cdot}(p_1{+}p_3)p_2{\cdot}(p_2{+}p_3)g_{\mu\nu}{+}p_1{\cdot}(p_1{+}p_3)p_{2\nu}(p_2{+}p_3)_\mu$$
$$-p_1{\cdot}(p_2{+}p_3)p_{2\mu}(p_1{+}p_3)_\nu{+}p_1{\cdot}(p_1{+}p_3)p_2{\cdot}(p_1{+}p_3)g_{\mu\nu}{-}p_1{\cdot}(p_2{+}p_3)p_{2\nu}(p_1{+}p_3)_\mu$$
$$+p_{1\nu}p_{2\mu}(p_1{+}p_3){\cdot}(p_2{+}p_3){-}p_{1\nu}p_2{\cdot}(p_1{+}p_3)(p_2{+}p_3)_\mu{+}p_{1\nu}p_2{\cdot}(p_2{+}p_3)(p_1{+}p_3)_\mu].$$

It is straightforward to verify (and expected from symmetry) that the fourth term involves the same expression but with p_1 and p_2 interchanged. Adding all these terms together, and simplifying the algebra we find, for the contribution of the 'third' and 'fourth' terms to $H_{\mu\nu}$, the result

$$\frac{-2}{(p_1\cdot p_3)(p_2\cdot p_3)}[2g_{\mu\nu}\{(p_1\cdot p_2)^2+(p_1\cdot p_2)(p_1\cdot p_3)+(p_1\cdot p_2)(p_2\cdot p_3)\}$$
$$+2p_{1\mu}p_{1\nu}(p_2\cdot p_3)+2p_{2\mu}p_{2\nu}(p_1\cdot p_3)$$
$$-\{(p_2\cdot p_3)+(p_1\cdot p_3)\}(p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu})$$
$$-(p_1\cdot p_2)\{(p_{2\mu}p_{3\nu}+p_{2\nu}p_{3\mu})+(p_{1\mu}p_{3\nu}+p_{1\nu}p_{3\mu})+2(p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu})\}],$$

which after some further algebra can be written in terms of the $L_{\mu\nu}$ tensors as

$$\frac{(p_1\cdot p_2)}{(p_1\cdot p_3)(p_2\cdot p_3)}[2L_{\mu\nu}(p_1,p_2)+L_{\mu\nu}(p_2,p_3)+L_{\mu\nu}(p_1,p_3)]$$
$$+\frac{1}{(p_2\cdot p_3)}[-L_{\mu\nu}(p_2,p_2)+L_{\mu\nu}(p_1,p_2)]$$
$$+\frac{1}{(p_1\cdot p_3)}[-L_{\mu\nu}(p_1,p_1)+L_{\mu\nu}(p_1,p_2)].$$

When the contributions from the 'first' and 'second' terms are included, we arrive at (14.57). [*Note that the factor* '8' *in (14.55) should be omitted, and that in the first term on the right hand side of (14.57) the factor* '$\frac{1}{p_1\cdot p_2}$' *should be* '$\frac{1}{p_1\cdot p_3}$'.]

14.6 In the CMS of particles 2 and 3 let us write p_1, p_2 and p_3 as

$$\begin{aligned} p_1 &= (q,\, \boldsymbol{q}) \ \text{ where } q = |\boldsymbol{q}|, \\ p_2 &= (p, -\boldsymbol{p}) \ \text{ where } p = |\boldsymbol{p}|, \\ p_3 &= (p,\, \boldsymbol{p}), \end{aligned}$$

where as usual all particles are massless. Then we have

$$s_{23} = (p_2 + p_3)^2 = 4p^2 = Q^2(1 - x_1)$$

$$s_{31} = (p_3 + p_1)^2 = 2qp(1 - \cos\tilde{\theta}) = Q^2(1 - x_2)$$

and

$$s_{12} = (p_1 + p_2)^2 = 2pq(1 + \cos\tilde{\theta}) = Q^2(1 - x_3).$$

It follows that

$$s_{31} + s_{12} = 4qp = Q^2 - s_{23}$$

using (14.60). Hence

$$Q^2(x_2 - x_3) = (Q^2 - s_{23})\cos\tilde{\theta}$$

which implies

$$x_2 - x_3 = x_1 \cos\tilde{\theta}.$$

Combining this with $x_3 = 2 - x_1 - x_2$ leads to (14.67).

14.7 [*Note that* 'expression (14.63)' *in the question should read* 'expression (14.62)'.] We have

$$\begin{aligned} &\frac{s_{13}}{s_{23}} + \frac{s_{23}}{s_{13}} + \frac{2Q^2 s_{12}}{s_{23}s_{13}} \\ &= \frac{(1-x_2)}{(1-x_1)} + \frac{(1-x_1)}{(1-x_2)} + \frac{2(1-x_3)}{(1-x_1)(1-x_2)} \\ &= \frac{(1-x_1)^2 + (1-x_2)^2 + 2[1-(2-x_1-x_2)]}{(1-x_1)(1-x_2)} \\ &= \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \end{aligned}$$

as required.

Chapter 15

15.1 We have

$$\Pi_\gamma^{[2]}(q^2) = 8e^2\mathrm{i}\int_0^1 \mathrm{d}x\, x(1-x)\int \frac{\mathrm{d}^4k'}{(2\pi)^4}\frac{1}{(k'^2-\Delta_\gamma+\mathrm{i}\epsilon)^2}.$$

We note that this is very similar to the quantity $\Pi_\mathrm{C}^{[2]}(q^2)$ calculated in section 10.3.1: the x-integral contains an additional function $x(1-x)$; Δ is replaced by Δ_γ; and there is an overall factor $(8e^2/g^2)$. Equation (15.11) then follows directly from equation (10.51), noting the misprint in the latter.

15.2 Equation (15.27) is

$$\frac{\mathrm{d}e_\mu}{e_\mu^3} = \frac{\mathrm{d}\mu}{12\pi^2\mu}$$

which integrates to

$$-\frac{1}{2e_\mu^2} = \frac{1}{12\pi^2}\ln\mu + c$$

where the constant c is determined from the condition that $e_\mu = e_M$ when $\mu = M$. This leads to (15.28).

15.3 We have

$$\begin{aligned}
\alpha_\mathrm{s}(|q^2|) &= \frac{\alpha_\mathrm{s}(\mu^2)}{1+\alpha_\mathrm{s}(\mu^2)b\ln(|q^2|/\mu^2)} \\
&= \frac{1}{b(\frac{1}{b\alpha_\mathrm{s}(\mu^2)} - \ln\mu^2 + \ln(|q^2|))} \\
&\equiv \frac{1}{b(-\ln\Lambda_\mathrm{QCD}^2 + \ln(|q^2|))}
\end{aligned}$$

from which (15.54) follows.

15.4 We give two ways of approaching this. First, consider equation (15.42). Near the zero of $\beta(\alpha_\mathrm{s})$ at $\alpha_\mathrm{s} = \alpha_\mathrm{s}^*$ shown in figure 15.7(b), we can write

$$\beta(\alpha_\mathrm{s}) \approx A(\alpha_\mathrm{s}(|q^2|) - \alpha_\mathrm{s}^*)$$

where $A > 0$. Thus near α_s^* equation (15.42) becomes

$$\frac{\partial \alpha_s(|q^2|)}{\partial \ln(|q^2|)} \approx A(\alpha_s(|q^2|) - \alpha_s^*). \qquad (1)$$

The denominator of the left hand side of (1) is negative, as $q^2 \to 0$. Consider a value of q^2 such that $\alpha_s(|q^2|) > \alpha_s^*$; the right hand side of (1) is then positive, so that the change in $\alpha_s(|q^2|)$ as q^2 moves towards zero will be negative, tending to pull $\alpha_s(|q^2|)$ towards α_s^*. On the other hand, if we start from a value of q^2 such that $\alpha_s(|q^2|) < \alpha_s^*$, the right hand side of (1) will be negative, and so $\alpha_s(|q^2|)$ will increase as q^2 goes to zero, and will get pulled towards α_s^*. In either case, $\alpha_s(|q^2|)$ ends up at α_s^* as $q^2 \to 0$.

A second approach is to look at an explicit function β with the behaviour shown in figure 15.7(b), for example

$$\beta(\alpha_s) = B\alpha_s(\alpha_s - \alpha_s^*).$$

where $B > 0$. Note that then, for small $\alpha_s, \beta \approx -B\alpha_s^*\alpha_s$, which is negative, while for $\alpha_s \approx \alpha_s^*$, $\beta \approx B\alpha_s^*(\alpha_s - \alpha_s^*)$, which has a positive slope. For definiteness we choose $\alpha_\mu > \alpha_s^*$; a similar analysis can be done for $\alpha_\mu < \alpha_s^*$. Integrating equation (15.40) we obtain

$$B\alpha_s^* \ln(|q^2|/\mu^2) = \ln\left(\frac{\alpha_s(|q^2|) - \alpha_s^*}{\alpha_s(|q^2|)}\,\frac{\alpha_\mu}{\alpha_\mu - \alpha_s^*}\right).$$

It follows that as $q^2 \to 0$, we must have $\alpha_s(|q^2|) \to \alpha_s^*$.

15.5 The gauge non-invariant part of (11.18) involves the integral

$$\int \frac{\mathrm{d}^4 k'}{(2\pi)^4}\left[\frac{2k'_\mu k'_\nu}{(k'^2 - \Delta_\gamma + \mathrm{i}\epsilon)^2} - \frac{g_{\mu\nu}}{(k'^2 - \Delta_\gamma + \mathrm{i}\epsilon)}\right].$$

The first term is evaluated by using equation (N.19) with $d = 4 - \epsilon$, giving

$$2g_{\mu\nu}\left[\frac{\mathrm{i}(-1)\Delta_\gamma}{(4\pi)^2}\,\frac{1}{2}\,\frac{\Gamma(-1+\epsilon/2)}{\Gamma(2)}\right].$$

The second term is evaluated using (N.9) and (N.12) giving

$$-g_{\mu\nu}\left[\frac{\mathrm{i}(-1)}{(4\pi)^2}\,\frac{\Gamma(-1+\epsilon/2)}{\Gamma(1)}\,\Delta_\gamma\right].$$

Since $\Gamma(1) = \Gamma(2) = 1$, these two contributions cancel.

15.6 We develop the right hand side of (15.76) using (15.78) and $x^\epsilon \approx 1 + \epsilon \ln x + \ldots$:

$$\frac{1}{(4\pi)^{(2-\epsilon/2)}} \frac{\Gamma(\epsilon/2)}{\Gamma(2)} \left(\frac{1}{\Delta_\gamma}\right)^{\epsilon/2} = \frac{1}{(4\pi)^2}(4\pi)^{\epsilon/2}\Gamma(\epsilon/2)(\Delta_\gamma)^{-\epsilon/2}$$

$$\begin{aligned}
&= \frac{1}{(4\pi)^2}(1 + \epsilon/2 \ln(4\pi) + \ldots)\left(\frac{2}{\epsilon} - \gamma + \ldots\right)(1 - \epsilon/2 \ln(\Delta_\gamma) + \ldots) \\
&= \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ldots\right)(1 + \epsilon/2 \ln(4\pi) - \epsilon/2 \ln(\Delta_\gamma) + \ldots) \\
&= \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} + \ln(4\pi) - \ln(\Delta_\gamma) - \gamma + \ldots\right)
\end{aligned}$$

as required.

15.7 The dimensions of the fields can be determined from the usual requirement that the Action (in d dimensions) is dimensionless. This is easily done from the kinetic terms. Consider a bosonic field ϕ, for example, with a kinetic term

$$\int \mathrm{d}^d x\, \partial_\mu \phi \partial^\mu \phi.$$

Let d_ϕ be the mass dimension of ϕ; then that of $\partial_\mu \phi \partial^\mu \phi$ is $2d_\phi + 2$, while that of the integration measure is $-d$. It follows that we must have

$$d_\phi = \frac{(d-2)}{2}.$$

Similarly, for a fermionic field ψ with kinetic term $\bar{\psi} \mathrm{i} \not{\partial} \psi$, we find

$$d_\psi = \frac{(d-1)}{2}.$$

Now consider the QED interaction

$$e \int \bar{\psi} \not{A} \psi \, \mathrm{d}^d x.$$

If the mass dimension of e is d_e, for this to be dimensionless requires

$$d_e + 2(d-1)/2 + d/2 - 1 - d = 0,$$

and hence

$$d_e = (4-d)/2 = \epsilon/2.$$

15.8 We can illustrate the effect of changing the renormalization scheme from MS to $\overline{\rm MS}$ by considering the QED case. We first evaluate $\Pi_\gamma^{[2]}(q^2)$ using dimensional regularization. From (15.74), (15.75), and (15.79) we obtain

$$\Pi_\gamma^{[2]}(q^2) = -b_{\rm QED}\, \alpha \left[\frac{2}{\epsilon} - \gamma + \ln 4\pi - \ln \Delta_\gamma\right]$$

where (cf (15.37)) $b_{\rm QED} = 1/3\pi$. We want to consider the effect of carrying out different subtractions on $\Pi_\gamma^{[2]}(q^2)$. Referring to (11.29), (11.30) and (11.31), we see that the subtraction effected in (11.32) - which is the same as (15.13) - is directly related to the choice of $Z_3^{[2]}$. Thus if we define

$$Z_3^{[2]}(\rm MS) = 1 - b_{\rm QED}\, \alpha 2/\epsilon$$

then we shall be subtracting from $\Pi_\gamma^{[2]}(q^2)$ just the singular part, proportional to $2/\epsilon$. On the other hand, if we define

$$Z_3^{[2]}(\overline{\rm MS}) = 1 - b_{\rm QED}\, \alpha(2/\epsilon - \gamma + \ln 4\pi)$$

then we shall also be subtracting the extra constant term $b\alpha(\gamma - \ln 4\pi)$ from $\Pi_\gamma^{[2]}(q^2)$. In the MS scheme, then, we relate bare and renormalized α's by

$$\alpha({\rm MS}) = \left(1 - b_{\rm QED}\, \alpha \left[\frac{2}{\epsilon}\right]\right) \alpha_0$$

while in the $\overline{\rm MS}$ scheme we have

$$\alpha(\overline{\rm MS}) = \left(1 - b_{\rm QED}\, \alpha \left[\frac{2}{\epsilon} - \gamma + \ln 4\pi\right]\right) \alpha_0.$$

It follows that

$$\alpha(\overline{\rm MS}) = [1 + b_{\rm QED}\, \alpha(\gamma - \ln 4\pi)]\alpha({\rm MS}),$$

where in fact the 'α' inside the bracket may consistently be replaced by α(MS) to this order. Consider now QCD: the essential difference is that the β function is now negative for small values of its argument (compare (15.37) and (15.49)), which effectively means that the 'b' parameter for QCD has the opposite sign from that in QED (compare for example (15.45) with (15.51)). Hence we have

$$\alpha_{\rm s}(\overline{\rm MS}) = [1 + b\, \alpha_{\rm s}({\rm MS})(\ln 4\pi - \gamma)]\alpha_{\rm s}({\rm MS}),$$

where b is given by (15.50) and is the same b as in (15.81) and (15.82). Hence, referring to (15.80), we have

$$c_1 = b \ln(4\pi - \gamma)$$

and (15.83) (*correcting* '$4/\pi$' *to* '4π') follows from (15.82).

15.9 The parton-level process for (15.91) is (virtual) photon + quark $\to$ gluon + quark, whereas in the calculation of problem 14.4 it is (virtual) photon $\to$ gluon + quark + antiquark. In forming the cross section, we average over initial state colours, and sum over final state colours. Hence for we need a factor of 1/3 in forming the average over the colours for the initial state quark, as compared with the simple sum over the clours of the final state antiquark.

15.10 Using (15.109) and (15.116), we need to evaluate

$$\begin{aligned}
\gamma^n_{\mathrm{qq}} &= -\frac{16}{3}\int_0^1 \mathrm{d}z \left(\frac{z^{n-1}-1}{z-1}\right)(1+z^2) \\
&= -\frac{16}{3}\int_0^1 \mathrm{d}z \left[(z^{n-2}+z^{n-3}+\ldots 1)+(z^n+z^{n-1}+\ldots z^2\right] \\
&= -\frac{16}{3}\left[\frac{z^{n+1}}{n+1}+\frac{z^n}{n}+2\left(\frac{z^{n-1}}{n-1}+\frac{z^{n-2}}{n-2}+\ldots\frac{z^3}{3}\right)+\frac{z^2}{2}+z\right]_0^1 \\
&= -\frac{16}{3}\left[\frac{1}{n+1}-\frac{1}{n}+2\left(\frac{1}{n}+\frac{1}{n-1}+\ldots\frac{1}{3}+\frac{1}{2}\right)+\frac{1}{2}\right]
\end{aligned}$$

which reduces to (15.117).

Chapter 16

16.1 Let's write

$$\frac{1}{N_1}\sum_{n_1=0}^{N_1-1} \mathrm{e}^{2\pi \mathrm{i} n_1(\nu_1-\nu_2)/N_1} = \frac{1}{N_1}\sum_{n_1=0}^{N_1-1} \mathrm{e}^{\mathrm{i} n_1 \delta}$$

where

$$\delta = 2\pi(\nu_1-\nu_2)/N_1.$$

The summation is a simple GP, whose sum is

$$\frac{1 - \mathrm{e}^{\mathrm{i}\delta N_1}}{1 - \mathrm{e}^{\mathrm{i}\delta}}. \quad \text{(S)}$$

Now

$$N_1 \delta = 2\pi(\nu_1 - \nu_2).$$

Consider the case $\nu_1 \neq \nu_2$: then $\nu_1 - \nu_2$ is an integer, so that the numerator of (S) vanishes, and moreover the maximum value of $\nu_1 - \nu_2$ is $N_1 - 1$, so that the denominator does not vanish. Hence (S) is zero for $\nu_1 \neq \nu_2$. On the other hand, when $\nu_1 = \nu_2$, we have $\delta = 0$, and all the N_1 terms in the summation are equal to 1, and (bearing in mind the normalizing factor $1/N_1$) the result is 1, thus verifying (16.9).

16.2 We have

$$\begin{aligned}
&\frac{1}{2}\int \mathrm{d}x\, m^2 \phi^2(x) \\
&= \frac{1}{2} m^2 a \sum_{n_1} \sum_{\nu_1,\nu_2} \frac{1}{N_1 a} \mathrm{e}^{2\pi \mathrm{i}\nu_1 n_1/N_1} \tilde{\phi}(\nu_1) \mathrm{e}^{2\pi \mathrm{i}\nu_2 n_1/N_1} \tilde{\phi}(\nu_2) \\
&= \frac{1}{2} m^2 \sum_{\nu_1 \nu_2} \left[\frac{1}{N_1} \sum_{n_1} \mathrm{e}^{2\pi n_1(\nu_1+\nu_2)/N_1} \right] \tilde{\phi}(\nu_1)\tilde{\phi}(\nu_2) \\
&= \frac{1}{2} m^2 \sum_{\nu_1,\nu_2} \delta_{\nu_1,-\nu_2} \tilde{\phi}(\nu_1)\tilde{\phi}(\nu_2) \quad \text{using (16.9)} \\
&= \frac{1}{2} m^2 \sum_{\nu_1} \tilde{\phi}(\nu_1)\tilde{\phi}(-\nu_1)
\end{aligned}$$

as in (16.10). In (16.13) the argument 'ν_1' has been replaced by the equivalent variable $k_{\nu_1} = 2\pi\nu_1/L$.

16.3 The mass term in (16.17) is essentially the same as in the bosonic case, equation (16.10), except that the complex conjugation in $\bar{\psi}$ will mean that the factor '$\exp[2\pi \mathrm{i}\nu_1 n_1/N_1]$' will be replaced by '$\exp[-2\pi \mathrm{i}\nu_1 n_1/N_1]$', and hence the δ function is δ_{ν_1,ν_2} rather than $\delta_{\nu_1,-\nu_2}$, so that $\tilde{\bar{\psi}}$ and $\tilde{\psi}$ are both evaluated at k_{ν_1} (*note that $\tilde{\psi}(-k_{\nu_1})$ in (16.19) should be replaced by $\tilde{\psi}(k_{\nu_1})$*).

Now consider the derivative term in (16.17), *noting first that the overall factor '$\frac{1}{2}$' should be replaced by '$\frac{a}{2}$'*. This term is

$$\frac{a}{2}\sum_{n_1}\frac{1}{a}\left[\sum_{\nu_1,\nu_2}\frac{1}{N_1 a}\mathrm{e}^{-2\pi \mathrm{i}\nu_1 n_1/N_1}\bar{\tilde{\psi}}(\nu_1)\gamma_1^{\mathrm{E}}\mathrm{e}^{2\pi \mathrm{i}\nu_2(n_1+1)/N_1}\tilde{\psi}(\nu_2)\right.$$
$$\left.-\frac{1}{N_1 a}\mathrm{e}^{-2\pi \mathrm{i}\nu_1(n_1+1)/N_1}\bar{\tilde{\psi}}(\nu_1)\gamma_1^{\mathrm{E}}\mathrm{e}^{2\pi \mathrm{i}\nu_2 n_2/N_1}\tilde{\psi}(\nu_2)\right]$$
$$=\frac{1}{2}\sum_{\nu_1,\nu_2}\frac{1}{N_1 a}\left[\sum_{n_1}\mathrm{e}^{2\pi \mathrm{i}(\nu_2-\nu_1)n_1/N_1}\right]\left\{\mathrm{e}^{2\pi \mathrm{i}\nu_2/N_1}\bar{\tilde{\psi}}(\nu_1)\gamma_1^{\mathrm{E}}\tilde{\psi}(\nu_2)-\mathrm{e}^{-2\pi \mathrm{i}\nu_1/N_1}\bar{\tilde{\psi}}(\nu_1)\gamma_1^{\mathrm{E}}\tilde{\psi}(\nu_2)\right\}$$
$$=\frac{1}{2}\sum_{\nu_1,\nu_2}\frac{1}{N_1 a}\delta_{\nu_1,\nu_2}\left\{\mathrm{e}^{2\pi \mathrm{i}\nu_2/N_1}\bar{\tilde{\psi}}(\nu_1)\gamma_1^{\mathrm{E}}\tilde{\psi}(\nu_2)-\mathrm{e}^{-2\pi \mathrm{i}\nu_1/N_1}\bar{\tilde{\psi}}(\nu_1)\gamma_1^{\mathrm{E}}\tilde{\psi}(\nu_2)\right\}$$
$$=\frac{1}{2a}\sum_{\nu_1}\left(\mathrm{e}^{2\pi \mathrm{i}\nu_1/N_1}-\mathrm{e}^{-2\pi \mathrm{i}\nu_1/N_1}\right)\bar{\tilde{\psi}}(\nu_1)\gamma_1^{\mathrm{E}}\tilde{\psi}(\nu_1)$$
$$=\frac{\mathrm{i}}{a}\sin(k_{\nu_1}a)\bar{\tilde{\psi}}(k_{\nu_1})\gamma_1^{\mathrm{E}}\tilde{\psi}(k_{\nu_1}),$$

as in (16.19) (with the same correction to '$\tilde{\psi}(k_{\nu_1})$'), noting that $k_{\nu_1}a = 2\pi\nu_1 a/(N_1 a) = 2\pi\nu_1/N_1$.

16.4 (16.25) gives

$$\mathcal{O}(x,y)=\phi^{\dagger}(x)\exp\left[\mathrm{i}e\int_y^x A(x')\,\mathrm{d}x'\right]\phi(y).$$

Placing this in (16.27) we obtain

$$\lim_{\mathrm{d}x\to 0}\left[\frac{\mathcal{O}(x,x+\mathrm{d}x)-\mathcal{O}(x,x)}{\mathrm{d}x}\right]$$
$$=\lim_{\mathrm{d}x\to 0}[\phi^{\dagger}(x)\exp\{\mathrm{i}e\int_{x+\mathrm{d}x}^x A(x')\,\mathrm{d}x'\}\phi(x+\mathrm{d}x)-\phi^{\dagger}(x)\phi(x)]/\mathrm{d}x$$
$$=\lim_{\mathrm{d}x\to 0}[\phi^{\dagger}(x)\exp\{-\mathrm{i}eA(x)\mathrm{d}x\}(\phi(x)+\frac{\mathrm{d}\phi}{\mathrm{d}x}\mathrm{d}x)-\phi^{\dagger}\phi]/\mathrm{d}x$$
$$=\lim_{\mathrm{d}x\to 0}[\phi^{\dagger}(x)(1-\mathrm{i}eA(x)\mathrm{d}x)(\phi+\frac{\mathrm{d}\phi}{\mathrm{d}x}\mathrm{d}x)-\phi^{\dagger}\phi]/\mathrm{d}x$$
$$=-\mathrm{i}e\phi^{\dagger}(x)A(x)\phi(x)+\phi^{\dagger}(x)\frac{\mathrm{d}\phi}{\mathrm{d}x},$$

as in (16.28).

16.5 We evaluate (16.59) as follows. We have

$$I^1 = \int \exp\left\{-\frac{m}{2\epsilon}[(q^2-q^1)^2+(q^1-q^i)^2]\right\} \mathrm{d}q^1.$$

The exponent is

$$\begin{aligned}
&-\frac{m}{2\epsilon}[(q^2)^2-2q^2q^1+(q^1)^2+(q^1)^2-2q^1q^i+(q^i)^2]\\
&=-\frac{m}{\epsilon}[(q^1)^2-q^1(q^2+q^i)+(q^2)^2/2+(q^i)^2/2]\\
&=-\frac{m}{\epsilon}[\{q^1-(q^2+q^i)/2\}^2+(q^2)^2/2+(q^i)^2/2-(q^2+q^i)^2/4]\\
&=-\frac{m}{\epsilon}[(q^{1\prime})^2+(q^2-q^i)^2/4]
\end{aligned}$$

where $q^{1\prime} = q^1 - (q^2+q^i)/2$. Hence, making the change of variable $q^1 \to q^{1\prime}$ in the integral, I^1 becomes

$$\begin{aligned}
I^1 &= \exp\left[\frac{-m}{4\epsilon}(q^2-q^i)^2\right]\int \exp[-m(q^{1\prime})^2/\epsilon]\,\mathrm{d}q^{1\prime}\\
&= \left(\frac{\pi\epsilon}{m}\right)^{\frac{1}{2}}\exp\left[\frac{-m}{4\epsilon}(q^2-q^i)^2\right]
\end{aligned}$$

using (16.54), and we have verified (16.60).
The evaluation of I^2 (see (16.61)) is similar. We have

$$I^2 = \int \exp\left\{-\frac{m}{4\epsilon}(q^2-q^i)^2-\frac{m}{2\epsilon}(q^3-q^2)^2\right\} \mathrm{d}q^2.$$

The exponent is

$$\begin{aligned}
&-\frac{m}{4\epsilon}[(q^2)^2-2q^2q^i+(q^i)^2]-\frac{m}{2\epsilon}[(q^3)^2-2q^3q^2+(q^2)^2]\\
&=-\frac{3m}{4\epsilon}(q^2)^2+\frac{m}{\epsilon}q^2(q^3+q^i/2)-\frac{m}{4\epsilon}(q^i)^2-\frac{m}{2\epsilon}(q^3)^2\\
&=-\frac{3m}{4\epsilon}\left\{(q^2)^2-\frac{4}{3}q^2(q^3-q^i/2)+\frac{1}{3}(q^i)^2+\frac{2}{3}(q^3)^2\right\}\\
&=-\frac{3m}{4\epsilon}\left\{[q^2-\frac{2}{3}(q^3+q^i/2)]^2+\frac{1}{3}(q^i)^2+\frac{2}{3}(q^3)^2-\frac{4}{9}(q^3+q^i/2)^2\right\}\\
&=-\frac{3m}{4\epsilon}\left\{[q^2-\frac{2}{3}(q^3+q^i/2)]^2+\frac{2}{9}(q^3-q^i)^2\right\}.
\end{aligned}$$

We change the variable of integration to $q^{2\prime} = q^2 - \frac{2}{3}(q^3+q^i/2)$ and use (16.54) to give (16.62).

16.6 The partition function is (cf (16.91))

$$Z_N = \sum_{\{s_n\}} \exp\left[\sum_{n=1}^{N-1} K_n s_n s_{n+1}\right],$$

where $K_n = J_n/k_{\rm B}T$. We need to evaluate the exponential for each configuration $\{s_n\}$ of the spin variables, and then add up the results for all the configurations. One configuration is of course $\{+ + + + +....+\}$, where this means that every s_n has the value +1. Another is $\{- + + + + + ...+\}$, in which $s_1 = -1$ and all the others equal +1. Since each spin can take one of two values (+1 or -1), and there are N spins, there are clearly 2^N configurations altogether.
Consider the evaluation of the exponential, which has exponent

$$K_1 s_1 s_2 + K_2 s_2 s_3 + K_3 s_3 s_4 + \ldots K_{N-1} s_{N-1} s_N.$$

In the configuration $\{+ + + + + + ...+\}$, this is clearly

$$K_1 + K_2 + K_3 + \ldots K_{N-1},$$

while in the configuration $\{- + + + + + ...+\}$ it is

$$-K_1 + K_2 + K_3 + \ldots K_{N-1}.$$

But the first of these is also the value of the exponent in the configuration $\{- - - - -....-\}$ in which all the spins have the value -1, since all the products $s_j s_{j+1}$ will then be equal to +1. Similarly, the configuration $\{+ - - - - - - - - \ldots -\}$ will lead to the same value of the exponent in the second case, because here also $s_1 s_2 = -1$, and all other products =+1. In short, each product of spin variables $s_j s_{j+1}$ takes the value +1 twice, once when both s_j and s_{j+1} equal +1, and once when they both equal -1; similarly, each product takes the value -1 twice, when one variable is +1 and the other is -1, and vice versa. Thus the possible values of the exponent are $K_1 + K_2 + K_3 + \ldots K_{N-1}$, $-K_1 + K_2 + K_3 + \ldots K_{N-1}$, $K_1 - K_2 + K_3 + \ldots K_{N-1}$, $-K_1 - K_2 + K_3 + \ldots K_{N-1}$, ..., $-K_1 - K_2 - K_3 - \ldots - K_{N-1}$. There are 2^{N-1} such values, each appearing twice (corresponding to the 2^N configurations).

Thus the result of doing the sum over the spin configurations is

$$
\begin{aligned}
Z_N = \; & 2\Big[\mathrm{e}^{K_1+K_2+K_3+\ldots+K_{N-1}K_N} \\
& +\mathrm{e}^{-K_1+K_2+K_3+\ldots+K_{N-1}K_N} \\
& +\mathrm{e}^{K_1-K_2+K_3+\ldots+K_{N-1}K_N} \\
& +\mathrm{e}^{-K_1-K_2+K_3+\ldots+K_{N-1}K_N} \\
& +\ldots+\mathrm{e}^{-K_1-K_2-K_3-\ldots-K_{N-1}K_N}\Big];
\end{aligned}
$$

that is, Z_N is a sum of exponentials, each of which contains one of the 2^{N-1} possible sign sequences $(++++++\ldots+), (-+++++ \ldots+), \ldots(--------\ldots-)$. A little thought (and perhaps the consideration of the case $N=4$) shows that the sum of exponentials is in fact the product of $N-1$ factors, as follows:

$$Z_N = 2\left(\mathrm{e}^{K_1}+\mathrm{e}^{-K_1}\right)\left(\mathrm{e}^{K_2}+\mathrm{e}^{-K_2}\right)\left(\mathrm{e}^{K_3}+\mathrm{e}^{-K_3}\right)\ldots\left(\mathrm{e}^{K_{N-1}}+\mathrm{e}^{-K_{N-1}}\right),$$

from which the result follows.
For the case in which all the J_n's are equal to J, we have

$$Z_N = 2\left(2\cosh K\right)^{N-1}$$

where $K = J/k_\mathrm{B}T$. The free energy F is defined in terms of the partition function Z by

$$F = -k_\mathrm{B}T\ln Z,$$

and the entropy S is defined in terms of the free energy F by

$$S = -\frac{\partial F}{\partial T} = k_\mathrm{B}\ln Z + \frac{k_\mathrm{B}T}{Z}\frac{\partial Z}{\partial T}.$$

Evaluating this in the present case with $Z = Z_N$ we find

$$S = k_\mathrm{B}\ln 2 + k_\mathrm{B}(N-1)\ln(2\cosh K) - k_\mathrm{B}K(N-1)\tanh K.$$

In the limit $T\to\infty$, we have $K\to 0, \cosh K\to 1$ and $\tanh K\to 0$; hence in this case $S\to Nk_\mathrm{B}\ln 2$. For $T\to 0$, we have $K\to\infty$, $2\cosh K\to \mathrm{e}^K$, and $\tanh K\to 1$. So in this limit $S\to k_\mathrm{B}\ln 2$.
The condition $1\ll p\ll N$ means that the site 'p' is surrounded on

both sides by many spins, and we don't have to worry about 'edge effects' at the beginning or end of the line of spins. The average value $\langle s_p s_{p+1}\rangle$ is defined by (compare (16.74) and (16.88))

$$\langle s_p s_{p+1}\rangle = \frac{1}{Z_N}\sum_{\{s_n\}} s_p s_{p+1}\exp\left[\sum_{n=1}^{N-1} K_n s_n s_{n+1}\right].$$

On the other hand, we have

$$\begin{aligned}\frac{\partial Z_N}{\partial K_p} &= \frac{\partial}{\partial K_p}\sum_{\{s_n\}}\exp\left[\sum_{n=1}^{N-1} K_n s_n s_{n+1}\right]\\ &= \sum_{\{s_n\}} s_p s_{p+1}\exp\left[\sum_{n=1}^{N-1} K_n s_n s_{n+1}\right]\end{aligned}$$

from which the stated result for $\langle s_p s_{p+1}\rangle$ follows.
In the same way, when we differentiate the exponent in Z_N with respect to $K_p, K_{p+1}, \ldots K_{p+j-1}$ (*note that 'K_{p+j}' should be replaced by 'K_{p+j-1}'*), we bring down the product of factors

$$(s_p s_{p+1})(s_{p+1}s_{p+2})(s_{p+2}s_{p+3})\ldots(s_{p+j-2}s_{p+j-1})(s_{p+j-1}s_{p+j}).$$

In this product the repeated pairs $s^2_{p+1}, s^2_{p+2}, \ldots s^2_{p+j-1}$ may all be replaced by $+1$, leaving $s_p s_{p+j}$ as required.
From the earlier part of the question, we have

$$Z_N = 2\prod_{n=1}^{N-1}(\cosh K_n),$$

and so

$$\frac{\partial Z_N}{\partial K_p} = Z_N \tanh K_p,$$

(since the differentiation replaces $\cosh K_p$ by $\sinh K_p$), and generally the j derivatives will give

$$Z_N \tanh K_p \, \tanh K_{p+1} \, \tanh K_{p+2} \ldots \tanh K_{p+j}.$$

Hence when all the J's are equal, we have

$$\langle s_p s_{p+j}\rangle = (\tanh K)^j$$

which can be written as

$$\mathrm{e}^{-ja/\xi}$$

where

$$\xi = -\frac{a}{\ln[\tanh K)]}.$$

When $T \to \infty$, we have $K \to 0$ and $\tanh K \to K$, so that $\xi \to \frac{a}{|\ln K|}$ which tends to zero logarithmically. When $T \to 0$, we have $K \to \infty$ and $\tanh K \to 1 - 2\mathrm{e}^{-2K}$ so that $\xi \to (a/2)\mathrm{e}^{2K}$, as in (16.108).

Chapter 17

17.1 We take the cube of volume Ω to have sides of length L, so that $\Omega = L^3$. Periodic boundary conditions in each of the three spatial directions imply that the components of $\boldsymbol{k}$ take discrete values $k_x = 2\pi n_x/L, k_y = 2\pi n_y/L, k_z = 2\pi n_z/L$ (cf equation (E.11) in volume 1, and the paragraph following equation (16.6)), so that we may write $\boldsymbol{k} = 2\pi\boldsymbol{n}/L$. The three-dimensional generalization of equation (E.12) is then

$$\frac{1}{\Omega}\int_\Omega \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{x}}\mathrm{d}^3\boldsymbol{x} = \frac{1}{\Omega}\int_\Omega \mathrm{e}^{-\mathrm{i}2\pi\boldsymbol{n}/L\cdot\boldsymbol{x}}\mathrm{e}^{\mathrm{i}2\pi\boldsymbol{n}'/L\cdot\boldsymbol{x}}\mathrm{d}^3\boldsymbol{x} = \delta_{\boldsymbol{n},\boldsymbol{n}'},$$

where $\boldsymbol{k}' = 2\pi\boldsymbol{n}'/L$; we could, however, just as well (and somewhat more conveniently) write the right hand side as $\delta_{\boldsymbol{k},\boldsymbol{k}'}$.

Consider the first term of (17.29), which comes from the first term in (17.25). Inserting (17.28) into the latter, we have

$$\begin{aligned}
&\frac{1}{2m}\int \mathrm{d}^3\boldsymbol{x}\,\boldsymbol{\nabla}\hat{\phi}^\dagger\cdot\boldsymbol{\nabla}\hat{\phi} \\
&= \frac{1}{2m}\int \mathrm{d}^3\boldsymbol{x}\frac{1}{\Omega^{\frac{1}{2}}}\sum_{\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}(-\mathrm{i}\boldsymbol{k})\cdot\frac{1}{\Omega^{\frac{1}{2}}}\sum_{\boldsymbol{k}'}\hat{a}_{\boldsymbol{k}'}(\mathrm{i}\boldsymbol{k}')\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{x}}.
\end{aligned}$$

The integral over $\boldsymbol{x}$ of the product of the two plane waves gives $\Omega\delta_{\boldsymbol{k},\boldsymbol{k}'}$, by the result of the preceding paragraph; the delta function then allows the sum over $\boldsymbol{k}'$ to be performed, leading to the result

$$\frac{1}{2m}\sum_{\boldsymbol{k}} \boldsymbol{k}^2 \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}},$$

which is the same as the first term in (17.29).
Putting (17.28) into the second term of (17.25) gives

$$\frac{1}{2\Omega^2}\int\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{y}\, v(|\boldsymbol{x}-\boldsymbol{y}|) \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}'_1,\boldsymbol{k}'_2} \hat{a}^\dagger_{\boldsymbol{k}_1}\hat{a}^\dagger_{\boldsymbol{k}_2}\hat{a}_{\boldsymbol{k}'_2}\hat{a}_{\boldsymbol{k}'_1} \times$$
$$\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{y}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}'_2\cdot\boldsymbol{y}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}'_1\cdot\boldsymbol{x}}.$$

We introduce relative and centre of mass coordinates via

$$\boldsymbol{r} = \boldsymbol{x}-\boldsymbol{y} \quad \text{and} \quad \boldsymbol{R} = (\boldsymbol{x}+\boldsymbol{y})/2$$

which imples

$$\boldsymbol{x} = \boldsymbol{R}+\boldsymbol{r}/2 \quad \text{and} \quad \boldsymbol{y} = \boldsymbol{R}-\boldsymbol{r}/2,$$

and

$$\int\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{y} \to \int \mathrm{d}^3\boldsymbol{R}\int \mathrm{d}^3\boldsymbol{r}.$$

Substituting for $\boldsymbol{x}$ and $\boldsymbol{y}$ in terms of $\boldsymbol{R}$ and $\boldsymbol{r}$, the exponentials become

$$\begin{aligned}&\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot(\boldsymbol{R}+\boldsymbol{r}/2)}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot(\boldsymbol{R}-\boldsymbol{r}/2)}\mathrm{e}^{\mathrm{i}\boldsymbol{k}'_2\cdot(\boldsymbol{R}-\boldsymbol{r}/2)}\mathrm{e}^{\mathrm{i}\boldsymbol{k}'_1\cdot(\boldsymbol{R}+\boldsymbol{r}/2)}\\ &= \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}'_2-\boldsymbol{k}'_1)\cdot\boldsymbol{R}}\mathrm{e}^{-\mathrm{i}[(\boldsymbol{k}_1-\boldsymbol{k}'_1)+(\boldsymbol{k}'_2-\boldsymbol{k}_2)]\cdot\boldsymbol{r}/2}.\end{aligned}$$

The integral over $\boldsymbol{R}$ gives $\Omega\Delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}'_1-\boldsymbol{k}'_2)$, enforcing momentum conservation $\boldsymbol{k}_1+\boldsymbol{k}_2 = \boldsymbol{k}'_1+\boldsymbol{k}'_2$. It follows that $\boldsymbol{k}'_2-\boldsymbol{k}_2 = \boldsymbol{k}_1-\boldsymbol{k}'_1$, so that the exponentials involving $\boldsymbol{r}$ become just $\exp[-\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}'_1)\cdot\boldsymbol{r}]$, and the integral over $\boldsymbol{r}$ yields $\bar{v}(|\boldsymbol{k}_1-\boldsymbol{k}'_1|)$, verifying the second term in (17.29) (*note that modulus signs are required in the argument of* $\bar{v}$ *in (17.32) and in (17.29), and that the exponent in (17.32) should have a negative sign*).

17.2 We need to separate out the zero-momentum terms in (17.29). In the first term of (17.29), which doesn't involve the potential, the contribution from $\boldsymbol{k}=0$ in the summation vanishes, since $\epsilon_{\boldsymbol{k}=0}=0$. Hence

'$\sum_{\boldsymbol{k}}$' can here be replaced by '$\sum'_{\boldsymbol{k}}$'. So we have accounted for the first term on the right hand side of (17.36).

Consider now the second term in (17.29), which involves four momentum labels $\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}'_1$ and $\boldsymbol{k}'_2$, subject to the condition $\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}'_1 + \boldsymbol{k}'_2$. The zero-momentum pieces are as follows. (a) We can have all four momenta equal to zero, which gives a contribution $(1/2\Omega)\bar{v}(0)N_0^2$. (b) We can have $\boldsymbol{k}_1 = \boldsymbol{k}'_1 = 0$, $\boldsymbol{k}_2 = \boldsymbol{k}'_2 \neq 0$, which contributes $(1/2\Omega)\bar{v}(0)N_0\hat{a}^\dagger_{\boldsymbol{k}_2}\hat{a}_{\boldsymbol{k}_2}$, summed over $\boldsymbol{k}_2 \neq 0$; and also a similar term with $\boldsymbol{k}_2 = \boldsymbol{k}'_2 = 0$, $\boldsymbol{k}_1 = \boldsymbol{k}'_1 \neq 0$, which contributes the same (the sum over $\boldsymbol{k}_1$ is of course equivalent to the sum over $\boldsymbol{k}_2$).

Thus so far, from (a) and (b), we have

$$
\begin{aligned}
&\frac{1}{2\Omega}\bar{v}(0)N_0^2 + \frac{1}{\Omega}N_0\sum_{\boldsymbol{k}}{}'\,\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} \\
&= \frac{1}{2\Omega}\bar{v}(0)\{[N_0 + \sum_{\boldsymbol{k}}{}'\,\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}]^2 - [\sum_{\boldsymbol{k}}{}'\,\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}]^2\} \\
&\approx \frac{1}{2\Omega}\bar{v}(0)N^2,
\end{aligned}
$$

where in the last step we have used

$$
N = \sum_{\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} = N_0 + \sum_{\boldsymbol{k}}{}'\,\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}},
$$

and have neglected the term with no N_0 factor. We have therefore derived the second term on the right hand side of (17.35).

A third zero-momentum contribution in the interaction term in (17.29) is (c), in which $\boldsymbol{k}_2 = \boldsymbol{k}'_1 = 0$, $\boldsymbol{k}_1 = \boldsymbol{k}'_2 \neq 0$. This contributes

$$
\frac{1}{2\Omega}\sum_{\boldsymbol{k}_1}{}'\,\bar{v}(|\boldsymbol{k}_1|)N_0\hat{a}^\dagger_{\boldsymbol{k}_1}\hat{a}_{\boldsymbol{k}_1}.
$$

There is also a similar term in which $\boldsymbol{k}_1 = \boldsymbol{k}'_2 = 0$, $\boldsymbol{k}_2 = \boldsymbol{k}'_1 \neq 0$, which contributes the same. Thus these two terms together give (changing the summation variables to $\boldsymbol{k}$)

$$
\frac{N_0}{\Omega}\sum_{\boldsymbol{k}}{}'\,\bar{v}(k)\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}
$$

which accounts for the second term on the right hand side of (17.36), after replacing N_0 by N.
Finally, there is the term (d) in which $\boldsymbol{k}_1 = \boldsymbol{k}_2 = 0$, $\boldsymbol{k}_1' = -\boldsymbol{k}_2' \neq 0$, which contributes

$$\frac{1}{2\Omega}\sum_{\boldsymbol{k}_2'}{}' \bar{v}(|\boldsymbol{k}_2'|)N_0 \hat{a}_{\boldsymbol{k}_2'}\hat{a}_{-\boldsymbol{k}_2'},$$

which is the fourth term on the right hand side of (17.35), after replacing N_0 by N (and changing the dummy summation variable); and there is the term (e) in which $\boldsymbol{k}_1' = \boldsymbol{k}_2' = 0$, $\boldsymbol{k}_1 = -\boldsymbol{k}_2 \neq 0$ which contributes

$$\frac{1}{2\Omega}\sum_{\boldsymbol{k}_1}{}' \bar{v}(|\boldsymbol{k}_1|)N_0 \hat{a}^\dagger_{\boldsymbol{k}_1}\hat{a}^\dagger_{-\boldsymbol{k}_1},$$

which is the third term on the right hand side of (17.35), after replacing N_0 by N.

17.3 *Please note that on page 206 '$\bar{v}(\boldsymbol{l})$' should be replaced throughout by '$\bar{v}(l)$', where l is the modulus of $\boldsymbol{l}$.*
For the LHS of (17.42), we need to evaluate the commutator

$$[\sum_{\boldsymbol{k}}{}' E_k \hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \frac{1}{2}\sum_{\boldsymbol{k}}{}' \frac{N}{\Omega}\bar{v}(k)(\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}^\dagger_{-\boldsymbol{k}} + \hat{a}_{\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}),$$
$$\cosh\theta_l \hat{a}^\dagger_{\boldsymbol{l}} + \sinh\theta_l \hat{a}_{-\boldsymbol{l}}],$$

where we have omitted the constant term in (17.35) since this obviously commutes with all operators. Consider first the E_k terms. We have

$$[\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{l}}] = \hat{a}^\dagger_{\boldsymbol{k}}\delta_{\boldsymbol{k},\boldsymbol{l}},$$

and

$$[\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}, \hat{a}_{-\boldsymbol{l}}] = -\hat{a}_{\boldsymbol{k}}\delta_{\boldsymbol{k},-\boldsymbol{l}},$$

so that the E_k terms are

$$E_l \cosh\theta_l \hat{a}^\dagger_{\boldsymbol{l}} - E_l \sinh\theta_l \hat{a}_{-\boldsymbol{l}}.$$

For the terms multiplying N/Ω we use

$$[\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}^\dagger_{-\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{l}}] = 0,$$

and

$$[\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}^\dagger_{-\boldsymbol{k}}, \hat{a}_{-\boldsymbol{l}}] = -\hat{a}^\dagger_{\boldsymbol{k}}\delta_{\boldsymbol{k},\boldsymbol{l}} - \hat{a}^\dagger_{-\boldsymbol{k}}\delta_{\boldsymbol{k},-\boldsymbol{l}},$$

which leads to a term

$$-\frac{N}{\Omega}\bar{v}(l)\sinh\theta_l\hat{a}^\dagger_{\boldsymbol{l}},$$

together with

$$[\hat{a}_{\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}, \hat{a}_{-\boldsymbol{l}}] = 0$$

and

$$[\hat{a}_{\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{l}}] = \hat{a}_{\boldsymbol{k}}\delta_{\boldsymbol{k},-\boldsymbol{l}} + \hat{a}_{-\boldsymbol{k}}\delta_{\boldsymbol{k},\boldsymbol{l}},$$

which lead to the term

$$\frac{N}{\Omega}\bar{v}(l)\cosh\theta_l\hat{a}_{-\boldsymbol{l}}.$$

Thus altogether the LHS of (17.42) is

$$(E_l\cosh\theta_l - \frac{N}{\Omega}\bar{v}(l)\sinh\theta_l)\hat{a}^\dagger_{\boldsymbol{l}} + (-E_l\sinh\theta_l + \frac{N}{\Omega}\bar{v}(l)\cosh\theta_l)\hat{a}_{-\boldsymbol{l}};$$

equating coefficients of $\hat{a}^\dagger_{\boldsymbol{l}}$ on both sides of (17.42) then gives (17.43), and of $\hat{a}_{-\boldsymbol{l}}$ gives (17.44).

17.4 (a) Let $\hat{A} = \frac{1}{2}\theta(\hat{a}^2 - \hat{a}^{\dagger 2})$; then $\hat{U}_\lambda = \exp[\lambda\hat{A}]$. Now note that $\hat{A}^\dagger = -\hat{A}$ and so $(\hat{U}_\lambda)^\dagger = \exp[-\lambda\hat{A}] = (\hat{U}_\lambda)^{-1}$ and $\hat{U}_\lambda$ is unitary.
(b) We have

$$\frac{\mathrm{d}\hat{I}_\lambda}{\mathrm{d}\lambda} = \frac{\mathrm{d}\hat{U}_\lambda}{\mathrm{d}\lambda}\hat{a}\hat{U}_\lambda^{-1} + \hat{U}_\lambda\hat{a}\frac{\mathrm{d}\hat{U}_\lambda^{-1}}{\mathrm{d}\lambda}.$$

For $\mathrm{d}\hat{U}_\lambda/\mathrm{d}\lambda$ we note that

$$\hat{U}_\lambda = \exp[\lambda\hat{A}] = 1 + \lambda\hat{A} + \frac{1}{2}\lambda^2\hat{A}^2 + \ldots$$

and so

$$\begin{aligned}\frac{\mathrm{d}\hat{U}_\lambda}{\mathrm{d}\lambda} &= \hat{A} + \lambda\hat{A}^2 + \frac{\lambda^2}{2}\hat{A}^3 + \ldots \\ &= \hat{A}\exp[\lambda\hat{A}] = \exp[\lambda\hat{A}]\,\hat{A}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\mathrm{d}\hat{U}_\lambda^{-1}}{\mathrm{d}\lambda} &= -\hat{A} + \lambda\hat{A}^2 - \frac{\lambda^2}{2}\hat{A}^3 + \ldots \\ &= -\hat{A}\exp[-\lambda\hat{A}] = -\exp[-\lambda\hat{A}]\,\hat{A}.\end{aligned}$$

Hence

$$\frac{\mathrm{d}\hat{I}_\lambda}{\mathrm{d}\lambda} = \exp[\lambda\hat{A}]\{\frac{1}{2}\theta[(\hat{a}^2 - \hat{a}^{\dagger 2}), \hat{a}]\}\exp[-\lambda\hat{A}].$$

The $[\hat{a}^2, \hat{a}]$ part of the commutator is zero, and the other part is

$$[\hat{a}, \hat{a}^{\dagger 2}] = 2\hat{a}^\dagger.$$

Hence

$$\frac{\mathrm{d}\hat{I}_\lambda}{\mathrm{d}\lambda} = \exp[\lambda\hat{A}]\frac{1}{2}\theta\,2\hat{a}^\dagger\exp[-\lambda\hat{A}] = \theta\hat{J}_\lambda.$$

Similarly,

$$\frac{\mathrm{d}\hat{J}_\lambda}{\mathrm{d}\lambda} = \exp[\lambda\hat{A}]\{\frac{1}{2}\theta[(\hat{a}^2 - \hat{a}^{\dagger 2}), \hat{a}^\dagger]\}\exp[-\lambda\hat{A}].$$

This time the commutator gives $2\hat{a}$, so that

$$\frac{\mathrm{d}\hat{J}_\lambda}{\mathrm{d}\lambda} = \theta\hat{I}_\lambda$$

and the second result now follows.
(c) The general solution of the second order differential equation for $\hat{I}_\lambda$ is

$$\hat{I}_\lambda = \cosh\theta\lambda\,\hat{X} + \sinh\theta\lambda\,\hat{Y}$$

where $\hat{X}$ and $\hat{Y}$ are operators, to be determined by the initial conditions for $\hat{I}_\lambda$ (i.e. the conditions at $\lambda = 0$). We have

$$\hat{I}_{\lambda=0} = \hat{U}_{\lambda=0}\,\hat{a}\,\hat{U}_{\lambda=0}^{-1}$$

and

$$\hat{U}_{\lambda=0} = 1 = \hat{U}^{-1}_{\lambda=0}.$$

Hence $\hat{X} = \hat{a}$. Similarly, by considering $\mathrm{d}\hat{I}_\lambda/\mathrm{d}\lambda$ at $\lambda = 0$ we deduce $\hat{Y} = \hat{a}^\dagger$, and the result follows by putting $\lambda = 1$ in the expression for $\hat{I}_\lambda$.

17.5 We have, for the kinetic terms,

$$\partial_\mu \hat{\phi} = \frac{1}{\sqrt{2}}(\partial_\mu \hat{h})\exp[-\mathrm{i}\hat{\theta}/v] + \frac{1}{\sqrt{2}}(v + \hat{h})\frac{-\mathrm{i}\partial_\mu\hat{\theta}}{v}\exp[-\mathrm{i}\hat{\theta}/v]$$

and

$$\partial_\mu \hat{\phi}^\dagger = \frac{1}{\sqrt{2}}(\partial_\mu \hat{h})\exp[\mathrm{i}\hat{\theta}/v] + \frac{1}{\sqrt{2}}(v + \hat{h})\frac{\mathrm{i}\partial_\mu\hat{\theta}}{v}\exp[\mathrm{i}\hat{\theta}/v],$$

and so the terms which are quadratic in the fields are

$$\partial_\mu \hat{\phi}^\dagger \partial^\mu \hat{\phi} = \frac{1}{2}\partial_\mu \hat{h}\partial^\mu \hat{h} + \frac{1}{2}\partial_\mu \hat{\theta}\partial^\mu \hat{\theta}.$$

The potential terms (up to quadratic powers in the field $\hat{h}$) are

$$\begin{aligned}
&-\frac{1}{4}\lambda\frac{1}{4}(v + \hat{h})^4 + \frac{\mu^2}{2}(v + \hat{h})^2 \\
&= -\frac{1}{16}(v^4 + 4v^3\hat{h} + 6v^2\hat{h}^2 + \ldots) + \frac{\mu^2}{2}v^2 + \mu^2 v\hat{h} + \frac{\mu^2}{2}\hat{h}^2 \\
&= -\frac{\lambda v^4}{16} - \frac{\lambda}{4}v^3\hat{h} - \frac{3}{8}\lambda v^2\hat{h}^2 + \frac{\mu^2}{2}v^2 + \mu^2 v\hat{h} + \frac{\mu^2}{2}\hat{h}^2 + \ldots
\end{aligned}$$

But from (17.80) we have $v = 2\mu/\lambda^{\frac{1}{2}}$. Substituting this into the preceding expression, we see that the term linear in $\hat{h}$ cancels, and the constant and quadratic terms are as given in (17.85).

17.6 *Please note that the exponent in (17.107) should be*

$$\exp[-\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{\tau}/(2v)].$$

The kinetic terms are derived as in the preceding problem, but because the exponent contains $\hat{\boldsymbol{\theta}}/2$ rather than $\hat{\theta}$ the $\hat{\boldsymbol{\theta}}$ term has a factor 1/8 rather than 1/2; the $\hat{H}$ term is the same as the $\hat{h}$ term in (17.85). The potential terms are the same as in (17.85), with $\hat{h} \to \hat{H}$, leading to (17.108).

17.7 We begin by checking that, for instance,

$$\{\hat{\beta}_{\boldsymbol{k}}, \hat{\beta}_{\boldsymbol{k}'}\} = 0.$$

Substituting from the first equality in (17.111) (*replacing* $c_{\boldsymbol{k}}$ *by* $\hat{c}_{\boldsymbol{k}}$), the LHS is

$$\{u_k\hat{c}_{\boldsymbol{k}} - v_k\hat{c}^\dagger_{-\boldsymbol{k}}, u_{k'}\hat{c}_{\boldsymbol{k}'} - v_{k'}\hat{c}^\dagger_{-\boldsymbol{k}'}\}.$$

The anticommutator of two $\hat{c}$'s or two $\hat{c}^\dagger$'s vanishes, and we are left with

$$-u_kv_{k'}\{\hat{c}_{\boldsymbol{k}}, \hat{c}^\dagger_{-\boldsymbol{k}'}\} - v_ku_{k'}\{\hat{c}^\dagger_{-\boldsymbol{k}}, \hat{c}_{\boldsymbol{k}'}\}.$$

The anti-commutators give zero unless $\boldsymbol{k} = -\boldsymbol{k}'$. In this case, we are dealing with $\{\hat{\beta}_{\boldsymbol{k}}, \hat{\beta}_{-\boldsymbol{k}}\}$, which can be evaluated as

$$\begin{aligned}&\{u_k\hat{c}_{\boldsymbol{k}} - v_k\hat{c}^\dagger_{-\boldsymbol{k}}, u_k\hat{c}_{-\boldsymbol{k}} + v_k\hat{c}^\dagger_{\boldsymbol{k}} \\ &= u_kv_k\{\hat{c}_{\boldsymbol{k}}, \hat{c}^\dagger_{\boldsymbol{k}}\} - u_kv_k\{\hat{c}^\dagger_{-\boldsymbol{k}}, \hat{c}_{-\boldsymbol{k}}\} = 0.\end{aligned}$$

Similarly, $\{\hat{\beta}^\dagger_{\boldsymbol{k}}, \hat{\beta}^\dagger_{\boldsymbol{k}'}\} = 0$. On the other hand,

$$\begin{aligned}\{\hat{\beta}_{\boldsymbol{k}}, \hat{\beta}^\dagger_{\boldsymbol{k}'}\} &= \{u_k\hat{c}_{\boldsymbol{k}} - v_k\hat{c}^\dagger_{\boldsymbol{k}}, u_{k'}\hat{c}^\dagger_{\boldsymbol{k}'} - v_{k'}\hat{c}_{\boldsymbol{k}'}\} \\ &= u_ku_{k'}\{\hat{c}_{\boldsymbol{k}}, \hat{c}^\dagger_{\boldsymbol{k}'}\} + v_kv_{k'}\{\hat{c}^\dagger_{\boldsymbol{k}}, \hat{c}_{\boldsymbol{k}'}\} \\ &= (u_k^2 + v_k^2)\delta_{\boldsymbol{k},\boldsymbol{k}'}\end{aligned}$$

which has the desired form if $u_k^2 + v_k^2 = 1$.

17.8 We have

$$\begin{aligned}[\hat{\beta}^\dagger_{\boldsymbol{k}}\hat{\beta}_{\boldsymbol{k}}, \hat{\beta}^\dagger_{\boldsymbol{l}}] &= \hat{\beta}^\dagger_{\boldsymbol{k}}\hat{\beta}_{\boldsymbol{k}}\hat{\beta}^\dagger_{\boldsymbol{l}} - \hat{\beta}^\dagger_{\boldsymbol{l}}\hat{\beta}^\dagger_{\boldsymbol{k}}\hat{\beta}_{\boldsymbol{k}} \\ &= \hat{\beta}^\dagger_{\boldsymbol{k}}\{\delta_{\boldsymbol{k},\boldsymbol{l}} - \hat{\beta}^\dagger_{\boldsymbol{l}}\hat{\beta}_{\boldsymbol{k}}\} - \hat{\beta}^\dagger_{\boldsymbol{l}}\hat{\beta}^\dagger_{\boldsymbol{k}}\hat{\beta}_{\boldsymbol{k}} \\ &= \hat{\beta}^\dagger_{\boldsymbol{k}}\delta_{\boldsymbol{k},\boldsymbol{l}} - (\hat{\beta}^\dagger_{\boldsymbol{k}}\hat{\beta}^\dagger_{\boldsymbol{l}} + \hat{\beta}^\dagger_{\boldsymbol{l}}\hat{\beta}^\dagger_{\boldsymbol{l}})\hat{\beta}_{\boldsymbol{k}} \\ &= \hat{\beta}^\dagger_{\boldsymbol{k}}\delta_{\boldsymbol{k},\boldsymbol{l}},\end{aligned}$$

which leads to the result.

17.9 Consider first the free kinetic (first) term on the RHS of (17.109). To evaluate its contribution to (17.117) we need

$$\begin{aligned}
&[\sum_{\boldsymbol{k}} \epsilon_k \hat{c}^\dagger_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}}, \hat{c}^\dagger_{\boldsymbol{l}}] \\
&= \sum_{\boldsymbol{k}} \epsilon_k (\hat{c}^\dagger_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}} \hat{c}^\dagger_{\boldsymbol{l}} - \hat{c}^\dagger_{\boldsymbol{l}} \hat{c}^\dagger_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}}) \\
&= \sum_{\boldsymbol{k}} \epsilon_k (\hat{c}^\dagger_{\boldsymbol{k}} \delta_{\boldsymbol{k},\boldsymbol{l}} - \{\hat{c}^\dagger_{\boldsymbol{k}} \hat{c}^\dagger_{\boldsymbol{l}} + \hat{c}^\dagger_{\boldsymbol{l}} \hat{c}^\dagger_{\boldsymbol{k}}\} \hat{c}_{\boldsymbol{k}}) \\
&= \epsilon_l \hat{c}^\dagger_{\boldsymbol{l}}.
\end{aligned}$$

Similarly, we find

$$[\sum_{\boldsymbol{k}} \epsilon_k \hat{c}^\dagger_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}}, \hat{c}_{-\boldsymbol{l}}] = -\epsilon_l \hat{c}_{-\boldsymbol{l}}.$$

Thus the contribution of the first term in $\hat{H}_{\mathrm{BCS}}$ to (11.117) is

$$\cos\theta_l \, \epsilon_l \hat{c}^\dagger_{\boldsymbol{l}} + \sin\theta_l \, \epsilon_l \hat{c}_{-\boldsymbol{l}}.$$

For the contribution of the second term in $\hat{H}_{\mathrm{BCS}}$ we need, first, to evaluate the commutator

$$\begin{aligned}
&\sum_{\boldsymbol{k},\boldsymbol{k}'} [\hat{c}^\dagger_{\boldsymbol{k}'} \hat{c}^\dagger_{-\boldsymbol{k}'} \hat{c}_{-\boldsymbol{k}} \hat{c}_{\boldsymbol{k}}, \hat{c}^\dagger_{\boldsymbol{l}}] \\
&= \sum_{\boldsymbol{k},\boldsymbol{k}'} \{\hat{c}^\dagger_{\boldsymbol{k}'} \hat{c}^\dagger_{-\boldsymbol{k}'} [\hat{c}_{-\boldsymbol{k}} \hat{c}_{\boldsymbol{k}}, \hat{c}^\dagger_{\boldsymbol{l}}] + [\hat{c}^\dagger_{\boldsymbol{k}'} \hat{c}^\dagger_{-\boldsymbol{k}'}, \hat{c}^\dagger_{\boldsymbol{l}}] \hat{c}_{-\boldsymbol{k}} \hat{c}_{\boldsymbol{k}}\}.
\end{aligned}$$

It is easy to see that the second commutator vanishes. In evaluating the first, we have to be careful as regards the meaning of the summations. Consider, for definiteness, the case in which $\boldsymbol{l}$ is set equal to the unit vector $\boldsymbol{i}$. In the sum over $\boldsymbol{k}$, there will appear one term with $\boldsymbol{k} = \boldsymbol{i}$, and this contributes

$$\begin{aligned}
&[\hat{c}_{-\boldsymbol{i}} \hat{c}_{\boldsymbol{i}}, \hat{c}^\dagger_{\boldsymbol{i}}] \\
&= \hat{c}_{-\boldsymbol{i}} \hat{c}_{\boldsymbol{i}} \hat{c}^\dagger_{\boldsymbol{i}} - \hat{c}^\dagger_{\boldsymbol{i}} \hat{c}_{-\boldsymbol{i}} \hat{c}_{\boldsymbol{i}} \\
&= \hat{c}_{-\boldsymbol{i}} (1 - \hat{c}^\dagger_{\boldsymbol{i}} \hat{c}_{\boldsymbol{i}}) - \hat{c}^\dagger_{\boldsymbol{i}} \hat{c}_{-\boldsymbol{i}} \hat{c}_{\boldsymbol{i}} \\
&= \hat{c}_{-\boldsymbol{i}} - (\hat{c}_{-\boldsymbol{i}} \hat{c}^\dagger_{\boldsymbol{i}} + \hat{c}^\dagger_{\boldsymbol{i}} \hat{c}_{-\boldsymbol{i}}) \hat{c}_{\boldsymbol{i}} \\
&= \hat{c}_{-\boldsymbol{i}}.
\end{aligned}$$

One might worry about another term, in which $\boldsymbol{k} = -\boldsymbol{i}$, but of course if both of these are included then the total is just $\hat{c}_{-\boldsymbol{i}}\hat{c}_{\boldsymbol{i}} + \hat{c}_{\boldsymbol{i}}\hat{c}_{-\boldsymbol{i}}$ which vanishes; so only the one contribution is required. In general, then, we find

$$\sum_{\boldsymbol{k},\boldsymbol{k}'} [\hat{c}^{\dagger}_{\boldsymbol{k}'}\hat{c}^{\dagger}_{-\boldsymbol{k}'}\hat{c}_{-\boldsymbol{k}}\hat{c}_{\boldsymbol{k}}, \hat{c}^{\dagger}_{\boldsymbol{l}}] = \sum_{\boldsymbol{k}',\boldsymbol{k}} \hat{c}^{\dagger}_{\boldsymbol{k}'}\hat{c}^{\dagger}_{-\boldsymbol{k}'}\hat{c}_{-\boldsymbol{k}}\delta_{\boldsymbol{k},\boldsymbol{l}} = (\sum_{\boldsymbol{k}'} \hat{c}^{\dagger}_{\boldsymbol{k}'}\hat{c}^{\dagger}_{-\boldsymbol{k}'})\hat{c}_{-\boldsymbol{l}};$$

there is no term involving $\delta_{-\boldsymbol{k},\boldsymbol{l}}$. The contribution from the commutator of $\cos\theta_l\,\hat{c}^{\dagger}_{\boldsymbol{l}}$ and the second term in (17.109) is thus (relabelling the summation)

$$-V\cos\theta_l \sum_{\boldsymbol{k}} \hat{c}^{\dagger}_{\boldsymbol{k}}\hat{c}^{\dagger}_{-\boldsymbol{k}}\,\hat{c}_{-\boldsymbol{l}}.$$

Similarly, the contribution from the commutator of $-\sin\theta_l\,\hat{c}_{-\boldsymbol{l}}$ and the second term of (17.109) is

$$+V\sin\theta_l \sum_{\boldsymbol{k}} \hat{c}_{-\boldsymbol{k}}\hat{c}_{\boldsymbol{k}}\,\hat{c}^{\dagger}_{\boldsymbol{l}}.$$

The LHS of (17.117) is therefore

$$[\cos\theta_l\epsilon_l + V\sin\theta_l \sum_{\boldsymbol{k}} \hat{c}_{-\boldsymbol{k}}\hat{c}_{\boldsymbol{k}}]\hat{c}^{\dagger}_{\boldsymbol{l}} + [\sin\theta_l\epsilon_l - V\cos\theta_l \sum_{\boldsymbol{k}} \hat{c}^{\dagger}_{\boldsymbol{k}}\hat{c}^{\dagger}_{-\boldsymbol{k}}]\hat{c}_{-\boldsymbol{l}},$$

and equating the coefficients of $\hat{c}^{\dagger}_{\boldsymbol{l}}$ on both sides of (17.117) leads to (17.118), while equating coefficients of $\hat{c}_{-\boldsymbol{l}}$ leads to (17.119).

Chapter 18

18.1 (18.24) is the same as (12.103) (compare (12.109) and (18.23)), the derivation of which was set as problem 12.11. We shall verify (18.26), using the same method as given in the solution to problem 12.11.

We have

$$\hat{Q}_{i,5} = \int \hat{q}^{\dagger}_{\alpha a}(x)(\gamma_5)_{\alpha\beta}(\tau_i/2)_{ab}\hat{q}_{\beta b}(x)\,\mathrm{d}^3\boldsymbol{x},$$

and

$$\hat{Q}_{j,5} = \int \hat{q}^\dagger_{\gamma c}(y)(\gamma_5)_{\gamma\delta}(\tau_j/2)_{cd}\hat{q}_{\delta d}(y)\,\mathrm{d}^3\boldsymbol{y}.$$

The commutator on the left hand side of (18.26)) is

$$\frac{1}{4}(\tau_i)_{ab}(\tau_j)_{cd}(\gamma_5)_{\alpha\beta}(\gamma_5)_{\gamma\delta}\int \mathrm{d}^3\boldsymbol{x}\int \mathrm{d}^3\boldsymbol{y}\,(\hat{q}^\dagger_{\alpha a}(x)\hat{q}_{\beta b}(x)\hat{q}^\dagger_{\gamma c}(y)\hat{q}_{\delta d}(y)$$
$$-\hat{q}^\dagger_{\gamma c}(y)\hat{q}_{\delta d}(y)\hat{q}^\dagger_{\alpha a}(x)\hat{q}_{\beta b}(x)).$$

The first product of four field operators may be written as (choosing $x_0 = y_0$)

$$\begin{aligned}\hat{q}^\dagger_{\alpha a}(x)\hat{q}_{\beta b}(x)\hat{q}^\dagger_{\gamma c}(y)\hat{q}_{\delta d}(y) &= \hat{q}^\dagger_{\alpha a}(x)[-\hat{q}^\dagger_{\gamma c}(y)\hat{q}_{\beta b}(x) + \delta_{\beta\gamma}\delta_{bc}\delta^3(\boldsymbol{x}-\boldsymbol{y})]\hat{q}_{\delta d}(y)\\ &= -\hat{q}^\dagger_{\gamma c}(y)\hat{q}^\dagger_{\alpha a}(x)\hat{q}_{\delta d}(y)\hat{q}_{\beta b}(x) + \hat{q}^\dagger_{\alpha a}(x)\hat{q}_{\delta d}(y)\delta_{\beta\gamma}\delta_{bc}\delta^3(\boldsymbol{x}-\boldsymbol{y}).\end{aligned}$$

Similarly, the second product of four field operators is

$$\hat{q}^\dagger_{\gamma c}(y)\hat{q}^\dagger_{\alpha a}(x)\hat{q}_{\delta d}(y)\hat{q}_{\beta b}(x) - \hat{q}^\dagger_{\gamma c}(y)\hat{q}_{\beta b}(x)\delta_{\delta\alpha}\delta_{ad}\delta^3(\boldsymbol{x}-\boldsymbol{y}).$$

The first term here cancels with the first term of the preceding expression, so that our commutator becomes

$$\frac{1}{4}(\tau_i)_{ab}(\tau_j)_{cd}(\gamma_5)_{\alpha\beta}(\gamma_5)_{\gamma\delta}\int\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{y}\,[\hat{q}^\dagger_{\alpha a}(x)\hat{q}_{\delta d}(y)\delta_{\beta\gamma}\delta_{bc} - \hat{q}^\dagger_{\gamma c}(y)\hat{q}_{\beta b}(x)\delta_{\delta\alpha}\delta_{ad}]\,\delta^3(\boldsymbol{x}-\boldsymbol{y})$$

where $x_0 = y_0$. Now note that

$$(\tau_i)_{ab}(\tau_j)_{cd}\delta_{bc} = (\tau_i)_{ab}(\tau_j)_{bd} = (\tau_i\tau_j)_{ad}$$

while

$$(\tau_i)_{ab}(\tau_j)_{cd}\delta_{ad} = (\tau_j\tau_i)_{cb}.$$

Similarly

$$(\gamma_5)_{\alpha\beta}(\gamma_5)_{\gamma\delta}\delta_{\beta\gamma} = (\gamma_5\gamma_5)_{\alpha\delta} = \delta_{\alpha\delta}$$

and

$$(\gamma_5)_{\alpha\beta}(\gamma_5)_{\gamma\delta}\delta_{\delta\alpha} = (\gamma_5\gamma_5)_{\gamma\beta} = \delta_{\gamma\beta}.$$

Thus the commutator is

$$\frac{1}{4}\int \mathrm{d}^3\boldsymbol{x}\,[\hat{q}^\dagger_{\alpha a}(x)(\tau_i\tau_j)_{ad}\hat{q}_{\alpha d}(x) - \hat{q}^\dagger_{\beta c}(x)(\tau_j\tau_i)_{cb}\hat{q}_{\beta b}(x)].$$

In the second term, the indices c and b are both summed over, and can be replaced by a and d, respectively, as in the first term, and similarly β can be replaced by α. The expression

$$\hat{q}^\dagger_{\alpha a}(x)(\tau_i\tau_j)_{ad}\hat{q}_{\alpha d}(x)$$

is just

$$\hat{q}^\dagger(x)(\tau_i\tau_j)\hat{q}(x)$$

with the explicit isospin and (Dirac) spinor indices inserted, and similarly for the $\tau_j\tau_i$ term, so the result is

$$\int \mathrm{d}^3\boldsymbol{x}\, \hat{q}^\dagger(x)[\tau_i/2, \tau_j/2]\hat{q}(x) = \mathrm{i}\epsilon_{ijk}\int \mathrm{d}^3\boldsymbol{x}\, \hat{q}^\dagger(x)(\tau_k/2)\hat{q}(x)$$

using (12.28), which is precisely the right hand side of (18.26), as required.

18.2 For (18.28) we have

$$\begin{aligned}
[\hat{Q}_{i,\mathrm{R}}, \hat{Q}_{j,\mathrm{R}}] &= \frac{1}{4}[\hat{Q}_i + \hat{Q}_{1,5}, \hat{Q}_j + \hat{Q}_{j,5}] \\
&= \frac{1}{4}\{[\hat{Q}_i, \hat{Q}_j] + [\hat{Q}_i, \hat{Q}_{j,5}] + [\hat{Q}_{i,5}, \hat{Q}_j] + [\hat{Q}_{i,5}, \hat{Q}_{j,5}]\} \\
&= \frac{1}{4}\{\mathrm{i}\epsilon_{ijk}\hat{Q}_k + \mathrm{i}\epsilon_{ijk}\hat{Q}_{k,5} - \mathrm{i}\epsilon_{jik}\hat{Q}_{k,5} + \mathrm{i}\epsilon_{ijk}\hat{Q}_k\} \\
&= \mathrm{i}\epsilon_{ijk}\frac{1}{2}(\hat{Q}_k + \hat{Q}_{k,5}) = \mathrm{i}\epsilon_{ijk}\hat{Q}_{k,\mathrm{R}},
\end{aligned}$$

using (18.24)-(18.26) and the antisymmetry of the ϵ symbol. Similar steps lead to (18.29) and (18.30).

18.3 The Lagrangian of (18.12) is

$$\bar{\hat{q}}\mathrm{i}\hat{\not{D}}\hat{q},$$

in the doublet notation of (18.18). We write $\hat{q}$ in terms of $\hat{q}_\mathrm{R}$ and $\hat{q}_\mathrm{L}$ via

$$\hat{q} = \left(\frac{1+\gamma_5}{2}\right)\hat{q} + \left(\frac{1-\gamma_5}{2}\right)\hat{q} \equiv \hat{q}_\mathrm{R} + \hat{q}_\mathrm{L}.$$

On the other hand,

$$\bar{\hat{q}}_{\mathrm{L}} = \left[\left(\frac{(1-\gamma_5)}{2}\right)\hat{q}\right]^{\dagger}\gamma_0 = \hat{q}^{\dagger}\left(\frac{1-\gamma_5}{2}\right)\gamma_0 = \hat{q}^{\dagger}\gamma_0\left(\frac{1+\gamma_5}{2}\right),$$

and similarly for $\bar{\hat{q}}_{\mathrm{R}}$. Now consider

$$\bar{\hat{q}}_{\mathrm{L}}\gamma^{\mu}\hat{q}_{\mathrm{R}} = \hat{q}^{\dagger}\gamma_0\left(\frac{1+\gamma_5}{2}\right)\gamma^{\mu}\left(\frac{1+\gamma_5}{2}\right)\hat{q} = \hat{q}^{\dagger}\gamma_0\gamma^{\mu}\left(\frac{1-\gamma_5}{2}\right)\left(\frac{1+\gamma_5}{2}\right)\hat{q} = 0,$$

and similarly for $\bar{\hat{q}}_{\mathrm{R}}\gamma^{\mu}\hat{q}_{\mathrm{L}}$. So the only surviving terms in $\bar{\hat{q}}\mathrm{i}\hat{\not{D}}\hat{q}$ are as in (18.39).

18.4 The invariant amplitude is

$$\begin{aligned}
\mathcal{M} &= -G_{\mathrm{F}}\cos\theta_{\mathrm{C}}\bar{u}_{\nu}(p_2)\gamma_{\mu}(1-\gamma_5)v(p_1)p^{\mu}f_{\pi} \\
&= -G_{\mathrm{F}}\cos\theta_{\mathrm{C}}f_{\pi}\bar{u}_{\nu}(p_2)(\not{p}_1+\not{p}_2)(1-\gamma_5)v(p_1) \\
&= -G_{\mathrm{F}}\cos\theta_{\mathrm{C}}f_{\pi}\bar{u}_{\nu}(p_2)(1+\gamma_5)\not{p}_1 v(p_1) \\
&= G_{\mathrm{F}}\cos\theta_{\mathrm{C}}f_{\pi}m_{\mu}\bar{u}_{\nu}(p_2)(1+\gamma_5)v(p_1),
\end{aligned}$$

neglecting the neutrino mass. Using the formula for the $1 \to 2$ decay rate given in appendix L, and expression (K.3) for the two-body phase space given in appendix K, we obtain for the total rate

$$\Gamma = \sum_{\text{spins}}\int \frac{1}{2m_{\pi}}|\mathcal{M}|^2 \frac{\mathrm{d}^3\boldsymbol{p}_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3\boldsymbol{p}_2}{(2\pi)^3 2E_2}(2\pi)^4\delta^4(p-p_1-p_2).$$

The sum over spins gives

$$\mathrm{Tr}[(1+\gamma_5)v(p_1)\bar{v}(p_1)(1-\gamma_5)u(p_2)\bar{u}(p_2)] = \mathrm{Tr}[(1+\gamma_5)(\not{p}_1-m_{\mu})(1-\gamma_5)\not{p}_2].$$

The term in m_{μ} vanishes since $(1+\gamma_5)(1-\gamma_5)=0$, while the term in $\not{p}_1$ gives

$$\begin{aligned}
\mathrm{Tr}[(1+\gamma_5)\not{p}_1(1-\gamma_5)\not{p}_2] &= \mathrm{Tr}[\not{p}_1(1-\gamma_5)^2\not{p}_2] \\
&= 2\mathrm{Tr}[\not{p}_1\not{p}_2] - 2\mathrm{Tr}[\not{p}_1\gamma_5\not{p}_2] \\
&= 8p_1\cdot p_2
\end{aligned}$$

using equations (J.30) and (J.36). The spatial part of the four-dimensional delta function allows the integral over $\boldsymbol{p}_2$ (say) to be done. Since the matrix element and the phase space factors are Lorentz invariant, we can evaluate the integral in any frame, and it is obviously convenient to choose the rest frame of the pion, in which the three-momenta of the neutrino and the muon are equal and opposite. The remaining delta function is then

$$\delta(m_\pi - |\boldsymbol{p}| - (\boldsymbol{p}^2 + m_\mu^2)^{1/2})$$

where $\boldsymbol{p}$ is the three-momentum of the muon, while $\mathrm{d}^3\boldsymbol{p}_1$ reduces to $4\pi|\boldsymbol{p}|^2\mathrm{d}|\boldsymbol{p}|$; also, we have $E_1 = (m_\mu^2 + \boldsymbol{p}^2)^{1/2}$ and $E_2 = |\boldsymbol{p}|$. To perform the integral over $|\boldsymbol{p}|$, we write

$$W = |\boldsymbol{p}| + (m_\mu^2 + \boldsymbol{p}^2)^{1/2}.$$

Then

$$\frac{\mathrm{d}W}{\mathrm{d}|\boldsymbol{p}|} = 1 + \frac{|\boldsymbol{p}|}{(m_\mu^2 + \boldsymbol{p}^2)^{1/2}} = \frac{W}{E_1}.$$

Hence

$$\mathrm{d}|\boldsymbol{p}| = \frac{E_1}{W}\mathrm{d}W$$

and the delta function is $\delta(m_\pi - W)$. Putting all these results together gives

$$\Gamma = \frac{1}{\pi m_\pi^2} G_\mathrm{F}^2 f_\pi^2 m_\mu^2 \cos^2\theta_\mathrm{C}\, p_1 \cdot p_2\, |\boldsymbol{p}|.$$

For $p_1 \cdot p_2$ we have

$$p^2 = m_\pi^2 = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = m_\mu^2 + 2p_1 \cdot p_2$$

whence

$$p_1 \cdot p_2 = (m_\pi^2 - m_\mu^2)/2.$$

To determine $|\boldsymbol{p}|$ we use the energy-conservation relation

$$m_\pi = |\boldsymbol{p}| + (m_\mu^2 + \boldsymbol{p}^2)^{1/2}.$$

Taking $|\boldsymbol{p}|$ to the LHS and squaring, we find

$$|\boldsymbol{p}| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}.$$

The result (18.53) then follows.

18.5 We give two ways of approaching this problem.

First, we can use the equations of motion for $\hat{q}$ and $\bar{\hat{q}}$ to calculate $\partial_\mu \hat{j}^\mu_{i,5}(x)$ directly, as follows:

$$\partial_\mu \hat{j}^\mu_{i,5}(x) = \partial_\mu(\bar{\hat{q}}(x)\gamma^\mu\gamma_5\frac{\tau_i}{2}\hat{q}(x)) = (\partial_\mu\bar{\hat{q}}(x))\gamma^\mu\gamma_5\frac{\tau_i}{2}\hat{q}(x) - \bar{\hat{q}}(x)\gamma_5\frac{\tau_i}{2}\gamma^\mu(\partial_\mu\hat{q}(x)).$$

The equation of motion for $\hat{q}$ is $\mathrm{i}\hat{\not{D}}\hat{q} = \boldsymbol{m}\hat{q}$, which can be written as

$$\gamma^\mu\partial_\mu\hat{q}(x) = -\mathrm{i}g_\mathrm{s}\gamma^\mu\boldsymbol{\lambda}/2\cdot\boldsymbol{A}_\mu\hat{q} - \mathrm{i}\boldsymbol{m}\hat{q}.$$

For $\partial_\mu\bar{\hat{q}}(x)$, we take the dagger of the last equation, multiply from the right by γ^0, and use $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$ (see (8.59)) to obtain

$$(\partial_\mu\bar{\hat{q}}(x))\gamma^\mu = \mathrm{i}g_\mathrm{s}\bar{\hat{q}}\gamma^\mu\boldsymbol{\lambda}/2\cdot\boldsymbol{A}_\mu + \mathrm{i}\bar{\hat{q}}\boldsymbol{m}$$

(this equation is how the second equation in (18.62) should be read). Hence

$$\partial_\mu \hat{j}^\mu_{i,5} = (\mathrm{i}g_\mathrm{s}\bar{\hat{q}}\gamma^\mu\boldsymbol{\lambda}/2\cdot\boldsymbol{A}_\mu + \mathrm{i}\bar{\hat{q}}\boldsymbol{m})\gamma_5\frac{\tau_i}{2}\hat{q} - \bar{\hat{q}}\gamma_5\frac{\tau_i}{2}(-\mathrm{i}g_\mathrm{s}\gamma^\mu\boldsymbol{\lambda}/2\cdot\boldsymbol{A}_\mu - \mathrm{i}\boldsymbol{m}\hat{q}).$$

Using $\gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 0$, we see that the terms in $\boldsymbol{A}_\mu$ cancel, leaving the result (18.64).

An alternative approach introduces a general method for determining the divergence of a symmetry current when a term or terms which violate the symmetry explicitly have been added to the Lagrangian (such as the mass terms in this case). Going back to (12.111), consider the variation of a Lagrangian under the infinitesimal transformation

$$\delta\hat{\psi}_r = -\mathrm{i}\epsilon_\alpha T^\alpha_{rs}\hat{\psi}_s,$$

where we have generalized (12.111) to include several parameters ϵ_α, as required in the general case. The variation is

$$\begin{aligned}
\delta\hat{\mathcal{L}} &= \frac{\partial\hat{\mathcal{L}}}{\partial\hat{\psi}_r}\delta\psi_r + \frac{\partial\hat{\mathcal{L}}}{\partial(\partial^\mu\hat{\psi}_r)}\partial^\mu(\delta\hat{\psi}_r) \\
&= \partial^\mu\left(\frac{\partial\hat{\mathcal{L}}}{\partial(\partial^\mu\hat{\psi}_r)}\right)\delta\hat{\psi}_r + \frac{\partial\hat{\mathcal{L}}}{\partial(\partial^\mu\hat{\psi}_r)}\partial^\mu(\delta\hat{\psi}_r) \\
&= \partial^\mu\left(\frac{\partial\hat{\mathcal{L}}}{\partial(\partial^\mu\hat{\psi}_r)}\right)(-\mathrm{i}\epsilon_\alpha T^\alpha_{rs}\hat{\psi}_{rs}) + \frac{\partial\hat{\mathcal{L}}}{\partial(\partial^\mu\hat{\psi}_r)}(-\mathrm{i}\epsilon_\alpha T^\alpha_{rs}\partial^\mu\hat{\psi}_s)
\end{aligned}$$

where the second line has used the E-L equations of motion. On the other hand, the corresponding symmetry current is

$$\hat{j}^{\alpha}_{\mu} = -\mathrm{i}\frac{\partial\hat{\mathcal{L}}}{\partial(\partial^{\mu}\hat{\psi}_r)}T^{\alpha}_{rs}\hat{\psi}_s,$$

so that

$$\partial^{\mu}\hat{j}^{\alpha}_{\mu} = -\mathrm{i}\partial^{\mu}\left(\frac{\partial\hat{\mathcal{L}}}{\partial(\partial^{\mu}\hat{\psi}_r)}\right)T^{\alpha}_{rs}\hat{\psi}_{rs} - \mathrm{i}\frac{\partial\hat{\mathcal{L}}}{\partial(\partial^{\mu}\hat{\psi}_r)}T^{\alpha}_{rs}\partial^{\mu}\hat{\psi}_s$$

from which it follows that

$$\delta\hat{\mathcal{L}} = \epsilon_{\alpha}\partial^{\mu}\hat{j}^{\alpha}_{\mu}.$$

Clearly, if the Lagrangian is invariant, then $\delta\hat{\mathcal{L}} = 0$ and $\partial^{\mu}\hat{j}^{\alpha}_{\mu} = 0$. But if not (for instance, if an explicit symmetry-breaking term has been added to an originally invariant Lagrangian) then the divergence of the current is given directly in terms of the non-invariant part of $\hat{\mathcal{L}}$ by

$$\partial^{\mu}\hat{j}^{\alpha}_{\mu} = \frac{\partial(\delta\hat{\mathcal{L}})}{\partial\epsilon_{\alpha}}.$$

In the present case, the field transformations are

$$\delta\hat{q} = -\mathrm{i}\boldsymbol{\eta}\cdot\boldsymbol{\tau}/2\gamma_5\hat{q}$$

and

$$\delta\hat{\bar{q}} = -\mathrm{i}\hat{\bar{q}}\gamma_5\boldsymbol{\eta}\cdot\boldsymbol{\tau}/2.$$

The non-invariant part of the Lagrangian is the mass term $-\hat{\bar{q}}\boldsymbol{m}\hat{q}$, whose variation is

$$\delta\hat{\mathcal{L}} = \mathrm{i}\hat{\bar{q}}\gamma_5(\boldsymbol{\eta}\cdot\boldsymbol{\tau}/2)\boldsymbol{m}\hat{q} + \mathrm{i}\hat{\bar{q}}\boldsymbol{m}(\boldsymbol{\eta}\cdot\boldsymbol{\tau}/2)\gamma_5\hat{q}$$

and then the formula

$$\frac{\partial(\delta\hat{\mathcal{L}})}{\partial\eta_i} = \partial_{\mu}\hat{j}^{\mu}_{i,5}$$

leads immediately to (18.64).

18.6 (i) We first remark that the transformations (18.78)-(18.80) are global rather than local (that is, the parameters $\boldsymbol{\eta}$ don't depend on x). Hence $\bar{\hat{\psi}}\mathrm{i}\,\partial\!\!\!/\hat{\psi}$ transforms to

$$\begin{aligned}\hat{\psi}^\dagger(1+\mathrm{i}\boldsymbol{\eta}\cdot\boldsymbol{\tau}/2\gamma_5)\gamma^0\mathrm{i}\,\partial\!\!\!/(1-\mathrm{i}\boldsymbol{\eta}\cdot\boldsymbol{\tau}/2\gamma_5)\hat{\psi} &= \bar{\hat{\psi}}\mathrm{i}\,\partial\!\!\!/(1+\mathrm{i}\boldsymbol{\eta}\cdot\boldsymbol{\tau}/2\gamma_5)(1-\mathrm{i}\boldsymbol{\eta}\cdot\boldsymbol{\tau}/2\gamma_5)\hat{\psi} \\ &= \bar{\hat{\psi}}\mathrm{i}\,\partial\!\!\!/\hat{\psi},\end{aligned}$$

showing that this (first) term in (18.72) is invariant; as usual, only terms of first order in the infinitesimal parameter $\boldsymbol{\eta}$ have been retained, and we have used $\{\gamma_5,\gamma^0\}=\{\gamma_5,\gamma^\mu\}=0$. It is also clear - since the gradients don't act on the $\boldsymbol{\eta}$'s - that if we prove that $\hat{\sigma}^2+\hat{\boldsymbol{\pi}}^2$ is invariant, it will follow that $\frac{1}{2}\partial_\mu\hat{\sigma}\partial^\mu\hat{\sigma}+\frac{1}{2}\partial_\mu\hat{\boldsymbol{\pi}}\cdot\partial^\mu\hat{\boldsymbol{\pi}}$ is also invariant.
(ii) $\hat{\sigma}^2+\hat{\boldsymbol{\pi}}^2$ transforms to

$$\begin{aligned}&(\hat{\sigma}-\boldsymbol{\eta}\cdot\hat{\boldsymbol{\pi}})^2+(\hat{\boldsymbol{\pi}}+\boldsymbol{\eta}\hat{\sigma})^2\\ &=\hat{\sigma}^2-2\boldsymbol{\eta}\cdot\hat{\boldsymbol{\pi}}\hat{\sigma}+\hat{\boldsymbol{\pi}}^2+2\boldsymbol{\eta}\cdot\hat{\boldsymbol{\pi}}\hat{\sigma}=\hat{\sigma}^2+\hat{\boldsymbol{\pi}}^2\end{aligned}$$

showing that it is invariant; we have used the fact that the $\hat{\sigma}$ and $\hat{\boldsymbol{\pi}}$ fields commute.
(iii) We are left with the terms multiplying $g_{\pi\mathrm{NN}}$. The first, $\mathrm{i}\bar{\hat{\psi}}\gamma_5\tau_i\hat{\psi}\hat{\pi}_i$, transforms to

$$\begin{aligned}&\mathrm{i}\hat{\psi}^\dagger(1+\mathrm{i}\boldsymbol{\eta}\cdot\boldsymbol{\tau}/2)\gamma^0\gamma_5\tau_i(1-\mathrm{i}\boldsymbol{\eta}\cdot\boldsymbol{\tau}/2\gamma_5)\hat{\psi}(\hat{\pi}_i+\eta_i\hat{\sigma})\\ &=\mathrm{i}\bar{\hat{\psi}}\gamma_5\boldsymbol{\tau}\hat{\psi}\cdot\hat{\boldsymbol{\pi}}+\mathrm{i}\bar{\hat{\psi}}\gamma_5\boldsymbol{\eta}\cdot\boldsymbol{\tau}\hat{\psi}\hat{\sigma}\\ &-\hat{\psi}^\dagger(\boldsymbol{\eta}\cdot\boldsymbol{\tau}/2)\gamma_5\gamma^0\gamma_5\tau_i\hat{\psi}\hat{\pi}_i+\hat{\psi}^\dagger\gamma^0\gamma_5\tau_i(\boldsymbol{\eta}\cdot\boldsymbol{\tau}/2)\gamma_5\hat{\psi}\hat{\pi}_i. \qquad \text{(A)}\end{aligned}$$

Using $\gamma_5^2=1$ and $\gamma_5\gamma^0\gamma_5=-\gamma^0$, the last two terms here combine to give

$$\begin{aligned}\bar{\hat{\psi}}(\boldsymbol{\eta}\cdot\boldsymbol{\tau}/2)\tau_i\hat{\psi}\hat{\pi}_i+\bar{\hat{\psi}}\tau_i(\boldsymbol{\eta}\cdot\boldsymbol{\tau}/2)\hat{\psi}\hat{\pi}_i &= \bar{\hat{\psi}}\frac{1}{2}(\tau_j\tau_i+\tau_i\tau_j)\hat{\psi}\eta_j\hat{\pi}_i\\ &=\bar{\hat{\psi}}\hat{\psi}\boldsymbol{\eta}\cdot\hat{\boldsymbol{\pi}}, \qquad \text{(B)}\end{aligned}$$

where we have used $\tau_i\tau_j+\tau_j\tau_i=2\delta_{ij}$ (see problem 4.4(a) for the same relation written for the σ matrices). The second term multiplying $g_{\pi\mathrm{NN}}$ is $\bar{\hat{\psi}}\hat{\psi}\hat{\sigma}$ which transforms to

$$\begin{aligned}&\hat{\psi}^\dagger(1+\mathrm{i}\boldsymbol{\eta}\cdot\boldsymbol{\tau}/2\gamma_5)\gamma^0(1-\mathrm{i}\boldsymbol{\eta}\cdot\boldsymbol{\tau}/2\gamma_5)\hat{\psi}(\hat{\sigma}-\boldsymbol{\eta}\cdot\hat{\boldsymbol{\pi}})\\ &=\bar{\hat{\psi}}\hat{\psi}\hat{\sigma}-\bar{\hat{\psi}}\hat{\psi}\boldsymbol{\eta}\cdot\hat{\boldsymbol{\pi}}-\mathrm{i}\bar{\hat{\psi}}\gamma_5\boldsymbol{\eta}\cdot\boldsymbol{\tau}\hat{\psi}\hat{\sigma}.\end{aligned}$$

The first term here combines with $i\bar{\hat{\psi}}\gamma_5\boldsymbol{\tau}\hat{\psi}\cdot\hat{\boldsymbol{\pi}}$ of (A) to give the original interaction term involving $g_{\pi NN}$, while the other two terms cancel with the second term in (A) and with that in (B). Hence this interaction term is also invariant. The other interactions are included in the invariant $(\hat{\sigma}^2 + \hat{\boldsymbol{\pi}}^2)^2$.

18.7 *Note that in this problem the parameter μ^2 in (18.72) is taken to be negative, so that the potential has the form of (18.83) with the 'new' μ^2 positive. Also note that in the second line of equation (18.88), μ^2 should be replaced by $-\mu^2$, with $\mu^2 > 0$.*
The terms involving $g_{\pi NN}$, and the gradients, are straightforward. The remaining terms (from $-V(\boldsymbol{\pi}, \sigma)$) are (we work with the classical potential for convenience - hats can be added afterwards)

$$\frac{1}{2}\mu^2(v_\sigma^2 + 2v_\sigma\sigma' + \sigma'^2 + \boldsymbol{\pi}^2) - \frac{1}{4}\lambda(v_\sigma^2 + 2v_\sigma\sigma' + \sigma'^2 + \boldsymbol{\pi}^2)^2$$
$$= \frac{1}{2}\mu^2 v_\sigma^2 + \mu^2 v_\sigma\sigma' + \frac{1}{2}\mu^2(\sigma'^2 + \boldsymbol{\pi}^2)$$
$$-\frac{1}{4}\lambda\{(v_\sigma^2 + 2v_\sigma\sigma')^2 + 2(v_\sigma^2 + 2v_\sigma\sigma')(\sigma'^2 + \boldsymbol{\pi}^2) + (\sigma'^2 + \boldsymbol{\pi}^2)^2\}.$$

Multiplying out, we see that the term linear in σ' cancels (using $\lambda v_\sigma^2 = \mu^2$), as do the terms proportional to $(\sigma'^2 + \boldsymbol{\pi}^2)$. The term in σ'^2 is

$$-\lambda v_\sigma^2 \sigma'^2 = -\mu^2\sigma'^2,$$

as in (the corrected form of) equation (18.88). The remaining terms in (18.88) are easily seen to be present.

18.8 We use the result for $\partial_\mu \hat{j}^\mu_{i,5}$ given in the 'alternative' solution to problem 18.5:

$$\partial_\mu \hat{j}^\mu_{i,5} = \frac{\partial(\delta\hat{\mathcal{L}})}{\partial\eta_i}.$$

Here, the term in $\hat{\mathcal{L}}$ which breaks the chiral symmetry explicitly is $c\hat{\sigma}$. Under (18.80) this produces a change in $\hat{\mathcal{L}}$ given by

$$\delta\hat{\mathcal{L}} = -c\eta_i\hat{\pi}_i.$$

The result follows immediately.

18.9 The invariant amplitude for $\pi^0 \to \gamma\gamma$ is

$$\mathcal{M} = \frac{e^2}{4\pi^2 f_\pi} \epsilon^{\nu\lambda\alpha\beta} \epsilon^*_{1\nu}(k_1, \lambda_1) \epsilon^*_{2\lambda}(k_2, \lambda_2) k_{1\alpha} k_{2\beta},$$

where k_1, k_2 are the 4-momenta of the γ's, λ_1, λ_2 are their polarizations, and $\nu, \lambda, \alpha, \beta$ are Lorentz indices. The rate is

$$\Gamma = \frac{1}{2m_\pi} \sum_{\lambda_1,\lambda_2} \int |\mathcal{M}|^2 \frac{1}{2} \frac{\mathrm{d}^3 \boldsymbol{k}_1}{2E_1(2\pi)^3} \frac{\mathrm{d}^3 \boldsymbol{k}_2}{2E_2(2\pi)^3} (2\pi)^4 \delta^4(p - k_1 - k_2),$$

where E_1, E_2 are the energies of the γ's, p is the 4-momentum of the pion, and the extra factor of $1/2$ arises from the identity of the γ's (see page 389 of volume 1, just after the formula for the $1 \to 2$ decay rate). When we form $|\mathcal{M}|^2$, we get the rather complicated-looking expression

$$\epsilon^{\nu\lambda\alpha\beta} \epsilon^{\rho\sigma\gamma\delta} \epsilon^*_{1\nu}(k_1, \lambda_1) \epsilon^*_{2\lambda}(k_2, \lambda_2) \epsilon_{1\rho}(k_1, \lambda_1) \epsilon_{2\sigma}(k_2, \lambda_2) k_{1\alpha} k_{2\beta} k_{1\gamma} k_{2\delta}.$$

The sums over the polarization labels are done using (see section 8.6.3)

$$\sum_{\lambda_1} \epsilon^*_{1\nu}(k_1, \lambda_1) \epsilon_{1\rho}(k_1, \lambda_1) = -g_{\nu\rho},$$

and similarly for the sum on λ_2, so that the previous expression reduces to

$$\epsilon^{\nu\lambda\alpha\beta} \epsilon^{\rho\sigma\gamma\delta} g_{\nu\rho} g_{\lambda\sigma} k_{1\alpha} k_{2\beta} k_{1\gamma} k_{2\delta}. \qquad \text{(C)}$$

We now need a formula for the product of two four-dimensional ϵ symbols, when both their first two indices are 'contracted' via the g symbols. The required formula is

$$\epsilon^{\nu\lambda\alpha\beta} \epsilon^{\rho\sigma\gamma\delta} g_{\nu\rho} g_{\lambda\sigma} = -2(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma})$$

which may be checked by examining special cases (note that $\epsilon_{0123} = +1$); see also problem 20.7. Inserting this result in (C) gives

$$-2((k_1)^2 (k_2)^2 - (k_1 \cdot k_2)^2) = 2(k_1 \cdot k_2)^2,$$

where we have used the masslessness conditions $k_1^2 = k_2^2 = 0$. Squaring the 4-momentum conservation condition gives

$$p^2 = m_\pi^2 = (k_1 + k_2)^2 = 2k_1 \cdot k_2$$

and hence $k_1 \cdot k_2 = m_\pi^2/2$. The phase space integrals are most conveniently done in the rest frame of the pion (as in problem 18.4), the energy part of the delta function then becoming just $\delta(m_\pi - 2|\boldsymbol{k}|)$, where $\boldsymbol{k}$ is the 3-momentum of one of the photons in this frame. Putting the details together leads to (18.101).

Taking $\alpha = 1/137$, $f_\pi = 93$ MeV, and $m_\pi = 135$ MeV, we find $\Gamma = 7.64 \times 10^{-6}$ MeV. The lifetime is given by $\hbar/\Gamma$, and using $\hbar = 6.58 \times 10^{-22}$ MeV s, we find the value 8.6×10^{-17} s for the π^0 lifetime (the value quoted in the data tables is $(8.4 \pm 0.6) \times 10^{-17}$ s).

18.10 Labelling the polarizations of the photons by λ_1 and λ_2, equations (18.97) and (18.102) give

$$\begin{aligned}
&\int \mathrm{d}^4x \, \langle k_1, \lambda_1; k_2, \lambda_2 | \partial_\mu \hat{j}^\mu_{3,5}(x)|0\rangle \mathrm{e}^{-\mathrm{i}q\cdot x} \\
&= (2\pi)^4 \delta^4(q - k_1 - k_2) \epsilon^*_{1\nu}(k_1, \lambda_1) \epsilon^*_{2\rho}(k_2, \lambda_2) \mathrm{i} q_\mu \mathcal{M}^{\mu\nu\rho} \\
&= \int \mathrm{d}^4x \, \mathrm{e}^{-\mathrm{i}q\cdot x} \, \langle k_1, \lambda_1; k_2, \lambda_2 | \frac{-e^2}{32\pi^2} \epsilon^{\alpha\nu\beta\rho} (\partial_\alpha \hat{A}_\nu(x) - \partial_\nu \hat{A}_\alpha(x))(\partial_\beta \hat{A}_\rho(x) - \partial_\rho \hat{A}_\beta(x))|0\rangle.
\end{aligned}$$

Since the 4-dimensional ϵ symbol is antisymmetric under the interchange of any pair of indices, we can replace

$$\epsilon^{\alpha\nu\beta\rho} (\partial_\alpha \hat{A}_\nu(x) - \partial_\nu \hat{A}_\alpha(x))(\partial_\beta \hat{A}_\rho(x) - \partial_\rho \hat{A}_\beta(x))$$

by

$$4\epsilon^{\alpha\nu\beta\rho} (\partial_\alpha \hat{A}_\nu(x))(\partial_\beta \hat{A}_\rho(x)).$$

To evaluate the matrix element, we need

$$\hat{A}_\nu(x) = \sum_\lambda \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3(2E_k)^{1/2}} [\hat{\alpha}_\lambda(k)\epsilon_\nu(k,\lambda)\mathrm{e}^{-\mathrm{i}k\cdot x} + \hat{\alpha}^\dagger_\lambda(k)\epsilon^*_\nu(k,\lambda)\mathrm{e}^{\mathrm{i}k\cdot x}]$$

where $E_k = |\boldsymbol{k}|$ (see (7.101)); $\hat{A}_\rho(x)$ has a similar expansion but with an integration variable $\boldsymbol{k}'$ and a polarization label λ'. Also,

$$\langle k_1, \lambda_1 | = (2E_1)^{1/2} \langle 0 | \hat{\alpha}_{\lambda_1}(k_1)$$

and similarly for $\langle k_2, \lambda_2|$ (see (8.160)). In the resulting vev, the only surviving term is the one with an equal number of creation and destruction operators, namely

$$\langle 0 | \hat{\alpha}_{\lambda_1}(k_1) \hat{\alpha}_{\lambda_2}(k_2) \hat{\alpha}^\dagger_\lambda(k) \hat{\alpha}^\dagger_{\lambda'}(k')|0\rangle$$

which can be evaluated using (7.105), but replacing $-g_{\lambda\lambda'}$ on the RHS by $\delta_{\lambda\lambda'}$, since for real photons only the spatial (and in fact transverse) polarizations are present. This gives

$$\delta_{\lambda\lambda_1}\delta_{\lambda'\lambda_2}(2\pi)^6\delta^3(\boldsymbol{k}-\boldsymbol{k}_1)\delta^3(\boldsymbol{k}'-\boldsymbol{k}_2)+\delta_{\lambda\lambda_2}\delta_{\lambda'\lambda_1}(2\pi)^6\delta^3(\boldsymbol{k}-\boldsymbol{k}_2)\delta^3(\boldsymbol{k}'-\boldsymbol{k}_1).$$

The matrix element is therefore

$$\begin{aligned}&\int \mathrm{d}^4x\,\mathrm{e}^{-\mathrm{i}q\cdot x}(4E_1E_2)^{1/2}\,\frac{-e^2}{32\pi^2}\,4\epsilon^{\alpha\nu\beta\rho}\\ &\sum_\lambda\sum_{\lambda'}\int\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3(2E_k)^{1/2}}\,\frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3(2E_{k'})^{1/2}}\,\epsilon^*_\nu(k,\lambda)\epsilon^*_\rho(k',\lambda')\,\mathrm{i}k_\alpha\,\mathrm{i}k'_\beta\,\mathrm{e}^{\mathrm{i}(k+k')\cdot x}\\ &\{\delta_{\lambda\lambda_1}\delta_{\lambda'\lambda_2}(2\pi)^6\delta^3(\boldsymbol{k}-\boldsymbol{k}_1)\delta^3(\boldsymbol{k}'-\boldsymbol{k}_2)+\delta_{\lambda\lambda_2}\delta_{\lambda'\lambda_1}(2\pi)^6\delta^3(\boldsymbol{k}-\boldsymbol{k}_2)\delta^3(\boldsymbol{k}'-\boldsymbol{k}_1\}.\end{aligned}$$

Consider the first term in the $\{....\}$ bracket. The $\boldsymbol{k}$ and $\boldsymbol{k}'$ integrals can be done using the 3-dimensional δ-functions, which set $\boldsymbol{k}=\boldsymbol{k}_1$ and $\boldsymbol{k}'=\boldsymbol{k}_2$, and hence $E_k=E_1$ and $E_{k'}=E_2$. This term is therefore

$$\begin{aligned}&\int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}(k_1+k_2-q)\cdot x}\,\frac{e^2}{32\pi^2}\,4\epsilon^{\alpha\nu\beta\rho}\epsilon^*_{1\nu}(k_1,\lambda_1)\epsilon^*_{2\rho}(k_2,\lambda_2)k_{1\alpha}k_{2\beta}\\ &=(2\pi)^4\delta^4(k_1+k_2-q)\frac{e^2}{8\pi^2}\epsilon^{\alpha\nu\beta\rho}\epsilon^*_{1\nu}(k_1,\lambda_1)\epsilon^*_{2\rho}(k_2,\lambda_2)k_{1\alpha}k_{2\beta}.\end{aligned}$$

Similarly, the second term in the $\{....\}$ bracket gives

$$(2\pi)^4\delta^4(k_1+k_2-q)\frac{e^2}{8\pi^2}\epsilon^{\alpha\nu\beta\rho}\epsilon^*_{2\nu}(k_2,\lambda_2)\epsilon^*_{1\rho}(k_1,\lambda_1)k_{2\alpha}k_{1\beta}.$$

This is clearly the same as the previous term but with the labels '1' and '2' exchanged, as would be expected. The two terms are in fact identical, as must follow from Bose symmetry. We can check this by re-labelling, in the second term, the index ν as ρ, and v.v., and the index α as β, and v.v. The second term is then the same as the first, apart from the ϵ symbol which has now become $\epsilon^{\beta\rho\alpha\nu}$; but this equals $\epsilon^{\alpha\nu\beta\rho}$, since they are related by an even number of index interchanges. Hence finally the factor $e^2/8\pi^2$ becomes $e^2/4\pi^2$, and equating the result to the RHS of the first equation of this solution leads to (18.103).

Chapter 19

19.1 We have

$$
\begin{aligned}
&[(-k^2+M^2)g^{\nu\mu}+k^\nu k^\mu]\left(\frac{-g_{\mu\rho}+k_\mu k_\rho/M^2}{k^2-M^2}\right)\\
&= g^{\nu\mu}g_{\mu\rho}-g^{\nu\mu}k_\mu k_\rho/M^2-\frac{k^\nu k^\mu g_{\mu\rho}}{k^2-M^2}+\frac{k^\nu k^2 k_\rho/M^2}{k^2-M^2}\\
&= g^\nu_\rho-\frac{k^\nu k_\rho}{M^2}-\frac{k^\nu k_\rho}{k^2-M^2}+\frac{k^\nu k_\rho k^2/M^2}{k^2-M^2}.
\end{aligned}
$$

The last two terms here combine to give $k^\nu k_\rho/M^2$, and the result follows.

19.2 The required Lorentz transformation has the form (for the 4-vector (x^0, x^1, x^2, x^3))

$$
\begin{aligned}
x^{0\prime} &= \gamma(x^0+\beta x^3)\\
x^{3\prime} &= \gamma(x^3+\beta x^0)
\end{aligned}
$$

where $\beta = |\boldsymbol{k}|/E$ and $\gamma = E/M$. Under this transformation, the components of the 4-vector $k_{\text{rest}} = (M, \mathbf{0})$ transform to

$$k^0 = \gamma M = E$$

and

$$k^3 = \gamma\beta M = |\boldsymbol{k}|$$

as required. The components of $\epsilon^\mu(k_{\text{rest}}, \lambda = 0) = (0,0,0,1)$ transform to

$$\epsilon^0(k, \lambda = 0) = \gamma\beta = |\boldsymbol{k}|/M$$

and

$$\epsilon^3(k, \lambda = 0) = \gamma = E/M,$$

while the transverse components in the x^1 and x^2 directions are unaffected by this transformation along the x^3-axis.

19.3 We proceed by checking different values for the indices μ and ν explicitly. For example, consider $\mu = \nu = 0$. In this case,

$$\sum_{\lambda=0,\pm1} \epsilon^0(k,\lambda)\epsilon^{0*}(k,\lambda) = \epsilon^0(k,\lambda=0)\epsilon^{0*}(k,\lambda=0),$$

since from (19.13) and (19.17) the $\lambda = \pm 1$ polarization vectors have zero $\mu = 0$ component. Using (19.18) this gives

$$\sum_{\lambda=0,\pm1} \epsilon^0(k,\lambda)\epsilon^{0*}(k,\lambda) = \boldsymbol{k}^2/M^2.$$

Meanwhile, the RHS of (19.19), for $\mu = \nu = 0$, is

$$-1 + (k^0)^2/M^2 = \frac{(k^0)^2 - M^2}{M^2} = \boldsymbol{k}^2/M^2,$$

verifying the result for this choice of μ and ν. As another example, consider $\mu = 0, \nu = 3$. Here again only the $\lambda = 0$ polarization appears and we find

$$\frac{|\boldsymbol{k}|}{M}\frac{k^0}{M}$$

for the LHS of (19.19), which checks with the RHS. Other components check similarly.

19.4 We have

$$\hat{\phi} = \frac{1}{\sqrt{2}}(v+\hat{h})\mathrm{e}^{-\mathrm{i}\hat{\theta}/v} \qquad \text{and} \qquad \hat{\phi}^\dagger = \frac{1}{\sqrt{2}}(v+\hat{h})\mathrm{e}^{\mathrm{i}\hat{\theta}/v},$$

and so

$$\begin{aligned} \partial^\nu\hat{\phi} &= \frac{1}{\sqrt{2}}(v+\hat{h})\,(-\mathrm{i}\partial^\nu\hat{\theta}/v)\,\mathrm{e}^{-\mathrm{i}\hat{\theta}/v} + \frac{1}{\sqrt{2}}\,\partial^\nu\hat{h}\,\mathrm{e}^{-\mathrm{i}\hat{\theta}/v} \\ \partial\hat{\phi}^\dagger &= \frac{1}{\sqrt{2}}(v+\hat{h})\,(\mathrm{i}\partial^\nu\hat{\theta}/v)\,\mathrm{e}^{\mathrm{i}\hat{\theta}/v} + \frac{1}{\sqrt{2}}\,\partial^\nu\hat{h}\,\mathrm{e}^{\mathrm{i}\hat{\theta}/v}. \end{aligned}$$

Hence

$$\begin{aligned} \hat{j}^\nu_{\mathrm{em}} &= \mathrm{i}q(\hat{\phi}^\dagger\partial^\nu\hat{\phi} - (\partial^\nu\hat{\phi}^\dagger)\hat{\phi}) - 2q^2\hat{A}^\nu\hat{\phi}^\dagger\hat{\phi} \\ &= \frac{\mathrm{i}q}{2}[(v+\hat{h})\{-\mathrm{i}(v+\hat{h})\partial^\nu\hat{\theta}/v + \partial^\nu\hat{h}\} - \{\mathrm{i}(v+\hat{h})\partial^\nu\hat{\theta}/v + \partial^\nu\hat{h}\}(v+\hat{h})] \\ &\quad -q^2\hat{A}^\nu(v+\hat{h})^2. \end{aligned}$$

The terms in $\partial^\nu \hat{h}$ cancel, and the terms in $\hat{A}^\nu$ and $\partial^\nu \hat{\theta}$ (omitting products of fields) combine to give (19.46).

19.5 With the ansatz (19.55) the Lagrangian of (19.40) is

$$\hat{\mathcal{L}}_{\mathrm{H}} = \frac{1}{2}[\partial^\mu \hat{h} + \mathrm{i}q\hat{A}^\mu(v+\hat{h})]^\dagger[\partial_\mu \hat{h} + \mathrm{i}q\hat{A}_\mu(v+\hat{h})] - \frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - \frac{1}{16}\lambda(v+\hat{h})^4 + \frac{1}{2}\mu^2(v+\hat{h})^2.$$

The quadratic terms in the first [...][...] expression are

$$\frac{1}{2}\partial^\mu \hat{h}\partial_\mu \hat{h} + \frac{1}{2}q^2 v^2 \hat{A}^\mu \hat{A}_\mu,$$

while the potential terms reduce to $-\mu^2 \hat{h}^2$ as in problem 17.5.

19.6 With

$$\hat{\phi} = \frac{1}{\sqrt{2}}(v + \hat{\chi}_1 + \mathrm{i}\hat{\chi}_2)$$

we have

$$\partial_\mu \hat{\phi} = \frac{1}{\sqrt{2}}(\partial_\mu \hat{\chi}_1 + \mathrm{i}\partial_\mu \hat{\chi}_2)$$

and it is clear that

$$\partial_\mu \hat{\phi}^\dagger \partial^\mu \hat{\phi} = \frac{1}{2}(\partial_\mu \hat{\chi}_1 \partial^\mu \hat{\chi}_1 + \partial_\mu \hat{\chi}_2 \partial^\mu \hat{\chi}_2).$$

The non-gradient terms are

$$\begin{aligned}
&-\frac{1}{4}\lambda[\frac{1}{2}(v + \hat{\chi}_1 + \mathrm{i}\hat{\chi}_2)(v + \hat{\chi}_1 - \mathrm{i}\hat{\chi}_2)]^2 + \mu^2\frac{1}{2}[(v + \hat{\chi}_1)^2 + (\hat{\chi}_2)^2] \\
&= -\frac{\lambda}{16}[(v + \hat{\chi}_1)^2 + (\hat{\chi}_2)^2]^2 + \frac{1}{2}\mu^2(v^2 + 2v\hat{\chi}_1 + (\hat{\chi}_1)^2 + (\hat{\chi}_2)^2) \\
&= -\frac{\mu^2}{4v^2}[(v + \hat{\chi}_1)^4 + 2(v + \hat{\chi}_1)^2(\hat{\chi}_2)^2 + \ldots] + \frac{1}{2}\mu^2(v^2 + 2v\hat{\chi}_1 + (\hat{\chi}_1)^2 + (\hat{\chi}_2)^2) \\
&= -\frac{\mu^2}{4v^2}(v^4 + 4v^3\hat{\chi}_1 + 6v^2(\hat{\chi}_1)^2 + 2v^2(\hat{\chi}_2)^2 + \ldots) + \frac{1}{2}\mu^2(v^2 + 2v\hat{\chi}_1 + (\hat{\chi}_1)^2 + (\hat{\chi}_2)^2).
\end{aligned}$$

The linear term in $\hat{\chi}_1$ cancels, as does the term in $(\hat{\chi}_2)^2$, and we are left with $-\mu^2\hat{\chi}_1^2$, apart from an irrelevant constant.

19.7 When the ansatz (19.65) is inserted into the Lagrangian (19.40) and only terms which are quadratic in the fields are retained, we find - in addition to the usual terms - an additional one

$$M\hat{A}_\mu \partial^\mu \hat{\chi}_2,$$

which leads to the first term on the RHS of (19.67), for the equation of motion for $\hat{A}^\nu$. Treating (the negative of) this as an interaction term in the Lagrangian, the Feynman rule for the vertex shown in figure 19.3 is obtained by evaluating the matrix element

$$\langle \text{one } \chi_2 \text{ particle}|\mathrm{i}M\hat{A}_\mu \partial^\mu \hat{\chi}_2|\text{one A particle}\rangle$$

(see for example section 6.3) and is

$$M\epsilon_\mu k^\mu$$

where ϵ_μ is the polarization vector of the gauge quantum, and k is the 4-momentum of the outgoing χ_2 particle. Note that the vertex with an ingoing χ_2 particle and an outgoing gauge quantum will be the negative of this. Consider now the sum of the first two terms on the RHS of the equality in figure 19.4. Calling the gauge propagator $\Delta^{\mu\nu}$, as given by (19.68), these two terms are

$$\Delta^{\mu\nu} + \Delta^{\mu\lambda}(Mk_\lambda)\frac{\mathrm{i}}{k^2}(-Mk_\sigma)\Delta^{\sigma\lambda}.$$

Using (19.68) for Δ, we easily find

$$k_\sigma \Delta^{\sigma\lambda} = \mathrm{i}k^\nu/M^2.$$

Hence the sum of these two terms is

$$\Delta^{\mu\nu} + \Delta^{\mu\lambda}\, k_\lambda k^\nu/k^2 = \Delta^{\mu\lambda}(g^\nu_\lambda + k^\nu k_\lambda/k^2)$$

which is in fact the same as the first two terms on the RHS of (19.71), when expanded as a series (cf (19.70)). A similar analysis can be done for the third term, which is found to be identical to the second; this is essentially because

$$\frac{k^\nu k_\rho}{k^2}\frac{k^\rho k_\lambda}{k^2} = \frac{k^\nu k_\lambda}{k^2}.$$

The result then follows by analogy with the sum in (19.70).

19.8 The factor multiplying $\tilde{\hat{A}}^{\mu}$ is

$$[(-k^2 + M^2)g^{\mu\nu} + k^\mu k^\nu(1 - 1/\xi)].$$

Let us write the inverse of this as $Ag^{\mu\nu} + Bk^\mu k^\nu$ (cf (7.89)-(7.91)). Then we require

$$\begin{aligned} g^\mu_\rho &= (Ag^{\mu\nu} + Bk^\mu k^\nu)[(-k^2 + M^2)g_{\nu\rho} + k_\nu k_\rho(1 - 1/\xi)] \\ &= A(-k^2 + M^2)g^\mu_\rho + Ak^\mu k_\rho(1 - 1/\xi) + Bk^\mu k_\rho(-k^2 + M^2) + Bk^2 k^\mu k_\rho(1 - 1/\xi). \end{aligned}$$

Equating the coefficient of g^μ_ρ on both sides we get

$$A = -\frac{1}{k^2 - M^2},$$

while equating the coefficient of $k^\mu k_\rho$ on both sides, and using this result for A, gives

$$B = \frac{1 - \xi}{(k^2 - \xi M^2)(k^2 - M^2)}$$

as required.

19.9 The terms in $\partial_\mu \hat{H} \partial^\mu \hat{H}$ and in $\hat{H}^2$ are clearly the same as the corresponding terms in (19.56). The terms quadratic in the gradients of $\hat{W}$ fields will all arise straightforwardly from the 'Maxwell' term $-1/4\, \hat{\boldsymbol{F}}_{\mu\nu} \cdot \hat{\boldsymbol{F}}^{\mu\nu}$. To find the other quadratic pieces, we need to look at the covariant derivative acting on $\hat{\phi}$. With the ansatz (19.87) this is

$$(\mathrm{i}g\boldsymbol{\tau} \cdot \hat{W}_\mu/2 + \mathrm{i}g'\hat{B}_\mu/2)\hat{\phi} = \begin{pmatrix} \frac{\mathrm{i}g}{2}(\hat{W}_{1\mu} - \mathrm{i}\hat{W}_{2\mu})\frac{1}{\sqrt{2}}(v + \hat{h}) \\ \frac{-\mathrm{i}g}{2}\hat{W}_{3\mu}\frac{1}{\sqrt{2}}(v + \hat{h}) + \frac{\mathrm{i}g'}{2}\hat{B}_\mu\frac{1}{\sqrt{2}}(v + \hat{h}) \end{pmatrix}$$

Multiplying this by its Hermitian conjugate, and retaining only the parts quadratic in the fields, yields

$$\frac{1}{8}v^2 g^2(\hat{W}^2_{1\mu} + \hat{W}^2_{2\mu}) + \frac{1}{8}v^2(g\hat{W}_{3\mu} - g'\hat{B}_\mu)(g\hat{W}^\mu_3 - g'\hat{B}^\mu)$$

as required.

Chapter 20

20.1 We have

$$\bar{u}_{\mathrm{p}}\gamma^\mu u_{\mathrm{n}} = u_{\mathrm{p}}^\dagger(1, \boldsymbol{\alpha})u_{\mathrm{n}},$$

and the spinors have the form

$$u_{\mathrm{p}} = N\begin{pmatrix} \phi_{\mathrm{p}} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\phi_{\mathrm{p}} \end{pmatrix} \approx N\begin{pmatrix} \phi_{\mathrm{p}} \\ 0 \end{pmatrix}$$

for $|\boldsymbol{p}| \ll m$, and similarly for u_{n}. Then the '1' contribution in $\bar{u}_{\mathrm{p}}\gamma^\mu u_{\mathrm{n}}$ becomes just

$$\phi_{\mathrm{p}}^\dagger\phi_{\mathrm{n}}$$

which vanishes if the spin states of p and n are different, while it is easy to check that the '$\boldsymbol{\alpha}$' part is zero, in this approximation.

20.2 *Please note that there are two misprints in this question: a 'γ_5' has been omitted from both bilinears.* We consider the behaviour of $\bar{\hat{\psi}}_1(\boldsymbol{x}, t)\gamma^\mu\gamma_5\hat{\psi}_2(\boldsymbol{x}, t)$ under $\hat{\mathbf{P}}$. The $\mu = 0$ component is $\hat{\psi}_1^\dagger\gamma_5\hat{\psi}_2$, which is a pseudoscalar as shown in (20.33). The spatial components transform to

$$\hat{\psi}_1^\dagger(-\boldsymbol{x}, t)\gamma^0(\boldsymbol{\gamma}\gamma_5)\gamma^0\hat{\psi}_2(-\boldsymbol{x}, t) = \hat{\psi}^\dagger(-\boldsymbol{x}, t)(\boldsymbol{\gamma}\gamma_5)\hat{\psi}_2(-\boldsymbol{x}, t),$$

showing that they form the components of an axial vector (see (20.13)).

20.3 We require N such that

$$\begin{aligned} 2m &= N^2\left(\phi^\dagger \;\phi^\dagger\frac{(E-|\boldsymbol{p}|)}{m}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \phi \\ \frac{(E-|\boldsymbol{p}|)}{m}\phi \end{pmatrix} \\ &= N^2\left(\phi^\dagger \;\phi^\dagger\frac{(E-|\boldsymbol{p}|)}{m}\right)\begin{pmatrix} \frac{(E-|\boldsymbol{p}|)}{m}\phi \\ \phi \end{pmatrix} \\ &= 2N^2\frac{(E-|\boldsymbol{p}|)}{m}, \end{aligned}$$

taking ϕ to be normalized to $\phi^\dagger\phi = 1$ as usual. Hence

$$N^2 = \frac{m^2}{E-|\boldsymbol{p}|} = E + |\boldsymbol{p}|$$

as required.

20.4 The spinor v satisfies $(\not{p} + m)v = 0$, which is (in the representation used in section 20.4)

$$\begin{pmatrix} m & E + \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ E - \boldsymbol{\sigma}\cdot\boldsymbol{p} & m \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0,$$

which gives

$$\phi = -\frac{(E + \boldsymbol{\sigma}\cdot\boldsymbol{p})}{m}\chi.$$

Hence (noting what is said above (20.54)) we have

$$v(p, \lambda = +1) = N \begin{pmatrix} -\frac{(E-|\boldsymbol{p}|)}{m}\chi_- \\ \chi_- \end{pmatrix},$$

and N is such that

$$-2m = N^2 \left(-\frac{(E - |\boldsymbol{p}|)}{m}\chi_-^\dagger \; \chi_-^\dagger \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{(E-|\boldsymbol{p}|)}{m}\chi_- \\ \chi_- \end{pmatrix},$$

which leads to

$$N^2 = \frac{m^2}{E - |\boldsymbol{p}|} = E + |\boldsymbol{p}|.$$

$v(p, \lambda = +1)$ is then

$$\begin{aligned} v(p, \lambda = +1) &= \sqrt{E + |\boldsymbol{p}|} \begin{pmatrix} -\frac{(E-|\boldsymbol{p}|)}{m}\chi_- \\ \chi_- \end{pmatrix} \\ &= \begin{pmatrix} -\sqrt{E - |\boldsymbol{p}|}\chi_- \\ \sqrt{E + |\boldsymbol{p}|}\chi_- \end{pmatrix} \end{aligned}$$

as required. (20.55) is obtained similarly.

20.5 Under **P** the Dirac spinor u transforms to βu which, in the representation (20.38), implies

$$\phi \to \chi, \qquad \chi \to \phi,$$

where ϕ and χ are the components of u, as in (20.44). In addition, of course, $\boldsymbol{p} \to -\boldsymbol{p}$. On the other hand, under **C** we have (see (20.63))

$$\begin{pmatrix} \phi \\ \chi \end{pmatrix} \to \mathrm{i}\gamma^2 \begin{pmatrix} \phi \\ \chi \end{pmatrix}^* = \begin{pmatrix} -\mathrm{i}\sigma_2\chi^* \\ \mathrm{i}\sigma_2\phi^* \end{pmatrix},$$

so that under **C**

$$\phi \to -\mathrm{i}\sigma_2\chi^* \qquad \chi \to \mathrm{i}\sigma_2\phi^*,$$

and under **CP**

$$\phi \to \phi_{\mathbf{CP}} = \mathrm{i}\sigma_2\phi^*, \qquad \chi \to \chi_{\mathbf{CP}} = -\mathrm{i}\sigma_2\chi^*,$$

together with $\boldsymbol{p} \to -\boldsymbol{p}$. Consider now the first of equations (20.56),

$$E\phi_0 = \boldsymbol{\sigma}\cdot\boldsymbol{p}\phi_0. \qquad (20.56a)$$

If this is invariant under **CP**, it should be the case that

$$E(\phi_0)_{\mathbf{CP}} = -\boldsymbol{\sigma}\cdot\boldsymbol{p}(\phi_0)_{\mathbf{CP}}.$$

Take the complex conjugate of both sides of (20.56a) and multiply by $\mathrm{i}\sigma_2$ from the left; we obtain

$$E(\mathrm{i}\sigma_2\phi_0^*) = \sigma_2\boldsymbol{\sigma}^*\cdot\boldsymbol{p}(\mathrm{i}\phi_0^*).$$

Now note that $\sigma_1^* = \sigma_1, \sigma_2^* = -\sigma_2, \sigma_3^* = \sigma_3$, while $\sigma_2\sigma_1 = -\sigma_1\sigma_2$, and $\sigma_2\sigma_3 = -\sigma_3\sigma_2$. These relations imply that the previous equation can be written as

$$E(\mathrm{i}\sigma_2\phi_0^*) = -\boldsymbol{\sigma}\cdot\boldsymbol{p}(\mathrm{i}\sigma_2\phi_0^*)$$

which is just

$$E(\phi_0)_{\mathbf{CP}} = -\boldsymbol{\sigma}\cdot\boldsymbol{p}(\phi_0)_{\mathbf{CP}}.$$

as required. The **CP**-invariance of the second equation in (20.56) can be checked similarly.

20.6 (a) We have

$$\begin{aligned}
\gamma_5^2 &= (\mathrm{i})^2\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= +(\gamma^0)^2\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\
&= (\gamma^1)^2\gamma^2\gamma^3\gamma^2\gamma^3 \\
&= -\gamma^2\gamma^3\gamma^2\gamma^3 = (\gamma^2)^2(\gamma^3)^2 = 1.
\end{aligned}$$

Then

$$(1+\gamma_5)(1-\gamma_5) = 1-(\gamma_5)^2 = 0.$$

(b) Consider $\{\gamma_5, \gamma^0\}$:

$$\begin{aligned}\{\gamma_5, \gamma^0\} &= \mathrm{i}[\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 + (\gamma^0)^2\gamma^1\gamma^2\gamma^3] \\ &= \mathrm{i}[-(\gamma^0)^2\gamma^1\gamma^2\gamma^3 + (\gamma^0)^2\gamma^1\gamma^2\gamma^3] \\ &= 0, \end{aligned} \tag{3}$$

and similarly for the other components of γ^μ. The other results follow easily.

20.7 (a) There are only four non-zero possibilities:

$$\epsilon_{12}\epsilon_{12},\ \epsilon_{12}\epsilon_{21},\ \epsilon_{21}\epsilon_{12},\ \text{and}\ \epsilon_{21}\epsilon_{21},$$

of which the first and fourth have the value +1, and the other two have the value -1. Thus when $i = k$ and $j = l$ we must get +1, while when $i = l$ and $j = k$ we must get -1, and for all other possibilities (eg $i = j$) we must get zero. It can be easily checked that these requirements are met by the expression $\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$.
Setting $k = i$ and summing over i (using the summation convention) gives

$$(\delta_{ii}\delta_{jl} - \delta_{il}\delta_{ji}) = 2\delta_{jl} - \delta_{jl} = \delta_{jl}.$$

Setting $j = l$ and summing over j gives $\delta_{jj} = 2$.
(b) Multiplying out the determinant gives

$$\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}).$$

Setting $i = l$ and summing over i gives

$$\begin{aligned}\epsilon_{ijk}\epsilon_{imn} &= 3(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) + (\delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn}) \\ &= (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) \\ &= \begin{vmatrix} \delta_{jm} & \delta_{jn} \\ \delta_{km} & \delta_{kn} \end{vmatrix}\end{aligned}$$

as required. Setting now $j = m$ and summing over j gives

$$\epsilon_{ijk}\epsilon_{ijn} = \delta_{jj}\delta_{kn} - \delta_{jn}\delta_{kj} = 2\delta_{kn},$$

and finally setting $k = n$ and summing over k gives

$$\epsilon_{ijk}\epsilon_{ijk} = 2\delta_{kk} = 6.$$

(c) By analogy with the 2- and 3-dimensional cases, we expect that the product of two 4-dimensional ϵ's will have the form

$$\epsilon_{\alpha\beta\gamma\delta}\epsilon_{\lambda\mu\nu\rho} = S \begin{vmatrix} g_{\alpha\lambda} & g_{\alpha\mu} & g_{\alpha\nu} & g_{\alpha\rho} \\ g_{\beta\lambda} & g_{\beta\mu} & g_{\beta\nu} & g_{\beta\rho} \\ g_{\gamma\lambda} & g_{\gamma\mu} & g_{\gamma\nu} & g_{\gamma\rho} \\ g_{\delta\lambda} & g_{\delta\mu} & g_{\delta\nu} & g_{\delta\rho} \end{vmatrix},$$

where S is +1 or -1, to be determined by looking at a special case. [From covariance, we know that the only tensors available to construct the RHS are the g's, and we also know that the expression must be antisymmetric under interchange of any pair of the indices α, β, γ and δ of the first ϵ symbol, and also under interchange of any pair of the indices λ, μ, ν and ρ of the second ϵ symbol: the determinant automatically builds in these antisymmetry requirements (as in the previous cases in (a) and (b)).] The special case $\alpha = \lambda = 0, \beta = \mu = 1, \gamma = \nu = 2, \delta = \rho = 3$ yields

$$\epsilon_{0123}\epsilon_{0123} = 1 = S \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -S,$$

and hence $S = -1$.

We are now interested in successive 'contractions' of the indices between the two ϵ symbols. The first is

$$\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\alpha}{}_{\mu\nu\rho} = \epsilon_{\alpha\beta\gamma\delta}\epsilon_{\lambda\mu\nu\rho}g^{\alpha\lambda}.$$

Using the formula for the product of two ϵ's (with $S = -1$) we obtain

$$\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\alpha}{}_{\mu\nu\rho} = - \begin{vmatrix} g^{\alpha}_{\alpha} & g_{\alpha\mu} & g_{\alpha\nu} & g_{\alpha\rho} \\ g^{\alpha}_{\beta} & g_{\beta\mu} & g_{\beta\nu} & g_{\beta\rho} \\ g^{\alpha}_{\gamma} & g_{\gamma\mu} & g_{\gamma\nu} & g_{\gamma\rho} \\ g^{\alpha}_{\delta} & g_{\delta\mu} & g_{\delta\nu} & g_{\delta\rho} \end{vmatrix}.$$

Note, by the way, that we could just as well write δ^{α}_{β} for g^{α}_{β}, etc, as we have in fact done in (20.107). Consider the expansion of this determinant about the first row. The first term is

$$-g^{\alpha}_{\alpha} \begin{vmatrix} g_{\beta\mu} & g_{\beta\nu} & g_{\beta\rho} \\ g_{\gamma\mu} & g_{\gamma\nu} & g_{\gamma\rho} \\ g_{\delta\mu} & g_{\delta\nu} & g_{\delta\rho} \end{vmatrix}.$$

The value of g_α^α is 4 (remember, it is summed over $\alpha = 0,1,2,3$, and $g^{00} = g_0^0 = 1, g^{11} = -1 = -g_1^1$ etc.) The second term in the expansion of the 4×4 determinant is

$$+g_{\alpha\mu}\begin{vmatrix} g_\beta^\alpha & b_{\beta\nu} & g_{\beta\rho} \\ g_\gamma^\alpha & b_{\gamma\nu} & g_{\gamma\rho} \\ g_\delta^\alpha & b_{\delta\nu} & g_{\delta\rho} \end{vmatrix} = \begin{vmatrix} g_{\beta\mu} & g_{\beta\nu} & g_{\beta\rho} \\ g_{\gamma\mu} & g_{\gamma\nu} & g_{\gamma\rho} \\ g_{\delta\mu} & g_{\delta\nu} & g_{\delta\rho} \end{vmatrix}.$$

The third and fourth terms can be similarly worked out, and are found to be equal to the second term, so that the result for this 'once-contracted' product is

$$\epsilon_{\alpha\beta\gamma\delta}\epsilon^\alpha{}_{\mu\nu\rho} = -\begin{vmatrix} g_{\beta\mu} & g_{\beta\nu} & g_{\beta\rho} \\ g_{\gamma\mu} & g_{\gamma\nu} & g_{\gamma\rho} \\ g_{\delta\mu} & g_{\delta\nu} & g_{\delta\rho} \end{vmatrix}.$$

Proceeding in the same way, we find the twice-contracted result (compare (20.107))

$$\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\alpha\beta}{}_{\nu\rho} = -2\begin{vmatrix} g_{\gamma\nu} & g_{\gamma\rho} \\ g_{\delta\nu} & g_{\delta\rho} \end{vmatrix},$$

and from this the thrice-contracted result

$$\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\alpha\beta\gamma}{}_\rho = -6g_{\delta\rho}$$

and finally

$$\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\alpha\beta\gamma\delta} = -24.$$

20.8 The tensor $N_{\mu\nu}$ is defined in (20.97). Using (20.100), (20.101) and (20.102) we obtain

$$N_{\mu\nu} = 8[k'_\mu k_\nu + k'_\nu k_\mu + (q^2/2)g_{\mu\nu}] - 8\mathrm{i}\epsilon_{\alpha\beta\gamma\delta}k^\alpha g_\nu^\beta k'^\gamma g_\mu^\delta.$$

The term in ϵ can be re-written as

$$-8\mathrm{i}\epsilon_{\alpha\nu\gamma\mu}k^\alpha k'^\gamma = -8\mathrm{i}\epsilon_{\alpha\nu\beta\mu}k^\alpha k'^\beta = -8\mathrm{i}\epsilon_{\mu\nu\alpha\beta}k^\alpha k'^\beta,$$

as in (20.108), the last step following after making an even number of interchanges of the indices of the ϵ symbol. (20.109) follows immediately, and (20.113) is explained in the text. We now have to evaluate $N_{\mu\nu}E_{\text{eff}}^{\mu\nu}$. This is

$$\begin{aligned} N_{\mu\nu}E_{\text{eff}}^{\mu\nu} &= [8(k'_\mu k_\nu + k'_\nu k_\mu) + 4q^2 g_{\mu\nu} - 8\mathrm{i}\epsilon_{\mu\nu\alpha\beta}k^\alpha k'^\beta] \\ &\times\ [8p^\mu p^\nu + 2q^2 g^{\mu\nu} - 4\mathrm{i}\epsilon^{\mu\nu\gamma\delta}p_\gamma q_\delta]. \end{aligned}$$

Note now that the only term involving the ϵ symbols which survives is the one in which they are contracted with each other, since the other parts of $N_{\mu\nu}$ and $E^{\mu\nu}_{\text{eff}}$ are symmetric in μ and ν, while the ϵ's are antisymmetric. The non-ϵ part of the product '$N \cdot E$' is

$$128(p \cdot k')(p \cdot k) + 32p^2q^2 + 32q^2(k \cdot k') + 32(q^2)^2.$$

Since we are negelecting all masses, we have $p^2 = 0$, together with the relations (20.114)-(20.116). Using these we obtain for this part of '$N \cdot E$' the result

$$-32us - 16t^2 + 32t^2 = -32us + 16t^2 = -32us + 16(u+s)^2 = 16(u^2+s^2),$$

which is the first term in (20.119). The product of the two ϵ pieces gives

$$\begin{aligned}&-32\epsilon_{\mu\nu\alpha\beta}k^\alpha k'^\beta \epsilon^{\mu\nu\gamma\delta}p_\gamma q_\delta \\ &= 64k^\alpha k'^\beta p_\gamma q_\delta (g^\gamma_\alpha g^\delta_\beta - g^\delta_\alpha g^\gamma_\beta) = 64[(k \cdot p)(k' \cdot q) - (k \cdot q)(k' \cdot p)]\end{aligned}$$

where we have used the 'twice-contracted' result of problem 20.7(c). We have $k' \cdot q = k' \cdot (k - k') = k \cdot k' = -t/2$, and $k \cdot q = k \cdot (k - k') = -k \cdot k' = t/2$. Hence this contribution is

$$-16st + 16tu = 16t(u - s) = -16(u + s)(u - s) = 16(s^2 - u^2)$$

which is the second term in (20.119), the total for '$N \cdot E$' being $32s^2$ as in (20.121). Since $t = -2\boldsymbol{p}^2(1 - \cos\theta)$ and $s = 4\boldsymbol{p}^2$, where $\boldsymbol{p}$ is the CM 3-momentum and θ is the CM scattering angle, we have

$$\mathrm{d}\Omega = 2\pi \mathrm{d}\cos\theta = (4\pi/s)\mathrm{d}t$$

and hence (20.123) becomes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{G_{\mathrm{F}}^2}{\pi}.$$

The total cross section follows from integrating $\mathrm{d}\sigma/\mathrm{d}\Omega$ over θ, giving a factor of 4π and hence the result (20.124).

20.9 *Please note that there is an error in the given expression for $\mathcal{M}$. It should read*

$$\mathcal{M} = -G_{\mathrm{F}} \cos\theta_{\mathrm{C}} f_\pi p^\mu \bar{u}(\nu)\gamma_\mu(1-\gamma_5)v(\mathrm{e})$$

as in (18.52). The decay rate using this invariant amplitude has been calculated in problem 18.4, and the result leads to the given expression for the ratio of rates.

Consider now repeating the calculation for the positron mode, with the invariant amplitude

$$\mathcal{M}' = -G_{\mathrm{F}} \cos\theta_{\mathrm{C}} f_\pi p^\mu \bar{u}(\nu)\gamma_\mu(g_{\mathrm{V}}+g_{\mathrm{A}}\gamma_5)v(\mathrm{e}),$$

and taking $m_\nu \neq 0$. The spinor part is

$$\begin{aligned}
&-\bar{u}_\nu(p_2)(\not{p}_1+\not{p}_2)(g_{\mathrm{V}}+g_{\mathrm{A}}\gamma_5)v_{\mathrm{e}}(p_1)\\
&=-\bar{u}_\nu(p_2)(m_\nu+\not{p}_1)(g_{\mathrm{V}}+g_{\mathrm{A}}\gamma_5)v_{\mathrm{e}}(p_1)\\
&=-\bar{u}_\nu(p_2)[(m_\nu-m_{\mathrm{e}})g_{\mathrm{V}}+\gamma_5(m_\nu+m_{\mathrm{e}})g_{\mathrm{A}}]v_{\mathrm{e}}(p_1)\\
&=\bar{u}_\nu(p_2)[(m_{\mathrm{e}}-m_\nu)g_{\mathrm{V}}-g_{\mathrm{A}}\gamma_5(m_{\mathrm{e}}+m_\nu)]v_{\mathrm{e}}(p_1).
\end{aligned}$$

The required Trace is then

$$\begin{aligned}
&\mathrm{Tr}\{[(m_{\mathrm{e}}-m_\nu)g_{\mathrm{V}}-g_{\mathrm{A}}\gamma_5(m_{\mathrm{e}}+m_\nu)](\not{p}_1-m_{\mathrm{e}})\\
&\times[(m_{\mathrm{e}}-m_\nu)g_{\mathrm{V}}+g_{\mathrm{A}}\gamma_5(m_{\mathrm{e}}+m_\nu)](\not{p}_2+m_\nu)]\}\\
&=\mathrm{Tr}\{[(m_{\mathrm{e}}-m_\nu)g_{\mathrm{V}}\not{p}_1-(m_{\mathrm{e}}-m_\nu)m_{\mathrm{e}}g_{\mathrm{V}}-g_{\mathrm{A}}\gamma_5(m_{\mathrm{e}}+m_\nu)\not{p}_1+g_{\mathrm{A}}\gamma_5 m_{\mathrm{e}}(m_{\mathrm{e}}+m_\nu)]\\
&\times[(m_{\mathrm{e}}-m_\nu)g_{\mathrm{V}}\not{p}_2+(m_{\mathrm{e}}-m_\nu)m_\nu g_{\mathrm{V}}+g_{\mathrm{A}}\gamma_5(m_{\mathrm{e}}+m_\nu)\not{p}_2+g_{\mathrm{A}}\gamma_5 m_\nu(m_{\mathrm{e}}+m_\nu)]\}.
\end{aligned}$$

Using the Trace theorems from section J.2 of appendix J of volume 1, the surviving terms are as follows:

$$(m_{\mathrm{e}}-m_\nu)^2 g_{\mathrm{V}}^2\,\mathrm{Tr}(\not{p}_1\not{p}_2) = (m_{\mathrm{e}}-m_\nu)^2 g_{\mathrm{V}}^2 4p_1\cdot p_2;$$

$$-(m_{\mathrm{e}}-m_\nu)^2 m_{\mathrm{e}} m_\nu g_{\mathrm{V}}^2\,\mathrm{Tr}(1) = -4(m_{\mathrm{e}}-m_\nu)^2 m_{\mathrm{e}} m_\nu g_{\mathrm{V}}^2;$$

$$-(m_{\mathrm{e}}+m_\nu)^2 g_{\mathrm{A}}^2\,\mathrm{Tr}(\gamma_5\not{p}_1\gamma_5\not{p}_2) = (m_{\mathrm{e}}+m_\nu)^2 g_{\mathrm{A}}^2 4p_1\cdot p_2;$$

$$(m_{\mathrm{e}}+m_\nu)^2 g_{\mathrm{A}}^2 m_{\mathrm{e}} m_\nu\,\mathrm{Tr}(\gamma_5^2) = 4(m_{\mathrm{e}}+m_\nu)^2 g_{\mathrm{A}}^2 m_{\mathrm{e}} m_\nu.$$

Also, we have

$$m_\pi^2 = p^2 = p_1^2 + 2p_1\cdot p_2 + p_2^2 = m_{\mathrm{e}}^2 + 2p_1\cdot p_2 + m_\nu^2$$

and so

$$2p_1 \cdot p_2 = m_\pi^2 - m_\mathrm{e}^2 - m_\nu^2.$$

Hence we find

$$\begin{aligned}\sum_{\text{spins}} |\mathcal{M}'|^2 &= 2(G_\mathrm{F}\cos\theta_\mathrm{C} f_\pi)^2[(m_\mathrm{e}-m_\nu)^2 g_\mathrm{V}^2 + (m_\mathrm{e}+m_\nu)^2 g_\mathrm{A}^2](m_\pi^2 - m_\mathrm{e}^2 - m_\nu^2) \\ &+ 4m_\mathrm{e}m_\nu[(m_\mathrm{e}+m_\nu)^2 g_\mathrm{A}^2 - (m_\mathrm{e}-m_\nu)^2 g_\mathrm{V}^2].\end{aligned}$$

It is interesting to note that the signs of g_V and g_A are irrelevant here. For the phase-space integration we use $\delta(m_\pi - W)$ where

$$W = (\boldsymbol{p}^2 + m_\mathrm{e}^2)^{1/2} + (\boldsymbol{p}^2 + m_\nu^2)^{1/2} \equiv E_1 + E_2$$

so that

$$\frac{\mathrm{d}W}{\mathrm{d}|\boldsymbol{p}|} = \frac{|\boldsymbol{p}|W}{E_1E_2}.$$

The phase-space integral can then be done via

$$|\boldsymbol{p}|\mathrm{d}|\boldsymbol{p}|\delta(m_\pi - W) = \frac{E_1E_2}{W}\mathrm{d}W\delta(m_\pi - W).$$

Finally, we have

$$m_\pi - (\boldsymbol{p}^2 + m_\nu^2)^{1/2} = (\boldsymbol{p}^2 + m_\mathrm{e}^2)^{1/2},$$

and by squaring this relation one finds

$$|\boldsymbol{p}| = [((m_\pi + m_\nu)^2 - m_\mathrm{e}^2)((m_\pi - m_\nu)^2 - m_\mathrm{e}^2)/(4m_\pi^2)]^{1/2}.$$

The ratio of the rates can then be evaluated. For $m_\nu \ll m_\mathrm{e}$, the sum over spins of the squared amplitude is approximately proportional to m_e^2, giving a strong suppression of the positron mode relative to the μ^+ mode, as before, independently of the numerical values of g_V and g_A.

20.10 (a) From (20.158) we have

$$\mathrm{d}\sigma^{(\nu)} = \frac{G_\mathrm{F}^2}{2}\frac{1}{4k \cdot p}4\pi M\, N_{\mu\nu}W_{(\nu)}^{\mu\nu}\frac{\mathrm{d}^3\boldsymbol{k}'}{2k'(2\pi)^3},$$

where $N_{\mu\nu}$ is given in (20.108), and $W_{(\nu)}^{\mu\nu}$ in (20.153). In the contraction '$N \cdot W_{(\nu)}$', the terms involving only one ϵ symbol vanish, using the fact that the non-ϵ pieces are symmetric in μ and ν, while ϵ is antisymmetric. There are two non-ϵ terms. The first is

$$8[(-2k\cdot k' - 2q^2)W_1^{(\nu)}] = 16k\cdot k' W_1^{(\nu)}.$$

Note also that, in the laboratory frame, $k\cdot k' = kk'(1-\cos\theta) = 4kk'\cos^2\theta$, where we are (rather loosely) using k and k' here for the moduli of the three-momentum parts of the four-vectors k and k'. This term is therefore

$$32kk'\sin^2(\theta/2)\, W_1^{(\nu)}.$$

The second non-ϵ term is

$$\frac{8}{M^2}(2(p\cdot k)(p\cdot k') + p^2q^2/2)W_2^{(\nu)}.$$

Here $p\cdot k = Mk$, and $p\cdot k' = Mk'$ (with the neglect of all masses k and k' are here equally well denoting energies), while

$$p^2q^2/2 = -M^2kk'(1-\cos\theta).$$

Hence this term is

$$16kk'\cos^2(\theta/2)\, W_2^{(\nu)}.$$

The '$\epsilon\cdot\epsilon$ term is

$$-\frac{8}{2M^2}\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\gamma\delta}k^\alpha k'^\beta p_\gamma q_\delta W_3^{(\nu)}.$$

Using the 'twice-contracted ϵ' formula of problem 20.7(c), this becomes

$$\frac{8}{M^2}(k\cdot p\, k'\cdot q - k\cdot q\, k'\cdot p)W_3^{(\nu)} = 16\frac{k+k'}{M}kk'\sin^2(\theta/2)\, W_3^{(\nu)}.$$

Thus altogether from '$N\cdot W$' we get

$$16kk'\{W_2^{(\nu)}\cos^2(\theta/2) + 2W_1^{(\nu)}\sin^2(\theta/2) + \frac{k+k'}{M}\sin^2(\theta/2)W_3^{(\nu)}\}.$$

Meanwhile, the phase-space factor, in the laboratory frame, is

$$\frac{1}{2k'}k'^2\mathrm{d}k' 2\pi\mathrm{d}\cos\theta/(8\pi^3).$$

To convert from $\mathrm{d}k'\mathrm{d}\cos\theta$ to $\mathrm{d}Q^2\mathrm{d}\nu$, we use

$$Q^2 = -(k-k')^2 = 2kk'(1-\cos\theta), \quad \nu = \frac{p\cdot q}{M} = (k-k'),$$

which implies

$$\mathrm{d}Q^2\mathrm{d}\nu = 2kk'\mathrm{d}k'\mathrm{d}\cos\theta.$$

Putting these results together leads to (20.159).
(b) We have

$$Q^2 = 2M\nu x, \qquad y = \nu/k,$$

so that

$$Q^2 = 2Mkxy, \qquad \nu = ky.$$

The Jacobian for the transformation $\mathrm{d}Q^2\mathrm{d}\nu \to \mathrm{d}x\mathrm{d}y$ is then (see also equation (9.19) and problem 9.2 in volume 1)

$$\begin{vmatrix} \frac{\partial Q^2}{\partial x} & \frac{\partial Q^2}{\partial y} \\ \frac{\partial \nu}{\partial x} & \frac{\partial \nu}{\partial y} \end{vmatrix} = \begin{vmatrix} 2Mky & 2Mkx \\ 0 & k \end{vmatrix} = 2Mk^2y.$$

Hence

$$\begin{aligned} \frac{\mathrm{d}^2\sigma^{(\nu)}}{\mathrm{d}x\mathrm{d}y} &= \frac{1}{2\pi}2MkG_{\mathrm{F}}^2\{k'y\cos^2(\theta/2)W_2^{(\nu)} + 2k'y\sin^2(\theta/2)W_1^{(\nu)} \\ &\quad + \frac{(k+k')}{M}yk'\sin^2(\theta/2)W_3^{(\nu)}\}. \end{aligned}$$

But $Q^2 = 4kk'\sin^2(\theta/2)$. Hence

$$\frac{k'\sin^2(\theta/2)}{M} = \frac{Q^2}{4Mk} = \frac{Q^2}{2M\nu}\frac{\nu}{2k} = \frac{xy}{2}$$

(as stated in the problem). In addition, $y(k+k') = \nu(k+k-ky)/k = 2\nu(1-y/2)$. Also,

$$\begin{aligned} k'y\cos^2(\theta/2) &= k'\frac{\nu}{k}\cos^2(\theta/2) = \frac{\nu}{k}k'(1-\sin^2(\theta/2)) = \frac{\nu k'}{k} - \frac{\nu}{k}\frac{Mxy}{2} \\ &= \nu(1-y) - \frac{\nu}{k}\frac{Mxy}{2}. \end{aligned}$$

Neglecting the term in M here (which amounts to taking $\theta \approx 0$ in the high-energy limit, in the laboratory system), and using $s = 2Mk$,

together with the Bjorken scaling behaviour, leads to the stated expression for $\mathrm{d}^2\sigma^{(\nu)}/\mathrm{d}x\mathrm{d}y$.
(c) Using the C-G relation we obtain

$$\begin{aligned}
\frac{\mathrm{d}^2\sigma^{(\nu)}}{\mathrm{d}x\mathrm{d}y} &= \frac{G_{\mathrm{F}}^2}{2\pi}s[F_2^{(\nu)}(1-y)+\frac{1}{2}F_2^{(\nu)}y^2+F_3^{(\nu)}(1-y/2)yx] \\
&= \frac{G_{\mathrm{F}}^2}{2\pi}sF_2^{(\nu)}[(1-y)+y^2/2+\frac{xF_3^{(\nu)}}{F_2^{(\nu)}}\frac{y(2-y)}{2}] \\
&= \frac{G_{\mathrm{F}}^2}{2\pi}sF_2^{(\nu)}\left(\frac{1+(1-y)^2}{2}+\frac{xF_3^{(\nu)}}{F_2^{(\nu)}}\frac{1-(1-y)^2}{2}\right).
\end{aligned}$$

20.11 Consider the process

$$\nu_\mu(k)+\bar{\mathrm{u}}(p)\to\mu^-+\bar{\mathrm{d}}(p')$$

where the symbols in brackets are the 4-momenta. This is equivalent to

$$\nu_\mu(k)+\mathrm{d}(-p')\to\mu^-(k')+\mathrm{u}(-p), \qquad \text{(R)}$$

for which the matrix element is (by analogy with (20.93), but using a slightly different labelling of the spinors)

$$-\mathrm{i}G_{\mathrm{F}}\bar{u}_\mu(k')\gamma_\mu(1-\gamma_5)u_{\nu_\mu}(k)\bar{v}_{\mathrm{u}}(p)\gamma^\mu(1-\gamma_5)v_{\mathrm{d}}(p').$$

The cross section is obtained by following the steps used for $\nu_\mu+\mathrm{e}^-\to\mu^-+\nu_{\mathrm{e}}$ in section 20.8 and problem 20.8. The only difference in the present case is that, in the second Trace in equation (20.95), the order of p and p' is reversed (see (R)); note that since we are neglecting masses, $u\bar{u}=v\bar{v}$. This means that, while the '$\nu_\mu-\mu^-$' tensor $N_{\mu\nu}$ stays the same, the antiquark tensor $E^{\mu\nu}$ is the same as (20.109) but with p and p' interchanged. Since their product multiplies the ϵ symbol, which is antisymmetric under the interchange of any pair of indices, it follows that the antiquark tensor can be written as in (20.109) but with a '+i' rather than '−i'. This in turn implies that the contribution from the '$\epsilon\cdot\epsilon$' term has the opposite sign from that in problem 20.8, so that instead of the expression $16(s^2+u^2)+16(s^2-u^2)=32s^2$ found there, in this case we have $16(s^2+u^2)-16(s^2-u^2)=32u^2$ - that is, s^2 is replaced by u^2 in the spin-summed squared amplitude. Hence, from (20.123) we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\nu\bar{\mathrm{q}}) = \frac{G_{\mathrm{F}}^2}{4\pi^2 s}u^2,$$

or equivalently from (20.125)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}(\nu\bar{\mathrm{q}}) = \frac{G_{\mathrm{F}}^2}{\pi}\frac{u^2}{s^2}.$$

Now

$$y = \frac{p\cdot q}{p\cdot k} = \frac{p\cdot(k-k')}{p\cdot k} = 1 - \frac{p\cdot k'}{p\cdot k} = 1 + \frac{u}{s},$$

and hence

$$\frac{u^2}{s^2} = (1-y)^2$$

and

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}(\nu\bar{\mathrm{q}}) = \frac{G_{\mathrm{F}}^2}{\pi}(1-y)^2$$

as required. Note that in the CM frame, $u = -2\boldsymbol{p}^2(1+\cos\theta)$, where $\boldsymbol{p}$ is the CM 3-momentum and θ is the CM scattering angle; it follows that the amplitude vanishes in the backward direction $\theta = \pi$, as discussed below figure 20.5.

(b) We have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2}(\nu\mathrm{q}) = \frac{G_{\mathrm{F}}^2}{\pi}.$$

Following steps analogous to those in section 9.2, we can write this as

$$\begin{aligned}\frac{\mathrm{d}^2\sigma}{\mathrm{d}Q^2\mathrm{d}\nu}(\nu\mathrm{q}) &= \frac{G_{\mathrm{F}}^2}{\pi}\delta(\nu - Q^2/2Mx) \\ &= \frac{G_{\mathrm{F}}^2}{\pi}\frac{x}{\nu}\delta(x - Q^2/2M\nu),\end{aligned}$$

using property (E.29) of the δ function. Converting to the variables x and y via the Jacobian known from problem 20.10(b), and using $s = 2Mk$, leads to

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}Q^2\mathrm{d}\nu}(\nu\mathrm{q}) = \frac{G_{\mathrm{F}}^2}{\pi}sx\delta(x - Q^2/2M\nu)$$

as required. The $\bar{\mathrm{q}}$ cross section follows similarly.
(c) From problem 20.10(c) we have

$$\begin{aligned}
\frac{\mathrm{d}^2\sigma^{(\nu)}}{\mathrm{d}x\mathrm{d}y} &= \frac{G_{\mathrm{F}}^2}{2\pi}s[F_2^{(\nu)}(1-y+y^2/2)+xF_3^{(\nu)}(y-y^2/2)] \\
&= \frac{G_{\mathrm{F}}^2}{2\pi}s[F_2^{(\nu)}+F_2^{(\nu)}(y^2/2-y)-xF_3^{(\nu)}(y^2/2-y)] \\
&= \frac{G_{\mathrm{F}}^2}{\pi}sx[(q(x)+\bar{q}(x))+2\bar{q}(x)(y^2/2-y)].
\end{aligned}$$

Equating coefficients of '1' and of '$y^2/2-y$' we find

$$F_2^{(\nu)} = 2x[q(x)+\bar{q}(x)],$$

and

$$F_2^{(\nu)} - xF_3^{(\nu)} = 4x\bar{q}(x),$$

which lead to

$$F_3^{(\nu)} = 2[q(x)+\bar{q}(x)],$$

and the indicated ratio for $xF_3^{(\nu)}(x)/F_2^{(\nu)}(x)$.

Chapter 21

21.1 (a) For $\boldsymbol{p} = |\boldsymbol{p}|(\sin\theta, 0, \cos\theta)$, we have

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

Setting

$$\phi_+ = \begin{pmatrix} a \\ b \end{pmatrix},$$

we have

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix},$$

which leads to

$$a(1-\cos\theta) = b\sin\theta, \quad \text{and} \quad a\sin\theta = b(1+\cos\theta).$$

Changing to half-angles, we see that both these relations become simply

$$a\sin\theta/2 = b\cos\theta/2,$$

so that a suitable (normalized) solution is

$$\phi_+ = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}.$$

Similarly (or simply using orthogonality) ϕ_- is

$$\phi_- = \begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}.$$

Hence u is

$$u = E^{1/2}\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \\ -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}. \qquad \text{(E)}$$

Since for massless particles the spinors u and v obey the same equation, the v spinors will have the same form (but the helicity labelling will be different - see page 288).
(b) For the spinor $u(\mu^-)$ we set $\theta = 0$ in part (a) and find

$$u(\mu^-) = E^{1/2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and so

$$(1-\gamma_5)u(\mu^-) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}.$$

For $v(\bar{\nu}_\mu)$ we take $\theta = \pi$ and obtain

$$v(\bar{\nu}_\mu) = E^{1/2}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

It is then simple to check that the $\mu = 0$ and $\mu = z$ or 3 components of $\bar{v}(\bar{\nu}_\mu)\gamma_\mu(1-\gamma_5)u(\mu^-)$ both vanish. On the other hand, noting that $\gamma_\mu = (\beta, -\boldsymbol{\gamma})$ and hence

$$\beta\gamma_x = -\begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix}, \qquad \text{and} \qquad \beta\gamma_y = -\begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix},$$

we find that the $\mu = x$ component is $-2E$ and the $\mu = y$ component is $2\mathrm{i}E$.

We now consider the other half of the matrix element. For $u(\mathrm{e}^-)$ we take (E), and for $v(\bar{\nu}_\mathrm{e})$ we take (E) with θ replaced by $\theta + \pi$, since this is the opposite direction to that of the electron in the CM frame. This gives

$$(1-\gamma_5)v(\bar{\nu}_\mathrm{e}) = \begin{pmatrix} 0 \\ 0 \\ -2\cos\theta/2 \\ -2\sin\theta/2 \end{pmatrix},$$

and

$$\begin{aligned} \beta\gamma^x(1-\gamma_5)v(\bar{\nu}_\mathrm{e}) &= \begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -2\cos\theta/2 \\ -2\sin\theta/2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 2\sin\theta/2 \\ 2\cos\theta/2 \end{pmatrix}. \end{aligned}$$

It follows that

$$u^\dagger(\mathrm{e}^-)\beta\gamma^x(1-\gamma_5)v(\bar{\nu}_\mathrm{e}) = 2E(\cos^2\theta/2 - \sin^2\theta/2) = 2E\cos\theta.$$

Similarly, we find

$$u^\dagger(\mathrm{e}^-)\beta\gamma^y(1-\gamma_5)v(\bar{\nu}_\mathrm{e}) = 2\mathrm{i}E(\cos^2\theta/2 + \sin^2\theta/2) = 2\mathrm{i}E.$$

We need to multiply the two $\mu = x$ matrix elements, and the two $\mu = y$ matrix elements, and add these two products together; this gives $-4E^2(1+\cos\theta)$.

21.2 Consider the term $\not{k}_1\not{p}_1\not{k}_1$ in the Trace. We have

$$\not{k}_1\not{p}_1\not{k}_1 = (-\not{p}_1\not{k}_1 + 2p_1\cdot k_1)\not{k}_1 = (-\not{p}_1 M_\mathrm{W}^2 + 2p_1\cdot k_1\not{k}_1).$$

We drop the term proportional to M_W^2, since the other term will contain an extra power of energy, by comparison. We can perform a similar manipulation by moving the $\not{k}_2$ from the beginning to the end of the Trace, and writing

$$\not{k}_2\not{p}_2\not{k}_2 \approx 2p_2\cdot k_2\not{k}_2.$$

We also have, to leading order in powers of the energy,

$$(p_1-k_1)^2 = (p_2-k_2)^2 \approx -2p_1\cdot k_1 = -2p_2\cdot k_2.$$

These steps reduce the Trace to

$$\frac{g^4}{M_\mathrm{W}^4}\mathrm{Tr}[(1-\gamma_5)(\not{p}_1-\not{k}_1)\not{k}_1(\not{p}_1-\not{k}_1)\not{k}_2] \approx -\frac{g^4}{M_\mathrm{W}^4}\mathrm{Tr}[(1-\gamma_5)\not{p}_1\not{k}_1\not{p}_2\not{k}_2],$$

where we have again discarded terms involving M_W^2. The '1' term in the Trace gives (using (J.31) from volume 1)

$$\begin{aligned}&-4\frac{g^4}{M_\mathrm{W}^4}[(p_1\cdot k_1)(p_2\cdot k_2) + (p_1\cdot k_2)(k_1\cdot p_2) - (p_1\cdot p_2)(k_1\cdot k_2)]\\ &= \frac{g^4}{M_\mathrm{W}^4}(s^2-u^2-t^2) = \frac{g^4}{M_\mathrm{W}^4}2tu,\end{aligned}$$

where we have also used (20.117). Since $t = -2E^2(1-\cos\theta)$ and $u = -2E^2(1+\cos\theta)$, we find that the spin-summed squared matrix element is proportional to $(g^4/M_\mathrm{W}^4)E^4(1-\cos^2\theta)$, as stated.
The 'γ_5' term in the Trace does not contribute. Evaluating it via (J.37) gives something proportional to

$$\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha k_1^\beta p_2^\gamma k_2^\delta;$$

but we can replace, say, k_2 here by $p_1 + p_2 - k_1$, and each of these 4-vectors already appears contracted with the ϵ symbol. It follows from the antisymmetry of ϵ that this contribution vanishes.

21.3 The sum of the amplitudes for figures 21.6(a) and 21.6(b) is (cf (21.29))

$$e^2\{\bar{v}(p_2)\not{\epsilon}_2 \frac{1}{\not{p}_1 - \not{k}_1 - m_{\mathrm{e}}}\not{\epsilon}_1 u(p_1) + \bar{v}(p_2)\not{\epsilon}_1 \frac{1}{\not{p}_1 - \not{k}_2 - m_{\mathrm{e}}}\not{\epsilon}_2 u(p_1)\}.$$

This in fact should vanish when the polarization vector of *either* photon is replaced by its momentum, since this is a consequence of electromagnetic gauge invariance, as discussed in section 8.6 and problem 8.15. We check this property under $\epsilon_1 \to k_1$. The first ('direct') amplitude, for figure 21.6(a), becomes

$$e^2\bar{v}(p_2)\not{\epsilon}_2 \frac{1}{\not{p}_1 - \not{k}_1 - m_{\mathrm{e}}}\not{k}_1 u(p_1). \qquad \text{(F)}$$

But we may replace $\not{k}_1 u(p_1)$ here by $(\not{p}_1 + \not{p}_2 - \not{k}_2)u(p_1)$ and then use the Dirac equation to get

$$(m + \not{p}_2 - \not{k}_2)u(p_1) = (m + \not{k}_1 - \not{p}_1)u(p_1) = -(\not{p}_1 - \not{k}_1 - m_{\mathrm{e}})u(p_1),$$

so that (F) becomes just

$$-e^2\bar{v}(p_2)\not{\epsilon}_2 u(p_1).$$

Similarly, in the second ('exchange') amplitude of figure 21.6(b), we use

$$\bar{v}(p_2)\not{k}_1 = \bar{v}(p_2)(\not{p}_1 + \not{p}_2 - \not{k}_2) = \bar{v}(p_2)(\not{p}_1 - m_{\mathrm{e}} - \not{k}_2)$$

to reduce it to

$$e^2\bar{v}(p_2)\not{\epsilon}_2 u(p_1);$$

and the sum of the two amplitudes vanishes. We can also do a similar calculation in which, from the start, both polarization vectors are replaced by the corresponding momenta.

Chapter 22

22.1 The part of (22.23) which involves the Z and photon fields is

$$-\bar{\hat{l}}_f[\frac{1}{2}g\tau_3(\cos\theta_{\rm W}\,\hat{\not{Z}}+\sin\theta_{\rm W}\hat{\not{A}})-\frac{1}{2}g'(-\sin\theta_{\rm W}\,\hat{\not{Z}}+\cos\theta_{\rm W}\hat{\not{A}})]\left(\frac{1-\gamma_5}{2}\right)\hat{l}_f$$
$$+\bar{\hat{l}}_f g'(-\sin\theta_{\rm W}\,\hat{\not{Z}}+\cos\theta_{\rm W}\hat{\not{A}})\left(\frac{1+\gamma_5}{2}\right)\hat{l}_f,$$

where we have used $y=-1$ for the left-handed doublets, and $y=-2$ for the right-handed (charged) singlets. For the charged ($Q=-1$) fields, then, remembering that the left-handed component has $\tau_3=-1$, the part involving $\hat{Z}_\mu$ is

$$\begin{aligned}
&-\hat{Z}_\mu\{\frac{1}{2}\bar{\hat{l}}_f\gamma^\mu[-g\cos\theta_{\rm W}+g'\sin\theta_{\rm W}]\left(\frac{1-\gamma_5}{2}\right)\hat{l}_f+g'\sin\theta_{\rm W}\bar{\hat{l}}_f\gamma^\mu\left(\frac{1+\gamma_5}{2}\right)\hat{l}_f\}\\
&=-\hat{Z}_\mu\frac{g}{\cos\theta_{\rm W}}\{\bar{\hat{l}}_f\gamma^\mu\frac{1}{2}(-\cos^2\theta_{\rm W}+\sin^2\theta_{\rm W})\left(\frac{1-\gamma_5}{2}\right)\hat{l}_f+\sin^2\theta_{\rm W}\bar{\hat{l}}_f\gamma^\mu\left(\frac{1+\gamma_5}{2}\right)\hat{l}_f\}\\
&=-\hat{Z}_\mu\frac{g}{\cos\theta_{\rm W}}\{\bar{\hat{l}}_f\gamma^\mu(-\frac{1}{2}+\sin^2\theta_{\rm W})\left(\frac{1-\gamma_5}{2}\right)+\sin^2\theta_{\rm W}\bar{\hat{l}}_f\gamma^\mu\left(\frac{1+\gamma_5}{2}\right)\hat{l}_f\}.
\end{aligned}$$

The terms in $\sin^2\theta_{\rm W}$ combine to give just $\sin^2\theta_{\rm W}\bar{\hat{l}}_f\gamma^\mu\hat{l}_f$, which can alternatively be written as $-\sin^2\theta_{\rm W}Q\bar{\hat{l}}_f\gamma^\mu\hat{l}_f$ for this case, while the '$-1/2$' can be written as t_3. This contribution therefore reduces to

$$-\hat{Z}_\mu\frac{g}{\cos\theta_{\rm W}}\{\bar{\hat{l}}_f\gamma^\mu t_3\left(\frac{1-\gamma_5}{2}\right)\hat{l}_f-\sin^2\theta_{\rm W}Q\bar{\hat{l}}_f\gamma^\mu\hat{l}_f\}.\quad\text{(M)}$$

For the neutral field with $\tau_3=+1$, the part involving $\hat{Z}_\mu$ is

$$\begin{aligned}
&-\hat{Z}_\mu\bar{\hat{l}}_f\gamma^\mu[\frac{g}{2}\cos\theta_{\rm W}+\frac{g'}{2}\sin\theta_{\rm W}]\left(\frac{1-\gamma_5}{2}\right)\hat{l}_f\\
&=-\hat{Z}_\mu\frac{g}{\cos\theta_{\rm W}}\frac{1}{2}\bar{\hat{l}}_f\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)\hat{l}_f.\quad\text{(N)}
\end{aligned}$$

In this case we can replace the '1/2' by t_3; (M) and (N) together then yield (22.36).

22.2 [*Note that '(22.56)' should read '(22.58)'*]. The rate is

$$\Gamma_{\rm W} = \frac{1}{2M_{\rm W}}\frac{1}{3}\sum_{\lambda,\text{spins}}\frac{g^2}{2}\int|\epsilon_\mu(p,\lambda)\bar{u}(p_1)\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)v(p_2)|^2\times$$
$$\frac{{\rm d}^3\boldsymbol{p}_1}{(2\pi)^3 2E_1}\frac{{\rm d}^3\boldsymbol{p}_2}{(2\pi)^3 2E_2}(2\pi)^4\delta^{(4)}(p-p_1-p_2),$$

where λ is the polarization label of the W, which has 4-momentum p, and p_1 and p_2 are the 4-momenta of the e^- and $\bar{\nu}_{\rm e}$. We neglect lepton masses. The '1/3' arises from averaging over the W polarizations. The total energy W is

$$W = 2|\boldsymbol{p}|,$$

where $\boldsymbol{p}$ is the CM momentum of the e^-, so that (in the CM) $E_1 = E_2 = |\boldsymbol{p}|$, and

$${\rm d}W = 2{\rm d}|\boldsymbol{p}|.$$

The phase space term then reduces to

$$\frac{|\boldsymbol{p}|^2({\rm d}W/2)4\pi}{4\pi^2 4|\boldsymbol{p}|^2}\delta(W-2|\boldsymbol{p}|) = (1/8\pi).$$

The spin-summed squared matrix element is

$$\frac{1}{3}\sum_\lambda \epsilon_\mu\epsilon^*_\nu\frac{1}{2}{\rm Tr}[\gamma^\mu \not{p}_2(1+\gamma_5)\gamma^\nu \not{p}_1].$$

The average over polarizations gives (from (21.29) - *note that* '$M_{\rm W}$' *should read* '$M_{\rm W}^2$')

$$\frac{1}{3}\left(-g_{\mu\nu}+\frac{p_\mu p_\nu}{M_{\rm W}^2}\right). \qquad \text{(P)}$$

In the Trace, the γ_5 part will give (see (J.37)) something antisymmetric in μ and ν, which will be contracted with the symmetric quantity (P), so it won't contribute. The non-γ_5 part of the Trace is (see (J.31))

$${\rm Tr}[\gamma^\mu \not{p}_2\gamma^\nu \not{p}_1] = 4[p_2^\mu p_1^\nu + p_1^\mu p_2^\nu - g^{\mu\nu}p_1\cdot p_2].$$

Contracting this with (P) gives

$$\frac{4}{3}[-2p_1\cdot p_2+4p_1\cdot p_2+2\frac{(p_1\cdot p)(p_2\cdot p)}{M_{\rm w}^2}-\frac{p^2}{M_{\rm w}^2}p_1\cdot p_2].$$

From $p=p_1+p_2$ we obtain (neglecting lepton masses) $p_1\cdot p_2=(p^2)/2=(M_{\rm W}^2)/2$, $p\cdot p_1=p\cdot p_2=(m_{\rm W}^2)/2$. Putting the pieces together yields (22.58).

22.3 We have

$$\hat{\phi}=\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}(v+\hat{H})\end{pmatrix},\quad \hat{\phi}_{\bf C}={\rm i}\tau_2\hat{\phi}^*=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}(v+\hat{H})\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}(v+\hat{H})\\ 0\end{pmatrix}.$$

So (22.125) becomes

$$\begin{aligned}\hat{\mathcal{L}}_{\psi\phi} &= a_{ij}(\bar{\hat{u}}_{{\rm L}i}\bar{\hat{d}}_{{\rm L}i})\begin{pmatrix}\frac{1}{\sqrt{2}}(v+\hat{H})\\ 0\end{pmatrix}\hat{u}_{{\rm R}j}\\ &+ b_{ij}(\bar{\hat{u}}_{{\rm L}i}\bar{\hat{d}}_{{\rm L}i})\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}(v+\hat{H})\end{pmatrix}\hat{d}_{{\rm R}j}+{\rm h.c.}\\ &= a_{ij}\frac{1}{\sqrt{2}}\bar{\hat{u}}_{{\rm L}i}(v+\hat{H})\hat{u}_{{\rm R}j}+b_{ij}\frac{1}{\sqrt{2}}\bar{\hat{d}}_{{\rm L}i}(v+\hat{H})\hat{d}_{{\rm R}j}+{\rm h.c},\end{aligned}$$

which is easily seen to be the same as (22.127), with the identifications (22.128).

22.4 First, note that from (22.130) and (22.131) we have (repeated indices are summed over)

$$\hat{u}_{{\rm L}i}=U^{({\rm u})\dagger}_{{\rm L}i\alpha}\hat{u}_{{\rm L}\alpha},$$

and so

$$\hat{u}^\dagger_{{\rm L}j}=\hat{u}^\dagger_{{\rm L}\beta}U^{({\rm u})}_{{\rm L}\beta j},$$

and similarly for the other fields. Consider now the first term on the RHS of (22.133), involving the 'L' fields. The neutral part of this is

$$\begin{aligned}&-(\bar{\hat{u}}_{{\rm L}\beta}U^{({\rm u})}_{{\rm L}\beta j}\ \bar{\hat{d}}_{{\rm L}\beta}U^{({\rm d})}_{{\rm L}\beta j})\gamma^\mu(g\frac{\tau_3}{2}\hat{W}_\mu+g'\frac{1}{2}y\hat{B}_\mu)\begin{pmatrix}U^{({\rm u})\dagger}_{{\rm L}j\alpha}\hat{u}_{{\rm L}\alpha}\\ U^{({\rm d})\dagger}_{{\rm L}j\alpha}\hat{d}_{{\rm L}\alpha}\end{pmatrix}\\ &=-\frac{g}{2}[\bar{\hat{u}}_{{\rm L}\beta}\hat{\not{W}}(U^{({\rm u})}_{{\rm L}\beta j}U^{({\rm u})\dagger}_{{\rm L}j\alpha})\hat{u}_{{\rm L}\alpha}-\bar{\hat{d}}_{{\rm L}\beta}\hat{\not{W}}(U^{({\rm d})}_{{\rm L}\beta j}U^{({\rm d})\dagger}_{{\rm L}j\alpha})\hat{d}_{{\rm L}\alpha}]\\ &-\frac{g'y}{2}[\bar{\hat{u}}_{{\rm L}\beta}\hat{\not{B}}(U^{({\rm u})}_{{\rm L}\beta j}U^{({\rm u})\dagger}_{{\rm L}j\alpha})\hat{u}_{{\rm L}\alpha}+\bar{\hat{d}}_{{\rm L}\beta}\hat{\not{B}}(U^{({\rm d})}_{{\rm L}\beta j}U^{({\rm d})\dagger}_{{\rm L}j\alpha})\hat{d}_{{\rm L}\alpha}].\end{aligned}$$

If the matrices $U^{(\mathrm{u})}$ and $U^{(\mathrm{d})}$ are unitary, then we have

$$(U^{(\mathrm{u})}_{\mathrm{L}\beta j}U^{(\mathrm{u})\dagger}_{\mathrm{L}j\alpha}) = \delta_{\beta\alpha}, \quad \text{and} \quad (U^{(\mathrm{d})}_{\mathrm{L}\beta j}U^{(\mathrm{d})\dagger}_{\mathrm{L}j\alpha}) = \delta_{\beta\alpha},$$

and these terms are diagonal in the mass basis (since $\alpha = \beta$). The 'R' terms are similar but simpler, since there is no '2×2 $\boldsymbol{\tau}$' structure, and we simply require

$$U^{(\mathrm{u})}_{\mathrm{R}\beta j}U^{(\mathrm{u})\dagger}_{\mathrm{R}j\alpha} = \delta_{\beta\alpha} \quad \text{and} \quad U^{(\mathrm{d})}_{\mathrm{R}\beta j}U^{(\mathrm{d})\dagger}_{\mathrm{R}j\alpha} = \delta_{\beta\alpha},$$

which are the unitarity conditions again.

22.5 *There are some errors in (22.149), (22.150) and (22.151). (22.149) should read*

$$\hat{\mathbf{C}}\hat{\mathbf{P}}(\hat{W}^a_0(\boldsymbol{x},t), \hat{W}^a_i(\boldsymbol{x},t))(\hat{\mathbf{C}}\hat{\mathbf{P}})^{-1} = (\hat{W}^{a\dagger}_0(-\boldsymbol{x},t), -\hat{W}^{a\dagger}_i(-\boldsymbol{x},t)),$$

where a is the SU(2) index, and $(0,i)$ *are the 4-vector indices. Also, the RHS of (22.150) and (22.151) should be be prefixed by a minus sign.* Making use of (22.148), we have

$$\hat{\mathbf{C}}\hat{\mathbf{P}}(\hat{u}^\dagger_{\mathrm{L}}\gamma^0 V_{\mathrm{ud}}\hat{d}_{\mathrm{L}})(\hat{\mathbf{C}}\hat{\mathbf{P}})^{-1} = \hat{u}^{\mathrm{T}}\gamma^{0\dagger}\gamma^{2\dagger}(-\mathrm{i})\gamma^0 V_{\mathrm{ud}}\gamma^\mu(\mathrm{i})\gamma^2\gamma^0\hat{d}^{\dagger\mathrm{T}}_{\mathrm{L}},$$

where we have not indicated exp[licitly the argument of the transformed fields, which is $(-\boldsymbol{x},t)$. In matrix terms, this is a single number and is hence equal to its transpose, apart from an additional minus sign needed when we interchange the fermion fields. Hence the RHS side is

$$-V_{\mathrm{ud}}\hat{d}^\dagger_{\mathrm{L}}\gamma^{0\mathrm{T}}\gamma^{2\mathrm{T}}\gamma^{\mu\mathrm{T}}\gamma^{0\mathrm{T}}\gamma^{2*}\gamma^{0*}\hat{u}_{\mathrm{L}}.$$

Now

$$\gamma^{2\mathrm{T}} = \gamma^2, \quad \gamma^{0\mathrm{T}} = \gamma^{0*} = \gamma^0, \quad \text{and} \quad \gamma^{2*} = -\gamma^2,$$

so our expression becomes

$$-V_{\mathrm{ud}}\bar{\hat{d}}_{\mathrm{L}}(\gamma^2\gamma^{\mu\mathrm{T}}\gamma^2)\hat{u}_{\mathrm{L}}. \quad \text{(V)}$$

For $\mu = 0$, $\gamma^{0\mathrm{T}} = \gamma^0$, and

$$\gamma^2\gamma^0\gamma^2 = -\gamma^0(\gamma^2)^2 = \gamma^0.$$

Similarly, for $\mu = 1, 3$, $\gamma^{(1,3)\mathrm{T}} = -\gamma^{(1,3)}$, and

$$\gamma^2\gamma^{(1,3)\mathrm{T}}\gamma^2 = -\gamma^2\gamma^{(1,3)}\gamma^2 = +\gamma^{(1,3)}(\gamma^2)^2 = -\gamma^{(1,3)}.$$

On the other hand, $\gamma^{2\mathrm{T}} = \gamma^2$, and

$$\gamma^2\gamma^{2\mathrm{T}}\gamma^2 = \gamma^2(\gamma^2)^2 = -\gamma^2.$$

Hence finally (V) is

$$-V_{\mathrm{ud}}\bar{\hat{d}}_{\mathrm{L}}(\gamma^0, -\gamma^i)\hat{u}_{\mathrm{L}},$$

and (with the corrections noted above) we obtain (22.150).

22.6 *Please note that the vacuum value of the Higgs field should read*

$$\langle 0|\hat{\phi}|0\rangle = \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}.$$

To obtain the photon and Z masses in this case, it is sufficient to place the vaccum value of $\hat{\phi}$ in the covariant derivative, which then becomes

$$[\partial_\mu + \begin{pmatrix} 0 \\ \mathrm{i}gf(\hat{W}^1_\mu - \mathrm{i}\hat{W}^2_\mu)/\sqrt{2} \\ -\mathrm{i}gf\hat{W}^3_\mu + \mathrm{i}g'f\hat{B}_\mu \end{pmatrix}],$$

where we have used the expressions for the 3×3 $\boldsymbol{t}$ matrices. It is clear that the same mixing of the $\hat{W}^3_\mu$ and $\hat{B}_\mu$ fields occurs here as it did in (19.88), for example, so that the unmixed fields $\hat{A}_\mu$ and $\hat{Z}_\mu$ will be defined as before, with the same $\sin\theta_{\mathrm{W}}$. With these definitions, the above covariant derivative on $\hat{\phi}$ becomes

$$[\partial_\mu + \begin{pmatrix} 0 \\ \mathrm{i}gf(\hat{W}^1_\mu - \mathrm{i}\hat{W}^2_\mu)/\sqrt{2} \\ -\mathrm{i}(g/\cos\theta_{\mathrm{W}})f\hat{Z}_\mu \end{pmatrix}].$$

Multiplying this by its Hermitian conjugate, as in ((22.8), we obtain the mass terms

$$\frac{g^2f^2}{2}(\hat{W}^1_\mu\hat{W}^{1\mu} + \hat{W}^2_\mu\hat{W}^{2\mu})$$

and

$$(g^2/\cos^2\theta_{\rm w})\hat{Z}_\mu\hat{Z}^\mu.$$

Hence

$$M_{\rm Z} = gf, \quad \text{and} \quad M_{\rm Z} = \sqrt{2}gf/\cos\theta_{\rm W}$$

and the relation between $M_{\rm W}$ and $M_{\rm Z}$ follows. From (22.44), the value of ρ in this model is $1/2$.

22.7 *Please note that there are two misprints in (22.183): the last line of the RHS should read:*

$$+\frac{g^2}{8}\hat{\boldsymbol{W}}_\mu^2(\hat{\sigma}^2+\hat{\boldsymbol{\pi}}^2).$$

When we form $(\hat{D}_\mu\hat{\phi})^\dagger(\hat{D}^\mu\hat{\phi})$ it's clear that the result has to be real, so that we need only write down the real terms in the product. These are

$$\begin{aligned}
&\frac{1}{2}(0\,1)\{\partial_\mu\hat{\sigma}\partial^\mu\hat{\sigma}+(\boldsymbol{\tau}\cdot\partial_\mu\hat{\boldsymbol{\pi}})(\boldsymbol{\tau}\cdot\partial^\mu\hat{\boldsymbol{\pi}})+ \quad (i)\\
&-\frac{g}{2}\partial_\mu\hat{\sigma}\hat{\boldsymbol{\pi}}\cdot\hat{\boldsymbol{W}}^\mu-\frac{g}{2}\hat{\boldsymbol{\pi}}\cdot\hat{\boldsymbol{W}}^\mu\partial_\mu\hat{\sigma} \quad (ii)\\
&+\frac{g}{2}\boldsymbol{\tau}\cdot\partial_\mu\hat{\boldsymbol{\pi}}\hat{\sigma}\boldsymbol{\tau}\cdot\hat{\boldsymbol{W}}^\mu+\frac{g}{2}\hat{\sigma}\boldsymbol{\tau}\cdot\hat{\boldsymbol{W}}_\mu\boldsymbol{\tau}\cdot\partial^\mu\hat{\boldsymbol{\pi}} \quad (iii)\\
&-\frac{g}{2}\boldsymbol{\tau}\cdot\partial_\mu\hat{\boldsymbol{\pi}}\boldsymbol{\tau}\cdot(\hat{\boldsymbol{W}}^\mu\times\hat{\boldsymbol{\pi}})-\frac{g}{2}\boldsymbol{\tau}\cdot(\hat{\boldsymbol{W}}_\mu\times\hat{\boldsymbol{\pi}})\boldsymbol{\tau}\cdot\partial^\mu\hat{\boldsymbol{\pi}} \quad (iv)\\
&+\frac{g^2}{4}\hat{\sigma}^2\boldsymbol{\tau}\cdot\hat{\boldsymbol{W}}_\mu\boldsymbol{\tau}\cdot\hat{\boldsymbol{W}}^\mu+\frac{g^2}{4}(\hat{\boldsymbol{\pi}}\cdot\hat{\boldsymbol{W}}_\mu)(\hat{\boldsymbol{\pi}}\cdot\hat{\boldsymbol{W}}^\mu) \quad (v)\\
&-\frac{g^2}{4}\hat{\sigma}\boldsymbol{\tau}\cdot\hat{\boldsymbol{W}}_\mu\boldsymbol{\tau}\cdot\hat{\boldsymbol{W}}^\mu\times\hat{\boldsymbol{\pi}}-\frac{g^2}{4}\hat{\sigma}\boldsymbol{\tau}\cdot\hat{\boldsymbol{W}}_\mu\times\hat{\boldsymbol{\pi}}\boldsymbol{\tau}\cdot\hat{\boldsymbol{W}}^\mu \quad (vi)\\
&+\frac{g^2}{4}(\boldsymbol{\tau}\cdot\hat{\boldsymbol{W}}_\mu\times\hat{\boldsymbol{\pi}})(\boldsymbol{\tau}\cdot\hat{\boldsymbol{W}}^\mu\times\hat{\boldsymbol{\pi}})\}\begin{pmatrix}0\\1\end{pmatrix}. \quad (vii)
\end{aligned}$$

We now use the identity (see problem 4.4(b) from volume 1)

$$(\boldsymbol{\tau}\cdot\boldsymbol{a})(\boldsymbol{\tau}\cdot\boldsymbol{b})=\boldsymbol{a}\cdot\boldsymbol{b}+\mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{a}\times\boldsymbol{b},$$

where a unit matrix is understood in the first term. Line (i) then reduces to

$$\frac{1}{2}\partial_\mu\hat{\sigma}\partial^\mu\hat{\sigma}+\frac{1}{2}\partial_\mu\hat{\boldsymbol{\pi}}\cdot\partial^\mu\hat{\boldsymbol{\pi}}. \quad \text{(a)}$$

The two terms in line (ii) add up to produce a total of

$$-\frac{g}{2}\partial_\mu\hat{\sigma}\hat{\boldsymbol{\pi}}\cdot\hat{\boldsymbol{W}}^\mu. \quad \text{(b)}$$

Using $\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$, the two terms in the third line contribute a total of

$$\frac{g}{2}\hat{\sigma}\partial_\mu\hat{\boldsymbol{\pi}}\cdot\hat{\boldsymbol{W}}^\mu, \quad \text{(c)}$$

while the two terms in line (iv) contribute a total of

$$-\frac{g}{2}\partial_\mu\hat{\boldsymbol{\pi}}\cdot(\hat{\boldsymbol{W}}^\mu\times\hat{\boldsymbol{\pi}}) = \frac{g}{2}\partial_\mu\hat{\boldsymbol{\pi}}\cdot(\hat{\boldsymbol{\pi}}\times\hat{\boldsymbol{W}}^\mu), \quad \text{(d)}$$

and the two terms in line (vi) contribute a total of

$$-\frac{g^2}{4}\hat{\sigma}\hat{\boldsymbol{W}}^\mu\cdot(\hat{\boldsymbol{W}}^\mu\times\hat{\boldsymbol{\pi}});$$

this last contribution, however, vanishes because two quantities in the triple scalar product in the internal space are identical. The first term in line (v) is just

$$\frac{g^2}{8}\hat{\sigma}^2\hat{\boldsymbol{W}}_\mu\cdot\hat{\boldsymbol{W}}^\mu. \quad \text{(e)}$$

Finally, the term in line (vii) is

$$\begin{aligned}
&\frac{g^2}{8}[(\hat{\boldsymbol{W}}_\mu\times\hat{\boldsymbol{\pi}})\cdot(\hat{W}^\mu\times\hat{\boldsymbol{\pi}})]\\
&=\frac{g^2}{8}[\hat{\boldsymbol{W}}_\mu\cdot(\hat{\boldsymbol{\pi}}\times(\hat{\boldsymbol{W}}^\mu\times\hat{\boldsymbol{\pi}}))]\\
&=\frac{g^2}{8}[\hat{\boldsymbol{W}}_\mu\cdot(\hat{\boldsymbol{W}}^\mu(\hat{\boldsymbol{\pi}}\cdot\hat{\boldsymbol{\pi}})-\hat{\boldsymbol{\pi}}(\hat{\boldsymbol{W}}^\mu\cdot\hat{\boldsymbol{\pi}}))]\\
&=\frac{g^2}{8}[(\hat{\boldsymbol{W}}_\mu\cdot\hat{\boldsymbol{W}}^\mu)(\hat{\boldsymbol{\pi}}\cdot\hat{\boldsymbol{\pi}})-(\hat{\boldsymbol{W}}_\mu\cdot\hat{\boldsymbol{\pi}})(\hat{\boldsymbol{W}}^\mu\cdot\hat{\boldsymbol{\pi}})]. \quad \text{(f)}
\end{aligned}$$

The second term of (f) cancels with the second term in line (v), and the first term in (f) combines with (e) to give

$$\frac{g^2}{8}\hat{\boldsymbol{W}}_\mu\cdot\hat{\boldsymbol{W}}^\mu(\hat{\sigma}^2+\hat{\boldsymbol{\pi}}\cdot\hat{\boldsymbol{\pi}}). \quad \text{(g)}$$

The contributions (a), (b), (c), (d) and (g) then give the (corrected) result for (22.183). Note that since all these terms do not involve the $\boldsymbol{\tau}$ matrices - that is, they multiply the unit matrix in internal space - they can be taken outside the matrix product, which simply reduces to

$$(0\ 1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1.$$

22.8 Taking the vertex from (22.136), and approximating V_{tb} by unity, the rate is

$$\begin{aligned}\Gamma_{\mathrm{t}} &= \frac{1}{2m_{\mathrm{t}}}\frac{1}{2}\sum_{\mathrm{spins},\lambda}\frac{g^2}{2}|\epsilon_\mu(k,\lambda)\bar{u}(p_2)\gamma^\mu\frac{1-\gamma_5}{2}u(p_1)|^2 \times \\ &\quad \frac{\mathrm{d}^3\boldsymbol{p}_2}{(2\pi)^3 2E_2}\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2E_k}(2\pi)^4\delta^{(4)}(p_1-p_2-k),\end{aligned}$$

where p_1 is the 4-momentum of the top, p_2 that of the bottom, and k, λ are the 4-momentum and polarization label of the W. The '1/2' before the spin sum arises from the average over the top spins. The sum over the fermion spins of the squared matrix element is

$$\begin{aligned}&\mathrm{Tr}[\gamma^\mu\frac{1-\gamma_5}{2}(\not{p}_1+m_{\mathrm{t}})\frac{1+\gamma_5}{2}\gamma^\nu(\not{p}_2+m_{\mathrm{b}})] \\ &= \mathrm{Tr}[\gamma^\mu\frac{1-\gamma_5}{2}\{\frac{1-\gamma_5}{2}\not{p}_1+\frac{1+\gamma_5}{2}m_{\mathrm{t}}\}\gamma^\nu(\not{p}_2+m_{\mathrm{b}})] \\ &= \mathrm{Tr}[\gamma^\mu\frac{1-\gamma_5}{2}\not{p}_1\gamma^\nu(\not{p}_2+m_{\mathrm{b}})] \\ &= \frac{1}{2}\mathrm{Tr}[\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2] = 2[p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - g^{\mu\nu}p_1\cdot p_2], \quad \text{(S)}\end{aligned}$$

where we have used (J.25), (J.31) and (J.36); the contribution from the term with four γ-matrices and a γ_5 will (from (J.37)) be antisymmetric in μ and ν, and will vanish when contracted with the symmetric quantity

$$\sum_\lambda \epsilon_\mu(k,\lambda)\epsilon_\nu^*(k,\lambda) = (-g_{\mu\nu}+k_\mu k_\nu/M_{\mathrm{W}}^2). \quad \text{(W)}$$

Contracting (S) with (W) gives

$$\begin{aligned}&2(-g_{\mu\nu}+k_\mu k_\nu/M_{\mathrm{W}}^2)(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - g_{\mu\nu}p_1\cdot p_2) \\ &= 2[-2p_1\cdot p_2 + 4p_1\cdot p_2 + 2(p_1\cdot k)(p_2\cdot k)/M_{\mathrm{W}}^2 - k^2 p_1\cdot p_2/M_{\mathrm{W}}^2] \\ &= 2[p_1\cdot p_2 + 2(p_1\cdot k)(p_2\cdot k)/M_{\mathrm{W}}^2]\end{aligned}$$

We evaluate the dot products by using the 4-momentum conservation equation,

$$p_1 = p_2 + k;$$

thus

$$p_1 \cdot k = (p_1^2 + k^2 - p_2^2)/2 = (m_\mathrm{t}^2 + M_\mathrm{W}^2 - m_\mathrm{b}^2)/2,$$
$$p_1 \cdot p_2 = (p_1^2 + p_2^2 - k^2)/2 = (m_\mathrm{t}^2 + m_\mathrm{b}^2 - M_\mathrm{W}^2)/2,$$

and

$$p_2 \cdot k = (p_1^2 - p_2^2 - k^2)/2 = (m_\mathrm{t}^2 - m_\mathrm{b}^2 - M_\mathrm{W}^2)/2.$$

In the CM frame, the total energy W is $W = E_2 + E_k = \sqrt{m_\mathrm{b}^2 + \boldsymbol{p}^2} + \sqrt{M_\mathrm{W}^2 + \boldsymbol{p}^2}$. So

$$\mathrm{d}W = \frac{|\boldsymbol{p}|(E_2 + E_k)}{E_2 E_k}\mathrm{d}|\boldsymbol{p}|.$$

Also,

$$|\boldsymbol{p}| = [\{(m_\mathrm{t} + M_\mathrm{W})^2 - m_\mathrm{b}^2\}\{(m_\mathrm{t} - M_\mathrm{W})^2 - m_\mathrm{b}^2\}]^{1/2}/2m_\mathrm{t}.$$

Performing the phase space integral in the now familiar way (using $\delta(m_\mathrm{t} - W)$), we find

$$\Gamma_\mathrm{t} = \frac{M_\mathrm{W}^2 G_\mathrm{F}}{4\pi m_\mathrm{t}^2\sqrt{2}}[(m_\mathrm{t}^2 + m_\mathrm{b}^2 - M_\mathrm{W}^2) + (m_\mathrm{t}^2 + M_\mathrm{W}^2 - m_\mathrm{b}^2)(m_\mathrm{t}^2 - M_\mathrm{W}^2 - m_\mathrm{b}^2)/M_\mathrm{W}^2]|\boldsymbol{p}|,$$

where we have used (22.29). From this we obtain the numerical values $\Gamma_\mathrm{W} = 1.65$ GeV, and $\tau_\mathrm{W} \sim 4 \times 10^{-25}$ s.

22.9 (a) For $\mathrm{H} \to f\bar{f}$ we use the vertex

$$-\mathrm{i}\frac{g}{2}\frac{m_f}{M_\mathrm{W}}\bar{u}(p_1)v(p_2)$$

where p_1, p_2 are the 4-momenta of the f and $\bar{f}$ respectively. The rate is

$$\Gamma(\mathrm{H} \to f\bar{f}) = \frac{1}{2M_\mathrm{H}}\sum_\mathrm{spins}\frac{g^2 m_f^2}{4M_\mathrm{W}^2}\mathrm{Tr}[v\bar{v}u\bar{u}]\,\frac{\mathrm{d}^3\boldsymbol{p}_1}{(2\pi)^3 2E_1}\,\frac{\mathrm{d}^3\boldsymbol{p}_2}{(2\pi)^3 2E_2}(2\pi)^4\delta^{(4)}(k - p_1 - p_2),$$

where k is the 4-momentum of the H. The trace is

$$\mathrm{Tr}[(\not{p}_2 - m_f)(\not{p}_1 + m_f)] = 4p_1 \cdot p_2 - 4m_f^2.$$

Since $k = p_1 + p_2$, we have

$$4p_1 \cdot p_2 = 2m_{\mathrm{H}}^2 - 4m_f^2.$$

Also, in the CM frame, $E_1 = E_2 = (m_f^2 + \boldsymbol{p}^2)^{1/2}$ and

$$W = 2(m_f^2 + \boldsymbol{p}^2)^{1/2}, \quad \mathrm{d}W = 2|\boldsymbol{p}|\mathrm{d}|\boldsymbol{p}|/E_1,$$

and

$$|\boldsymbol{p}| = (m_{\mathrm{H}}^2 - 4m_f^2)^{1/2}/2.$$

Putting these results together gives (22.202), with a factor of three in the case of quarks, from the colour multiplicity.
(b) For $\mathrm{H} \to \mathrm{W}^+\mathrm{W}^-$, the matrix element is

$$\mathrm{i}gM_{\mathrm{W}}\epsilon_1(k_1, \lambda_1) \cdot \epsilon_2(k_2, \lambda_2)$$

in an obvious notation. The rate is

$$\begin{aligned}\Gamma(\mathrm{H} \to \mathrm{W}^+\mathrm{W}^-) &= \frac{1}{2M_{\mathrm{H}}} g^2 M_{\mathrm{W}}^2 \sum_{\lambda_1,\lambda_2} (\epsilon_{1\mu}\epsilon_2^{\mu})(\epsilon_{1\nu}^*\epsilon_2^{*\nu}) \times \\ & \quad \frac{\mathrm{d}^3\boldsymbol{k}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\boldsymbol{k}_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(p - k_1 - k_2)\end{aligned}$$

where p is the 4-momentum of the H. The polarization sum is

$$\begin{aligned}&(-g_{\mu\nu} + k_{1\mu}k_{1\nu}/M_{\mathrm{W}}^2)(-g^{\mu\nu} + k_2^{\mu}k_2^{\nu}/M_{\mathrm{W}}^2) \\ &= (4 - k_2^2/M_{\mathrm{W}}^2 - k_1^2/M_{\mathrm{W}}^2 + (k_1 \cdot k_2)^2/M_{\mathrm{W}}^4) \\ &= (4 - 2 + (m_{\mathrm{H}}^2 - 2M_{\mathrm{W}}^2)^2/(4M_{\mathrm{W}}^4)) \\ &= \frac{m_{\mathrm{H}}^4}{4M_{\mathrm{W}}^4}(1 - 4M_{\mathrm{W}}^2/m_{\mathrm{H}}^2 + 12M_{\mathrm{W}}^4/m_{\mathrm{H}}^4).\end{aligned}$$

In the CM frame $E_1 = E_2 = (M_{\mathrm{W}}^2 + \boldsymbol{p}^2)^{1/2}$, and the total energy is $W = 2(M_{\mathrm{W}}^2 + \boldsymbol{p}^2)^{1/2}$, so that

$$\mathrm{d}|\boldsymbol{p}| = E_1 \mathrm{d}W/(2|\boldsymbol{p}|),$$

and

$$|\boldsymbol{p}| = (m_{\mathrm{H}}^2 - 4M_{\mathrm{W}}^2)^{1/2}/2.$$

Performing the phase space integral in the usual way, these results lead to (22.203). *Note that there is a misprint in (22.203): the first factor in round brackets should read*

$$\left(1 - \frac{4M_{\mathrm{W}}^2}{m_{\mathrm{H}}^2}\right)^{1/2}.$$