Solutions to problems in
"Gauge Theories in Particle Physics", Third Edition, volume 1:
"From Relativistic Quantum Mechanics to QED"

Chapter 2

2.1 The quantum numbers of the 'anything' must be $B = 1, Q = 0$ in both cases, together with $S = -1$ in the first case and $S = +1$ in the second; consider what are the lightest single-particle or two-particle states with these quantum numbers.

2.2 (i) 4-momentum conservation gives $p+k = p'+k'$ which may be written as $p' = (k-k')+p$; squaring both sides gives $p'^2 = q^2+2p\cdot(k-k')+M^2c^2$, and the result follows by evaluating the dot product in the frame such that $p = (Mc, \mathbf{0})$. For highly relativistic electrons $E \approx c|\boldsymbol{k}|, E' \approx c|\boldsymbol{k}'|$, whence $q^2c^2 = (E - E')^2 - c^2(\boldsymbol{k} - \boldsymbol{k}')^2 \approx -2EE' + 2EE' \cos\theta = -4EE' \sin^2\theta/2$. For elastic scattering $p'^2 = M^2c^2$, and hence from the first result $q^2 = -2M(E-E')$. Combining this with the (approximate) formula for q^2 in terms of θ gives the formula for E'.
(ii) $E' = 4.522$ GeV (negelecting the electron mass).
(iii) The invariant mass of the produced hadronic state is $(p'^2/c^2)^{1/2}$; using the first displayed equation and the (approximate) expression for q^2 in terms of θ gives 1238 MeV$/c^2$ (approximately equal to the mass of the $I = 3/2$ Δ resonance).
(iv) The term 'quasi-elastic peak' in this context means the value of E' corresponding to a collision between the incident electron and a single nucleon in the He nucleus, ignoring the binding energy of the struck nucleon (i.e. it is treated like elastic scattering). Since the binding energy is of order 10 MeV, very much less than the electron energy, this is expected to be a good first approximation to the kinematics. It gives $E' = 355.6$ MeV.

The struck nucleon is, however, moving by virtue of being bound inside the He nucleus (it has a bound state wavefunction and an associated momentum distribution). Rather than attempting a more realistic correction based on the nucleon wavefunction, we can get a fair idea of a typical nucleon momentum by using the 'uncertainty relation' estimate $p \sim \hbar/R$ where R is the nuclear radius. Here $R \sim 1.5$fm, and $p \sim 130\mathrm{MeV}/c$, which gives a struck nucleon speed of order $v/c \sim 0.14$. (Note that He is a tightly bound nucleus, and the formula for the nuclear radius $R = 1.1 \times (A)^{1/3}$ is not really applicable - of course, we are making rough estimates anyway.) Considering configurations with the outgoing electron moving parallel/antiparallel to the struck nucleon gives a typical shift in E' of order $\pm$ 50 MeV. Note that this is quite a bit bigger than the nucleon's binding energy - the relativistic transformation has amplified the effect.

2.3 (i) 5.1 eV.
(ii) (a) 1.29 (b) 0.75
(iii) $\langle \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \rangle = +1$ for $S = 1$, -3 for $S = 0$. Hyperfine splitting = 8.45×10^{-4} eV.
(iv) 0.57.

2.4 One-gluon exchange; confinement.
Ground state expected to be at the minimum of $E(r)$ as a function of r, i.e. at r_0 such that $dE(r)/dr|_{(r=r_0)} = 0$.
$E_{\mathrm{gr}}(\mathrm{c}\bar{\mathrm{c}}) \approx 3.23\mathrm{GeV}$.
Threshold for production of 'open charm' ($\mathrm{D}\bar{\mathrm{D}}$ states) opens at about 3.73 GeV.

2.5 $E_{\pm} = 2a\cos 2\theta \pm a$; $|+\rangle = \cos\theta|1\rangle + \sin\theta|2\rangle$, $|-\rangle = -\sin\theta|1\rangle + \cos\theta|2\rangle$.
System is in state $\cos\theta|+\rangle - \sin\theta|-\rangle$ at time $t = 0$; evolves to

$$\cos\theta \exp[\mathrm{i}(2a\cos 2\theta + a)t/\hbar]|+\rangle - \sin\theta \exp[\mathrm{i}(2a\cos 2\theta - a)t/\hbar]|-\rangle$$

at time t; amplitude of $|2\rangle$ in this state is

$$\cos\theta\sin\theta\{\exp[\mathrm{i}(2a\cos 2\theta + a)t/\hbar] - \exp[\mathrm{i}(2a\cos 2\theta - a)t/\hbar]\};$$

modulus squared of this gives result.

2.6 (iv) $d = 2$, $R = 1$mm: $M_{\mathrm{P},3+\mathrm{d}} \approx 1.54$ TeV.

Chapter 3

3.1 From the inverse transformation and the chain rule

$$\frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x^1}{\partial t'}\frac{\partial}{\partial x^1}$$

we find

$$\frac{\partial}{\partial t'} = \gamma\left[\frac{\partial}{\partial t} - v\left(-\frac{\partial}{\partial x^1}\right)\right]$$

and similarly for $-\frac{\partial}{\partial x^{1\prime}}$.

3.2 Six. $F^{21} = B^3, F^{32} = B^1, F^{13} = B^2; F^{10} = E^1, F^{20} = E^2, F^{30} = E^3$. Consider $\nu = 1$ component of (3.17); LHS is

$$\begin{aligned}\partial_\mu F^{\mu 1} &= \partial_2 F^{21} + \partial_3 F^{31} + \partial_0 F^{01} \\ &= \frac{\partial B^3}{\partial x^2} - \frac{\partial B^2}{\partial x^3} - \frac{\partial E^1}{\partial t} = (\boldsymbol{\nabla}\times\boldsymbol{B})^1 - \left(\frac{\partial \boldsymbol{E}}{\partial t}\right)^1\end{aligned}$$

which verifies the first component of (3.8).

3.3 It is advisable in such manipulations to let the whole expression act on an arbitrary function $F(x)$ say, which can be removed at the end. Thus

$$\begin{aligned}\hat{p}\mathrm{e}^{-\mathrm{i}qf(x)}F(x) &= -\mathrm{i}\frac{\partial}{\partial x}\{\mathrm{e}^{-\mathrm{i}qf(x)}F(x)\} \\ &= -\mathrm{i}(-\mathrm{i}q)\frac{\partial f}{\partial x}\mathrm{e}^{-\mathrm{i}qf(x)}F + \mathrm{e}^{-\mathrm{i}qf(x)} - \mathrm{i}\frac{\partial F(x)}{\partial x} \\ &= (-q)\frac{\partial f}{\partial x}\mathrm{e}^{-\mathrm{i}qf(x)}F + \mathrm{e}^{-\mathrm{i}qf(x)}\hat{p}F(x);\end{aligned}$$

multiplying this from the left by $\mathrm{e}^{\mathrm{i}qf(x)}$ gives the result after F is removed.

Chapter 4

4.1(b) $\rho = \mathrm{i}(\phi^*\dot{\phi} - \dot{\phi}^*\phi)$, $\boldsymbol{j} = \mathrm{i}^{-1}(\phi^*\boldsymbol{\nabla}\phi - (\boldsymbol{\nabla}\phi^*)\phi)$.
For $\phi = N\mathrm{e}^{-\mathrm{i}p\cdot x}$, $\rho = 2|N|^2 E$, $\boldsymbol{j} = 2|N|^2\boldsymbol{p}$, $j^\mu = 2|N|^2 p^\mu$.

4.2(ii) Consider for example $\beta\alpha_i = -\alpha_i\beta$. Multiply from the left by β to obtain $\alpha_i = -\beta\alpha_i\beta$. Now take the Trace and use $\mathrm{Tr}(\beta\alpha_i\beta) = \mathrm{Tr}(\alpha_i\beta^2) = \mathrm{Tr}(\alpha_i)$.

4.4(b) $(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b}) = \sigma_i a_i \sigma_j b_j \quad (\text{sum on } i \text{ and } j) \quad = \sigma_i\sigma_j a_i b_j$
$= (\delta_{ij}\mathbf{1} + \mathrm{i}\epsilon_{ijk}\sigma_k)a_i b_j = \boldsymbol{a}\cdot\boldsymbol{b}\mathbf{1} + \mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b})$.

4.6(i) $(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{u}})^2 = \mathbf{1}$. Hence $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{u}}\frac{1}{2}(1+\boldsymbol{\sigma}\cdot\hat{\boldsymbol{u}})\phi = \frac{1}{2}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{u}}+1)\phi$. Projection operator for $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{u}} = -1$ is $\frac{1}{2}(1-\boldsymbol{\sigma}\cdot\hat{\boldsymbol{u}})$.
(ii) Take

$$\phi_+ = \frac{N}{2}(1+\boldsymbol{\sigma}\cdot\hat{\boldsymbol{u}})\begin{pmatrix}1\\0\end{pmatrix} = \frac{N}{2}\begin{pmatrix}1+\cos\theta & \sin\theta\mathrm{e}^{-\mathrm{i}\phi}\\ \sin\theta\mathrm{e}^{\mathrm{i}\phi} & 1-\cos\theta\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = N\begin{pmatrix}\cos^2\theta/2\\ \sin\theta/2\cos\theta/2\mathrm{e}^{\mathrm{i}\phi}\end{pmatrix}$$

and choose $N = 1/(\cos\theta/2)$ for normalization. Possible ϕ_- is $\begin{pmatrix}-\sin\theta/2\mathrm{e}^{-\mathrm{i}\phi}\\ \cos\theta/2\end{pmatrix}$.

4.8 Consider j'_y for example. We have

$$j'_y = \omega'^\dagger\alpha_y\omega' = \omega^\dagger\mathrm{e}^{-\mathrm{i}\Sigma_x\alpha/2}\alpha_y\mathrm{e}^{\mathrm{i}\Sigma_x\alpha/2}\omega.$$

Now

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}\Sigma_x\alpha/2} &= 1+\mathrm{i}\Sigma_x\alpha/2+\frac{1}{2}(\mathrm{i}\Sigma_x\alpha/2)^2+\frac{1}{3!}(\mathrm{i}\Sigma_x\alpha/2)^3+\ldots\\
&= 1+\mathrm{i}\Sigma_x\alpha/2-\frac{1}{2}(\alpha/2)^2-\frac{1}{3!}\mathrm{i}\Sigma_x(\alpha/2)^3+\ldots\\
&= \cos\alpha/2+\mathrm{i}\Sigma_x\sin\alpha/2
\end{aligned}$$

as in (4.80). Hence

$$j'_y = \omega^\dagger[(\cos\alpha/2-\mathrm{i}\Sigma_x\sin\alpha/2)\alpha_y(\cos\alpha/2+\mathrm{i}\Sigma_x\sin\alpha/2)]\omega.$$

But

$$\Sigma_x\alpha_y = \begin{pmatrix}\sigma_x & 0\\ 0 & \sigma_x\end{pmatrix}\begin{pmatrix}0 & \sigma_y\\ \sigma_y & 0\end{pmatrix} = \begin{pmatrix}0 & \sigma_x\sigma_y\\ \sigma_x\sigma_y & 0\end{pmatrix} = \begin{pmatrix}0 & \mathrm{i}\sigma_z\\ \mathrm{i}\sigma_z & 0\end{pmatrix} = \mathrm{i}\alpha_z;$$

similarly $\alpha_y\Sigma_x = -\mathrm{i}\alpha_z$; and $\Sigma_x\alpha_y\Sigma_x = -\alpha_y\Sigma_x^2 = -\alpha_y$. Hence

$$\begin{aligned}
j'_y &= \omega^\dagger[\cos^2\alpha/2\ \alpha_y + 2\sin\alpha/2\cos\alpha/2\ \alpha_z - \sin^2\alpha/2\ \alpha_y]\omega\\
&= \omega^\dagger[\cos\alpha\ \alpha_y + \sin\alpha\ \alpha_z]\omega\\
&= \cos\alpha\ j_y + \sin\alpha\ j_z.
\end{aligned}$$

4.10 Under (4.83):

$$\bar{\psi}'\psi' = \psi'^{\dagger}\beta\psi' = \psi^{\dagger}\mathrm{e}^{-\mathrm{i}\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{n}}\theta/2}\beta\mathrm{e}^{\mathrm{i}\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{n}}\theta/2}\psi.$$

Now $\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \end{pmatrix}$, $\beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$, and $\boldsymbol{\Sigma}\beta = \beta\boldsymbol{\Sigma}$, so that

$$\mathrm{e}^{-\mathrm{i}\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{n}}\theta/2}\beta = \beta\mathrm{e}^{\mathrm{i}\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{n}}\theta/2}$$

and the exponentials cancel.
Under (4.90): use $\alpha_x\beta = -\beta\alpha_x$.

4.11 (a) $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$.
(b) Taking the dagger of $(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = 0$ gives $-\mathrm{i}(\partial_\mu\psi^\dagger)\gamma^{\mu\dagger} - m\psi^\dagger = 0$, or $\psi^\dagger(\mathrm{i}\gamma^{\mu\dagger}\overleftarrow{\partial_\mu} + m) = 0$ where the notation $\overleftarrow{\partial_\mu}$ means that the derivative acts on what is to the left of it, i.e. on $\psi^\dagger$. Multiply this equation from the right by $\gamma^0 = \beta$ and use $\gamma^{\mu\dagger}\beta = \beta\gamma^\mu$ (obvious for $\mu = 0$, check it for the space components); this gives $\bar{\psi}(\mathrm{i}\overleftarrow{\not{\partial}} + m) = 0$.
(c) $\partial_\mu(\bar{\psi}\gamma^\mu\psi) = (\bar{\psi}\overleftarrow{\not{\partial}})\psi + \bar{\psi}(\not{\partial}\psi) = -m\bar{\psi}\psi + m\bar{\psi}\psi = 0$.

4.13 $\hat{V}_{\mathrm{KG}} = \mathrm{i}q(\partial_\mu A^\mu + A^\mu\partial_\mu) - q^2 A_\mu A^\mu$.

4.14 (ii) $(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})(\boldsymbol{\sigma}\cdot\boldsymbol{A})\psi = \sigma_i\partial_i\sigma_j A_j\psi = \sigma_i\sigma_j\partial_i(A_j\psi)$
$= (\delta_{ij}\mathbf{1} + \mathrm{i}\epsilon_{ijk}\sigma_k)\partial_i(A_j\psi) = \partial_i(A_i\psi) + \mathrm{i}\epsilon_{ijk}\sigma_k\partial_i(A_j\psi)$
$= \boldsymbol{\nabla}\cdot(\boldsymbol{A}\psi) + \mathrm{i}\boldsymbol{\sigma}\cdot\{\boldsymbol{\nabla}\times(\boldsymbol{A}\psi)\}$.

4.15 (i) For a massive particle the direction of its momentum reverses on transforming to a frame moving parallel to, and faster than, it; this is not possible for a massless particle which moves at the speed of light.
(ii) We give the solution in a form which adapts to the general case. We have $(E - \sigma_x p_x)u_{\mathrm{R}} = 0$. Mulitiplying from the left by $(1 - \frac{1}{2}\sigma_x\epsilon_x)$ and inserting a unit operator (to first order in ϵ_x) it follows that
$(1 - \frac{1}{2}\sigma_x\epsilon_x)(E - \sigma_x p_x)(1 - \frac{1}{2}\sigma_x\epsilon_x)(1 + \frac{1}{2}\sigma_x\epsilon_x)u_{\mathrm{R}} = 0$.
But $(1 + \frac{1}{2}\sigma_x\epsilon_x)u_{\mathrm{R}} = u'_{\mathrm{R}}$, and
$(1 - \frac{1}{2}\sigma_x\epsilon_x)(E - \sigma_x p_x)(1 - \frac{1}{2}\sigma_x\epsilon_x) = E - \sigma_x p_x - E\sigma_x\epsilon_x + \epsilon_x p_x$
(to first order in ϵ_x), which is just $E' - \sigma_x p'_x$, whence $(E' - \sigma_x p'_x)u'_{\mathrm{R}} = 0$ as required.
(iv) See part (i). Since, for $m \neq 0$, we can reverse the helicity by a

Lorentz transformation, we expect that in a Lorentz-invariant theory mass terms will mix states of opposite helicity.
(v) let $B_{\mathrm{R}} = (1 + \frac{1}{2}\sigma_x \epsilon_x)$, $B_{\mathrm{L}} = (1 - \frac{1}{2}\sigma_x \epsilon_x)$, so that $u'_{\mathrm{R}} = B_{\mathrm{R}} u_{\mathrm{R}}$, $u'_{\mathrm{L}} = B_{\mathrm{L}} u_{\mathrm{L}}$. Equation (4.98) is $(E - \sigma_x p_x)\phi = m\chi$. So we have $B_{\mathrm{L}}(E - \sigma_x p_x) B_{\mathrm{L}} B_{\mathrm{L}}^{-1}\phi = m B_{\mathrm{L}}\chi$. But $B_{\mathrm{L}}^{-1} = B_{\mathrm{R}}$, and part (ii) showed that $B_{\mathrm{L}}(E - \sigma_x p_x) B_{\mathrm{L}} = E' - \sigma_x p'_x$. Hence it follows that $(E' - \sigma_x p'_x)\phi' = m\chi'$ if $\phi' = B_{\mathrm{R}}\phi$ and $\chi' = B_{\mathrm{L}}\chi$. Similarly for (4.99).

Chapter 5

5.1 Consider the term $\int_0^L \left(\frac{\partial\phi}{\partial x}\right)^2 \mathrm{d}x$. We have

$$\frac{\partial\phi}{\partial x} = \sum_{r=1}^{\infty} A_r(t)\left(\frac{r\pi}{L}\right)\sin\left(\frac{r\pi x}{L}\right).$$

The quantity '$\left(\frac{\partial\phi}{\partial x}\right)^2$' is the square of this entire series - i.e. it involves not only terms of the form

$$\left[A_1(t)\left(\frac{\pi}{L}\right)\sin\left(\frac{\pi x}{L}\right)\right]^2, \left[A_2(t)\left(\frac{2\pi}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\right]^2,$$

etc., but also all the 'cross' terms such as

$$A_1(t)\left(\frac{\pi}{L}\right)\sin\left(\frac{\pi x}{L}\right)\cdot A_2(t)\left(\frac{2\pi}{L}\right)\sin\left(\frac{2\pi x}{L}\right).$$

To make sure we are including all these cross terms, we use a *different* summation index for the two *independent* summations when writing the product:

$$\begin{aligned}
\int_0^L \left(\frac{\partial\phi}{\partial x}\right)^2 \mathrm{d}x &= \int_0^L \left[\sum_{r=1}^{\infty} A_r(t)\left(\frac{r\pi}{L}\right)\sin\left(\frac{r\pi x}{L}\right)\right] \\
&\times \left[\sum_{s=1}^{\infty} A_s(t)\left(\frac{s\pi}{L}\right)\sin\left(\frac{s\pi x}{L}\right)\right]\mathrm{d}x \\
&= \sum_{r,s=1}^{\infty} A_r(t)A_s(t)\left(\frac{r\pi}{L}\right)\left(\frac{s\pi}{L}\right)\int_0^L \sin\left(\frac{r\pi x}{L}\right)\sin\left(\frac{s\pi x}{L}\right)\mathrm{d}x. \quad (*)
\end{aligned}$$

We need to evaluate the integral:

$$\begin{aligned} &\textstyle\int_0^L \sin\left(\frac{r\pi x}{L}\right)\sin\left(\frac{s\pi x}{L}\right)\mathrm{d}x \\ &= \textstyle\int_0^L \frac{1}{2}\left\{\cos\left[(r-s)\frac{\pi x}{L}\right] - \cos\left[(r+s)\frac{\pi x}{L}\right]\right\}\mathrm{d}x \\ &= \textstyle\frac{1}{2}\left\{\frac{L}{\pi(r-s)}\sin\left[(r-s)\frac{\pi x}{L}\right] - \frac{L}{\pi(r+s)}\sin\left[(r+s)\frac{\pi x}{L}\right]\right\}_0^L. \end{aligned}$$

Here both r and s are positive integers, so the second term vanishes. The first does also, *except* when $r = s$, in which case it has the value $L/2$ as can be seen by evaluating the integral directly for $r = s$. So we have

$$\begin{aligned} \int_0^L \sin\left(\frac{r\pi x}{L}\right)\sin\left(\frac{s\pi x}{L}\right) &= L/2 \quad \text{for} \quad r = s \\ &= 0 \qquad \text{for} \quad r \neq s. \end{aligned}$$

This means that in the sums over r and s in $(*)$, only the term in which $r = s$ survives, *after* the integral over x has been done. So $(*)$ becomes

$$\sum_{r=1}^{\infty}(A_r(t))^2\left(\frac{r\pi}{L}\right)^2 L,$$

and

$$\int_0^L \frac{1}{2}\rho c^2\left(\frac{\partial\phi}{\partial x}\right)^2\mathrm{d}x = \left(\frac{L}{2}\right)\sum_{r=1}^{\infty}\frac{1}{2}\rho\omega_r^2 A_r^2.$$

Similarly for the $\dot{\phi}^2$ term.

5.2 $S = \int_0^{t_0}[\frac{1}{2}m\dot{x}^2 + mgx]\mathrm{d}t$.

$$\text{(a)} \quad x = at: \quad S_{(\mathrm{a})} = \int_0^{t_0}(\frac{1}{2}ma^2 + mgat)\mathrm{d}t = \frac{1}{2}ma^2t_0 + \frac{1}{2}mgat_0^2.$$

Choose 'a' so that end point of this trajectory coincides with end point of the Newtonian trajectory in (b), which is at $x = \frac{1}{2}gt_0^2$. So we set $at_0 = \frac{1}{2}gt_0^2$ i.e. $a = \frac{1}{2}gt_0$. Then $S_{(\mathrm{a})} = \frac{3}{8}mg^2t_0^3 = 0.375(mg^2t_0^3)$. (b): $S_{(\mathrm{b})} = \frac{1}{3}mg^2t_0^3 = 0.3333(mg^2t_0^3)$. (c): $b = \frac{1}{2}gt_0^{-1}$, $S_{(\mathrm{c})} = \frac{7}{20}mg^2t_0^3 = 0.35(mg^2t_0^3)$. So $S_{(\mathrm{b})} < S_{(\mathrm{c})} < S_{(\mathrm{a})}$.

5.3 (a) We want to verify that $\dot{\hat{q}} = -\mathrm{i}[\hat{q}, \hat{H}]$ is consistent with $\dot{\hat{q}} = \hat{p}/m$. We have

$$\begin{aligned}
[\hat{q}, \hat{H}] &= [\hat{q}, \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2] = \frac{1}{2m}[\hat{q}, \hat{p}^2] \\
&= \frac{1}{2m}\{\hat{p}\ [\hat{q}, \hat{p}] + [\hat{q}, \hat{p}]\ \hat{p}\} \ \text{ using } \ [\hat{A}, \hat{B}\hat{C}] = \hat{B}\ [\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\ \hat{C} \\
&= \frac{1}{2m}\{2\mathrm{i}\hat{p}\} = \frac{\mathrm{i}}{m}\hat{p}
\end{aligned}$$

whence $-\mathrm{i}[\hat{q}, \hat{H}] = \hat{p}/m$.

5.4 (a) From (5.61) and (5.62), $\hat{q} = (\hat{a} + \hat{a}^\dagger)/(2m\omega)^{\frac{1}{2}}$. So $\frac{1}{2}m\omega^2\hat{q}^2 = \frac{1}{4}\omega(\hat{a} + \hat{a}^\dagger)^2$. Similarly for the $\frac{1}{2m}\hat{p}^2$ term.
(b) $[\hat{a}, \hat{a}^\dagger] = 1$. $[(\hat{a}^\dagger\hat{a} + \frac{1}{2})\omega, \hat{a}] = \omega\ [\hat{a}^\dagger\hat{a}, \hat{a}] = \omega\ \{\hat{a}^\dagger\ [\hat{a}, \hat{a}] + [\hat{a}^\dagger, \hat{a}]\ \hat{a}\} = -\omega\hat{a}$.
(c) We first prove that $\hat{a}(\hat{a}^\dagger)^n - (\hat{a}^\dagger)^n\hat{a} = n\ (\hat{a}^\dagger)^{n-1}$, by induction: (i) it is true for $n = 1$; (ii) multiplying from the left by $\hat{a}^\dagger$ and using $\hat{a}^\dagger\hat{a} = \hat{a}\hat{a}^\dagger - 1$ in the first term proves the result for $n+1$ assuming it's true for n. Letting this result act on $|0\rangle$ we deduce $\hat{a}(\hat{a}^\dagger)^n|0\rangle = n(\hat{a}^\dagger)^{n-1}|0\rangle$. Hence $\langle 0|(\hat{a})^n(\hat{a}^\dagger)^n|0\rangle = \langle 0|(\hat{a})^{n-1}\hat{a}(\hat{a}^\dagger)^n|0\rangle = n\langle 0|(\hat{a})^{n-1}(\hat{a}^\dagger)^{n-1}|0\rangle$, and using induction again (or iterating this step another $n-1$ times) it follows that $\langle 0|(\hat{a})^n(\hat{a}^\dagger)^n|0\rangle = n!$, provided $|0\rangle$ is normalized.

$$\begin{aligned}
\text{(d)}\ \ \hat{n}\ |n\rangle = \frac{1}{\sqrt{n!}}\hat{a}^\dagger\hat{a}(\hat{a}^\dagger)^n|0\rangle &= \frac{1}{\sqrt{n!}}\hat{a}^\dagger\{n(\hat{a}^\dagger)^{n-1} + (\hat{a}^\dagger)^n\hat{a}\}|0\rangle \\
&= n\frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle = n|n\rangle.
\end{aligned}$$

5.6 (a) From (5.116),

$$\hat{\phi}(x,t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi\sqrt{2\omega}}[\hat{a}(k)\mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega t} + \hat{a}^\dagger(k)\mathrm{e}^{-\mathrm{i}kx + \mathrm{i}\omega t}]$$

and so $\hat{\pi}(y,t)$ is given by

$$\hat{\pi}(y,t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k'}{2\pi\sqrt{2\omega'}}[(-\mathrm{i}\omega')\hat{a}(k')\mathrm{e}^{\mathrm{i}k'y - \mathrm{i}\omega' t} + (\mathrm{i}\omega')\hat{a}^\dagger(k')\mathrm{e}^{-\mathrm{i}k'y + \mathrm{i}\omega' t}].$$

Note the use of a *different* integration variable in $\hat{\pi}(x,t)$, to ensure that the operators inside the integral for $\hat{\phi}$ are treated independently of those inside the integral for $\hat{\pi}$. In the commutator $[\hat{\phi}(x,t), \hat{\pi}(y,t)]$ there are therefore four types of term: $[\hat{a}(k), \hat{a}(k')]$, $[\hat{a}^\dagger(k), \hat{a}^\dagger(k')]$, which vanish by the second line of (5.117), and $[\hat{a}(k), \hat{a}^\dagger(k')]$, $[\hat{a}^\dagger(k), \hat{a}(k')]$. The first of these latter two yields

$$\int_{-\infty}^{\infty} \mathrm{d}k \int_{-\infty}^{\infty} \mathrm{d}k' \frac{1}{(2\pi)^2} \frac{1}{\sqrt{4\omega\omega'}} \mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega t}\ \mathrm{e}^{-\mathrm{i}k'y + \mathrm{i}\omega' t}(\mathrm{i}\omega') 2\pi \delta(k - k')$$

$$= \int_{-\infty}^{\infty} \mathrm{d}k \frac{1}{2\pi} \mathrm{i}\omega \frac{1}{2\omega} \mathrm{e}^{\mathrm{i}k(x-y)} = \frac{\mathrm{i}}{2}\delta(x-y)$$

using (E.25). The $[\hat{a}^\dagger(k), \hat{a}(k')]$ term gives the same.

(b)

$$[\hat{\phi}(\boldsymbol{x}_1, t_1), \hat{\phi}(\boldsymbol{x}_2, t_2)] = \int_{-\infty}^{\infty} \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2E}} \int_{-\infty}^{\infty} \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2E'}}$$

$$[(\hat{a}(k)\mathrm{e}^{-\mathrm{i}k\cdot x_1} + \hat{a}^\dagger(k)\mathrm{e}^{\mathrm{i}k\cdot x_1}), (\hat{a}(k')\mathrm{e}^{-\mathrm{i}k'\cdot x_2} + \hat{a}^\dagger(k')\mathrm{e}^{\mathrm{i}k'\cdot x_2})].$$

The surviving terms are the '$[\hat{a}, \hat{a}^\dagger]$' and '$[\hat{a}^\dagger, \hat{a}]$' ones which give

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3\sqrt{2E}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^3\sqrt{2E'}} [\mathrm{e}^{-\mathrm{i}k\cdot x_1}\ \mathrm{e}^{\mathrm{i}k'\cdot x_2}(2\pi)^3\delta^3(\boldsymbol{k} - \boldsymbol{k}')$$

$$-\mathrm{e}^{\mathrm{i}k\cdot x_1}\ \mathrm{e}^{-\mathrm{i}k'\cdot x_2}(2\pi)^3\delta^3(\boldsymbol{k} - \boldsymbol{k}')]$$

$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2E} [\mathrm{e}^{-\mathrm{i}k\cdot(x_1 - x_2)} - \mathrm{e}^{\mathrm{i}k\cdot(x_1 - x_2)}]$$

with $k \cdot (x_1 - x_2) = E(t_1 - t_2) - \boldsymbol{k} \cdot (\boldsymbol{x}_1 - \boldsymbol{x}_2)$. For $t_1 = t_2$ the integral is

$$\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2E} [\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)} - \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)}]$$

which vanishes since the integrand is an odd function of $\boldsymbol{k}$. So D is Lorentz invariant (see Appendix E) and vanishes when $t_1 = t_2$. It therefore also vanishes at all points which can be connected to '$t_1 = t_2$' by a Lorentz transformation. Noting that for $t_1 = t_2$ the invariant interval $(x_1 - x_2)^2$ is spacelike (i.e. is less than zero), we deduce that D vanishes for all x_1, x_2 such that $(x_1 - x_2)^2 < 0$.

5.7 We have

$$\begin{aligned}\hat\phi(x,t) &= \int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi\sqrt{2\omega}}[\hat a(k)\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}+\hat a^\dagger(k)\mathrm{e}^{-\mathrm{i}kx+\mathrm{i}\omega t}]\\ \frac{\partial\hat\phi}{\partial x} &= \int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi\sqrt{2\omega}}[\hat a(k)(\mathrm{i}k)\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}+\hat a^\dagger(k)(-\mathrm{i}k)\mathrm{e}^{-\mathrm{i}kx+\mathrm{i}\omega t}]\end{aligned}$$

and so

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}x\left(\frac{\partial\phi}{\partial x}\right)^2 &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{(2\pi)\sqrt{2\omega}}[\hat a(k)(\mathrm{i}k)\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}+\hat a^\dagger(k)(-\mathrm{i}k)\mathrm{e}^{-\mathrm{i}kx+\mathrm{i}\omega t}].\\ &\quad\int_{-\infty}^{\infty}\frac{\mathrm{d}k'}{(2\pi)\sqrt{2\omega'}}[\hat a(k')(\mathrm{i}k')\mathrm{e}^{\mathrm{i}k'x-\mathrm{i}\omega' t}+\hat a^\dagger(k')(-\mathrm{i}k')\mathrm{e}^{-\mathrm{i}k'x+\mathrm{i}\omega' t}].\end{aligned}$$

There are four terms when the brackets are multiplied out. For example, the first is

$$\frac{1}{2}\int\mathrm{d}x\int\frac{\mathrm{d}k}{2\pi\sqrt{2\omega}}\int\frac{\mathrm{d}k'}{2\pi\sqrt{2\omega'}}(\mathrm{i}k)(\mathrm{i}k')\hat a(k)\hat a(k')\mathrm{e}^{\mathrm{i}(k+k')x-\mathrm{i}(\omega+\omega')t}.$$

The integral over x yields $(2\pi)\delta(k+k')$; the k'-integral can then be done leading to

$$\frac{1}{2}\int\frac{\mathrm{d}k}{(2\pi)\sqrt{2\omega}}\frac{1}{2\pi}\frac{1}{\sqrt{2\omega}}(\mathrm{i}k)(-\mathrm{i}k)\hat a(k)\hat a(-k)\mathrm{e}^{-2\mathrm{i}\omega t}\qquad(**)$$

using $\omega'=|k'|=k=\omega$. The three other terms can be reduced similarly. Meanwhile,

$$\hat\pi(x,t)=\dot{\hat\phi}(x,t)=\int\frac{\mathrm{d}x}{2\pi\sqrt{2\omega}}[\hat a(k)(-\mathrm{i}\omega)\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}+\hat a^\dagger(k)(\mathrm{i}\omega)\mathrm{e}^{-\mathrm{i}kx+\mathrm{i}\omega t}]$$

and so

$$\begin{aligned}\frac{1}{2}\int\hat\pi^2\mathrm{d}x &= \frac{1}{2}\int\mathrm{d}x\int\frac{\mathrm{d}k}{2\pi\sqrt{2\omega}}[\hat a(k)(-\mathrm{i}\omega)\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}+\hat a^\dagger(k)(\mathrm{i}\omega)\mathrm{e}^{-\mathrm{i}kx+\mathrm{i}\omega t}].\\ &\quad\int\frac{\mathrm{d}k'}{2\pi\sqrt{2\omega'}}[\hat a(k')(-\mathrm{i}\omega')\mathrm{e}^{\mathrm{i}k'x-\mathrm{i}\omega' t}+\hat a^\dagger(k')(\mathrm{i}\omega')\mathrm{e}^{-\mathrm{i}k'x+\mathrm{i}\omega' t}].\end{aligned}$$

For the '$\hat a\hat a$' term the x-integral yields

$$\frac{1}{2}\int\frac{\mathrm{d}k}{[(2\pi)\sqrt{2\omega}]^2}(-\mathrm{i}\omega)^2\hat a(k)\hat a(-k)\mathrm{e}^{-2\mathrm{i}\omega t}$$

which cancels against $(**)$. Similarly, the '$\hat{a}^\dagger\hat{a}^\dagger$' terms cancel, and the '$\hat{a}\hat{a}^\dagger$' and '$\hat{a}^\dagger\hat{a}$' terms give (5.119).

5.8 (a) We have

$$\begin{aligned}\dot{\hat{\phi}}(x,t) &= -\mathrm{i}[\hat{\phi}(x,t),\hat{H}] \\ &= -\mathrm{i}[\hat{\phi}(x,t),\frac{1}{2}\int\{\hat{\pi}^2(y,t)+\left(\frac{\partial\hat{\phi}}{\partial y}\right)^2\}\mathrm{d}y].\end{aligned}$$

$\hat{\phi}$ commutes with $(\partial\hat{\phi}/\partial y)^2$, and the non-vanishing term on the RHS is

$$\begin{aligned}&-\mathrm{i}\frac{1}{2}[\hat{\phi}(x,t),\int\hat{\pi}^2(y,t)\mathrm{d}y] \\ =\ &\frac{-\mathrm{i}}{2}\int\{\hat{\pi}(y,t)[\hat{\phi}(x,t),\hat{\pi}(y,t)]+[\hat{\phi}(x,t),\hat{\pi}(y,t)]\hat{\pi}(y,t)\}\mathrm{d}y \\ =\ &\frac{-\mathrm{i}}{2}\int\{\hat{\pi}(y,t)\mathrm{i}\delta(x-y)+\mathrm{i}\delta(x-y)\hat{\pi}(y,t)\}\mathrm{d}y=\hat{\pi}(x,t)\end{aligned}$$

as required.

Chapter 6

6.1 The RHS is

$$\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}t_1\left\{\int_{-\infty}^{t_1}\mathrm{d}t_2\hat{f}(t_1)\hat{f}(t_2)+\int_{t_1}^{\infty}\mathrm{d}t_2\hat{f}(t_2)\hat{f}(t_1)\right\}. \quad \text{(A)}$$

The second term is

$$\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}t_1\int_{t_1}^{\infty}\mathrm{d}t_2\ \hat{f}(t_2)\hat{f}(t_1).$$

With the usual convention for double integrals, in this expression the integral over t_2 is understood to be performed first, holding t_1 fixed, and then t_1 is integrated over. We proceed by changing the order of integration in this double integral. In order not to make a mistake, it is a good idea to draw a diagram of the t_1 (horizontal) - t_2 (vertical) plane, with the region of integration shaded over: it is the whole of the region lying above the line $t_1 = t_2$. When we write the integral in the other order (i.e. doing first the t_1 integral and then the t_2 one) we must be careful to arrange the limits so that the required region is covered. The result is

$$\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t_2 \int_{-\infty}^{t_2} \mathrm{d}t_1 \; \hat{f}(t_2)\hat{f}(t_1).$$

We now rename t_1 as t_2, and t_2 as t_1, showing that this term is in fact equal to the first in (A).

6.2 In the rest frame of C,

$$m_{\mathrm{C}} = E_{\mathrm{A}} + E_{\mathrm{B}} = \sqrt{m_{\mathrm{A}}^2 + \boldsymbol{p}^2} + \sqrt{m_{\mathrm{B}}^2 + \boldsymbol{p}^2}.$$

Taking the term $\sqrt{m_{\mathrm{A}}^2 + \boldsymbol{p}^2}$ to the LHS, squaring, cancelling the $\boldsymbol{p}^2$, and squaring again we arrive at

$$(m_{\mathrm{A}}^2 + m_{\mathrm{C}}^2 - m_{\mathrm{B}}^2)^2 = 4m_{\mathrm{C}}^2(m_{\mathrm{A}}^2 + \boldsymbol{p}^2).$$

This gives $|\boldsymbol{p}|$ as required.

6.3 (iii)

$$\begin{aligned}
&\frac{\partial}{\partial t_1}[\theta(t_1 - t_2)\hat{\phi}(x_1, t_1)\hat{\phi}(x_2, t_2) + \theta(t_2 - t_1)\hat{\phi}(x_2, t_2)\hat{\phi}(x_1, t_1)] \\
&= \delta(t_1 - t_2)\hat{\phi}(x_1, t_1)\hat{\phi}(x_2, t_2) + \theta(t_1 - t_2)(\frac{\partial}{\partial t_1}\hat{\phi}(x_1, t_1))\hat{\phi}(x_2, t_2) \\
&-\delta(t_2 - t_1)\hat{\phi}(x_2, t_2)\hat{\phi}(x_1, t_1) + \theta(t_2 - t_1)\hat{\phi}(x_2, t_2)(\frac{\partial}{\partial t_1}\hat{\phi}(x_1, t_1))
\end{aligned}$$

The δ functions pick out only the point $t_1 = t_2$ in the terms multiplying them, and at this (equal-time) point the $\hat{\phi}$ fields commute according to (5.105), leading to the result (($\dot{\hat{\phi}}(x_1, t_1)$ is short for $\partial\hat{\phi}(x_1, t_1)/\partial t_1$).
(iv) Differentiating the result of (iii) with respect to t_1 we get

$$\begin{aligned}
\frac{\partial^2}{\partial t_1^2}\{T(\hat{\phi}(x_1, t_1)\hat{\phi}(x_2, t_2)\} \;&=\; \delta(t_1 - t_2)\hat{\pi}(x_1, t_1)\hat{\phi}(x_2, t_2) + \theta(t_1 - t_2)\ddot{\hat{\phi}}(x_1, t_1)\hat{\phi}(x_2, t_2) \\
&\quad -\delta(t_2 - t_1)\hat{\phi}(x_2, t_2)\hat{\pi}(x_1, t_1) + \theta(t_2 - t_1)\hat{\phi}(x_2, t_2)\ddot{\hat{\phi}}(x_1, t_1).
\end{aligned}$$

Again the δ functions force t_1 to equal t_2 in the terms multiplying them, which become

$$[\hat{\phi}(x_1, t_1), \hat{\phi}(x_2, t_2)] \; \delta(t_1 - t_2) = -\mathrm{i}\delta(x_1 - x_2)\delta(t_1 - t_2)$$

using (5.104) and (E.32). The remaining two terms are just $T\{\ddot{\hat{\phi}}(x_1,t_1)\hat{\phi}(x_2,t_2)\}$. Since the operator $-\partial^2/\partial x_1^2+m^2$ doesn't involve t_1, it 'passes through' the θ-functions in the T-product, acting just on $\hat{\phi}(x_1,t_1)$:

$$(-\frac{\partial^2}{\partial x_1^2}+m^2)T\{\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)\}=T\left\{\left((-\frac{\partial^2}{\partial x_1^2}+m^2)\hat{\phi}(x_1,t_1))\right)\hat{\phi}(x_2,t_2)\right\}.$$

Hence

$$\left(\frac{\partial^2}{\partial t_1^2}-\frac{\partial^2}{\partial x_1^2}+m^2\right)T\{\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)\}$$

$$=-\mathrm{i}\delta(x_1-x_2)\delta(t_1-t_2)+T\left\{\left((\frac{\partial^2}{\partial t_1^2}-\frac{\partial^2}{\partial x_1^2}+m^2)\hat{\phi}(x_1,t_1)\right)\hat{\phi}(x_2,t_2)\right\}.$$

The equation of motion for the free KG field $\hat{\phi}$ implies that the second term vanishes.

6.4

$$\langle 0|\hat{a}_{\mathrm{A}}(p'_{\mathrm{A}})\hat{\phi}_{\mathrm{A}}(x_1)|0\rangle=\langle 0|\hat{a}_{\mathrm{A}}(p'_{\mathrm{A}})\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2E_k}}[\hat{a}_{\mathrm{A}}(k)\mathrm{e}^{-\mathrm{i}k\cdot x_1}+\hat{a}^\dagger_{\mathrm{A}}(k)\mathrm{e}^{\mathrm{i}k\cdot x_1}]|0\rangle,$$

with $E_k=\sqrt{m_{\mathrm{A}}^2+\boldsymbol{k}^2}$. The $\hat{a}_{\mathrm{A}}(k)$ term gives zero on $|0\rangle$. To evaluate the '$\hat{a}_{\mathrm{A}}\hat{a}^\dagger_{\mathrm{A}}$' term, we use (6.46):

$$\hat{a}_{\mathrm{A}}(p'_{\mathrm{A}})\hat{a}^\dagger_{\mathrm{A}}(k)=(2\pi)^3\delta^3(\boldsymbol{p}'_{\mathrm{A}}-\boldsymbol{k})+\hat{a}^\dagger_{\mathrm{A}}(k)\hat{a}_{\mathrm{A}}(p'_{\mathrm{A}})$$

leading to

$$\begin{aligned}\langle 0|\hat{a}_{\mathrm{A}}(p'_{\mathrm{A}})\hat{\phi}_{\mathrm{A}}(x_1)|0\rangle &= \langle 0|\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2E_k}}\ \mathrm{e}^{\mathrm{i}k\cdot x_1}(2\pi)^3\delta^3(\boldsymbol{p}'_{\mathrm{A}}-\boldsymbol{k})|0\rangle\\ &= \frac{1}{\sqrt{2E'_{\mathrm{A}}}}\ \mathrm{e}^{\mathrm{i}p'_{\mathrm{A}}\cdot x_1},\end{aligned}$$

where $E'_{\mathrm{A}}=(m_{\mathrm{A}}^2+{\boldsymbol{p}'_{\mathrm{A}}}^2)^{1/2}$.

6.5

$$\begin{aligned}\langle 0|T\{\hat{\phi}_{\mathrm{C}}(x_1)\hat{\phi}_{\mathrm{C}}(x_2)\}|0\rangle &= \langle 0|\theta(t_1-t_2)\hat{\phi}_{\mathrm{C}}(x_1)\hat{\phi}_{\mathrm{C}}(x_2)+\theta(t_2-t_1)\hat{\phi}_{\mathrm{C}}(x_2)\hat{\phi}_{\mathrm{C}}(x_1)|0\rangle\\ &= \langle 0|\left\{\theta(t_1-t_2)\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega_k}}\ [\hat{a}_{\mathrm{C}}(k)\mathrm{e}^{-\mathrm{i}k\cdot x_1}+\hat{a}^\dagger_{\mathrm{C}}(k)\mathrm{e}^{\mathrm{i}k\cdot x_1}]\right.\end{aligned}$$

$$\times \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega_{k'}}}\,[\hat{a}_C(k')e^{-ik'\cdot x_2} + \hat{a}_C^\dagger(k')e^{ik'\cdot x_2}]|0\rangle + \text{similar term multiplying } \theta(t_2-t_1)\Big\}$$

where $k^0 = \omega_k = \sqrt{m_C^2 + \boldsymbol{k}^2}$ and $k'_0 = \omega'_k = \sqrt{m_C^2 + \boldsymbol{k}'^2}$. The conditions $\langle 0|\hat{a}_C^\dagger(k) = \hat{a}_C(k')|0\rangle = 0$ reduce the $\theta(t_1-t_2)$ term to

$$\langle 0|\theta(t_1-t_2)\int \frac{d^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega_k}} \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega_{k'}}}\,\hat{a}_C(k)\hat{a}_C^\dagger(k')e^{-ik\cdot x_1}\, e^{ik'\cdot x_2}|0\rangle.$$

Using

$$\hat{a}_C(k)\hat{a}_C^\dagger(k') = (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}') + \hat{a}_C^\dagger(k')\hat{a}_C(k)$$

this becomes

$$\theta(t_1-t_2)\int \frac{d^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega_k}} \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega_{k'}}}\,(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')e^{-ik\cdot x_1}\, e^{ik'\cdot x_2}$$
$$= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3 2\omega_k}\,\theta(t_1-t_2)e^{-i\omega_k(t_1-t_2)+i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}.$$

Similarly for the $\theta(t_2-t_1)$ term.

6.6 We write (6.91) as

$$(-ig)^2\int\int d^4x_1 d^4x_2 e^{i(p'_A-p_B)\cdot x_1}\, e^{i(p'_B-p_A)\cdot x_2}\, f(x_1-x_2)$$

where $f(x_1-x_2)$ is given by the RHS of (6.98). Introducing $x = x_1 - x_2$, $X = (x_1+x_2)/2$, the integral becomes

$$\int\int d^4x d^4X\; e^{i(p'_A-p_B)\cdot(X+x/2)}\, e^{i(p'_B-p_A)\cdot(X-x/2)}\, f(x)$$

since the Jacobian for the change of variables is unity (for instance, $dt_1dt_2 \to d(t_1-t_2)d(t_1+t_2)/2$, etc.) The only place X appears is in the exponent, allowing the X-integral to be done using the 4-dimensional version of (E.26); we then get

$$(2\pi)^4\delta^4(p'_A - p_B + p'_B - p_A)\int d^4x\; e^{i[(p'_A-p_B)-(p'_B-p_A)]\cdot x/2}\, f(x).$$

The 4-momentum δ-function ensures that in the exponent $p'_{\mathrm{A}} - p_{\mathrm{B}}$ is equal to $-(p'_{\mathrm{B}} - p_{\mathrm{A}})$, each being equal to q. Re-instating the factor $(-\mathrm{i}g)^2$ we obtain

$$(-\mathrm{i}g)^2(2\pi)^4\delta^4(p'_{\mathrm{A}} + p'_{\mathrm{B}} - p_{\mathrm{A}} - p_{\mathrm{B}}) \int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}q\cdot x}\ f(x)$$

which is (6.99). Substituting now the RHS of (6.98) for $f(x)$ and performing the x-integration gives a factor $(2\pi)^2\delta^4(q-k)$, and finally performing the k-integration gives (6.100).

6.7 The analogue of (6.91) for this contraction is

$$(-\mathrm{i}g)^2 \int\int \mathrm{d}^4x_1\mathrm{d}^4x_2\ \mathrm{e}^{\mathrm{i}(p'_{\mathrm{A}}+p'_{\mathrm{B}})\cdot x_1}\ \mathrm{e}^{-\mathrm{i}(p_{\mathrm{A}}+p_{\mathrm{B}})\cdot x_2}\ \langle 0|T(\hat{\phi}_{\mathrm{C}}(x_1)\hat{\phi}_{\mathrm{C}}(x_2))|0\rangle.$$

Introducing x and X as in problem 6.6, this becomes

$$(-\mathrm{i}g)^2(2\pi)^4\delta^4(p'_{\mathrm{A}} + p'_{\mathrm{B}} - p_{\mathrm{A}} - p_{\mathrm{B}}) \int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}(p_{\mathrm{A}}+p_{\mathrm{B}})\cdot x}\ f(x).$$

Inserting (6.98) for $f(x)$ and performing the x and then the k integrals leads to the same expression as (6.100) but with q replaced by $p_{\mathrm{A}} + p_{\mathrm{B}}$ - that is, (6.101).

6.8 In the CM frame,

$$p_{\mathrm{A}} = (\sqrt{m_{\mathrm{A}}^2 + \boldsymbol{k}^2}, \boldsymbol{k}),\quad p'_{\mathrm{B}} = (\sqrt{m_{\mathrm{A}}^2 + \boldsymbol{k}'^2}, \boldsymbol{k}'),$$

and $|\boldsymbol{k}| = |\boldsymbol{k}'|$ by energy conservation. Hence

$$u = (p_{\mathrm{A}} - p'_{\mathrm{B}})^2 = (0, (\boldsymbol{k} - \boldsymbol{k}'))^2 = -(\boldsymbol{k} - \boldsymbol{k}')^2 \leq 0.$$

6.9 In the frame in which B is initially at rest, $p_{\mathrm{B}} = (m_{\mathrm{B}}, \boldsymbol{0})$, $p_{\mathrm{A}} = (\sqrt{m_{\mathrm{A}}^2 + \boldsymbol{p}^2}, \boldsymbol{p})$ and

$$\begin{aligned}(p_{\mathrm{A}} \cdot p_{\mathrm{B}})^2 - m_{\mathrm{A}}^2 m_{\mathrm{B}}^2 &= (m_{\mathrm{B}}\sqrt{m_{\mathrm{A}}^2 + \boldsymbol{p}^2})^2 - m_{\mathrm{A}}^2 m_{\mathrm{B}}^2 \\ &= m_{\mathrm{B}}^2 \boldsymbol{p}^2.\end{aligned}$$

Hence $[(p_{\mathrm{A}} \cdot p_{\mathrm{B}})^2 - m_{\mathrm{A}}^2 m_{\mathrm{B}}^2]^{1/2} = m_{\mathrm{B}}|\boldsymbol{p}|$. But in this frame $m_{\mathrm{B}} = E_{\mathrm{B}}$, and $|\boldsymbol{p}| = E_{\mathrm{A}}|\boldsymbol{v}|$.

Chapter 7

7.2 (a) The calculation is similar to that in problem 5.7.

$$\hat{N}^0_\phi = \mathrm{i}\int(\phi^\dagger\dot{\hat{\phi}} - \hat{\phi}\dot{\hat{\phi}}^\dagger)\ \mathrm{d}^3\boldsymbol{x} \quad = \quad \mathrm{i}\int \mathrm{d}^3\boldsymbol{x}\left\{\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega}}\ (\hat{a}^\dagger(k)\mathrm{e}^{\mathrm{i}k\cdot x} + \hat{b}(k)\mathrm{e}^{-\mathrm{i}k\cdot x})\right.$$

$$\times \quad \int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega'}}\ (-\mathrm{i}\omega'\hat{a}(k')\mathrm{e}^{-\mathrm{i}k'\cdot x} + \mathrm{i}\omega'\hat{b}^\dagger(k')\mathrm{e}^{\mathrm{i}k'\cdot x})$$

$$\left. - \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega}}\ (\hat{a}(k)\mathrm{e}^{-\mathrm{i}k\cdot x} + \hat{b}^\dagger(k)\mathrm{e}^{\mathrm{i}k\cdot x}) \int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega'}}\ (\mathrm{i}\omega'\hat{a}^\dagger(k')\mathrm{e}^{\mathrm{i}k'\cdot x} - \mathrm{i}\omega'\hat{b}(k')\mathrm{e}^{-\mathrm{i}k'\cdot x})\right\}.$$

After multiplying out the brackets, the terms $\hat{a}^\dagger\hat{b}^\dagger$ and $\hat{a}\hat{b}$ cancel using $[\hat{a}^\dagger(k), \hat{b}^\dagger(k')] = [\hat{a}(k), \hat{b}(k')] = 0$. The term $\hat{a}^\dagger\hat{a}$ is

$$\int \mathrm{d}^3\boldsymbol{x} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega}} \int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega'}}\omega'\hat{a}^\dagger(k)\hat{a}(k')\mathrm{e}^{\mathrm{i}(\omega-\omega')t-\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}}.$$

The $\boldsymbol{x}$-integral gives $(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')$, which forces $\omega = \omega'$ and allows the $\boldsymbol{k}'$-integral to be done giving

$$\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2}\hat{a}^\dagger(k)\hat{a}(k).$$

The term $\hat{a}\hat{a}^\dagger$ gives the same when normally ordered (throwing away a divergent piece); similarly the $\hat{b}^\dagger\hat{b}$ terms.

7.3 The commutator is

$$[\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega}}\ (\hat{a}(k)\mathrm{e}^{-\mathrm{i}k\cdot x_1}+\hat{b}^\dagger(k)\mathrm{e}^{\mathrm{i}k\cdot x_2}), \int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega'}}\ (\hat{a}^\dagger(k')\mathrm{e}^{\mathrm{i}k\cdot x_2}+\hat{b}(k')\mathrm{e}^{-\mathrm{i}k\cdot x_2})].$$

The non-vanishing terms involve $[\hat{a}(k), \hat{a}^\dagger(k')]$ and $[\hat{b}^\dagger(k), \hat{b}(k')]$. The first of these gives

$$\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega}} \int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega'}}\ \mathrm{e}^{-\mathrm{i}k\cdot x_1}\ \mathrm{e}^{\mathrm{i}k'\cdot x_2}(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')$$

$$= \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega}\, \mathrm{e}^{-\mathrm{i}k\cdot(x_1-x_2)}$$

and the second gives

$$-\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega}\, \mathrm{e}^{\mathrm{i}k\cdot(x_1-x_2)}.$$

As long as $(x_1 - x_2)$ is spacelike ($(x_1 - x_2)^2 < 0$) we can transform $x_1 - x_2$ into $-(x_1 - x_2)$ by a continuous Lorentz transformation, and so the second term cancels the first for $(x_1 - x_2)^2 < 0$.

Similar manipulations show that the two terms in the commutator can be re-written as follows. The first is

$$\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega}\, \mathrm{e}^{-\mathrm{i}k\cdot(x_1-x_2)} = \langle 0|\hat{\phi}(x_1)\hat{\phi}^\dagger(x_2)|0\rangle$$

which has the interpretation that a particle is created at x_2 and destroyed at x_1. The second is

$$-\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega}\, \mathrm{e}^{\mathrm{i}k\cdot(x_1-x_2)} = -\langle 0|\hat{\phi}^\dagger(x_2)\hat{\phi}(x_1)|0\rangle$$

which corresponds to the creation of an antiparticle at x_1 and its destruction at x_2. Compare the comments following (7.32). Thus for the commutator to vanish in the spacelike region, the contributions of these two processes must cancel, and this requires the components of 'k' to be the same in each case - that is, the masses of the particle and antiparticle must be identical.

7.5 The LHS of the anticommutator (7.44) is

$$\left\{ \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega}} \sum_{s=1,2} (\hat{c}_s(k) u_\alpha(k,s) \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{d}_s^\dagger(k) v_\alpha(k,s) \mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}), \right.$$

$$\left. \int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega'}} \sum_{s'=1,2} (\hat{c}_{s'}^\dagger(k') u_\beta^\dagger(k',s') \mathrm{e}^{\mathrm{i}\omega' t - \mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{y}} + \hat{d}_{s'}(k') v_\beta^\dagger(k',s') \mathrm{e}^{-\mathrm{i}\omega' t + \mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{y}}) \right\}.$$

From (7.41) the only non-vanishing terms involve $\{\hat{c}_s(k), \hat{c}^\dagger_{s'}(k')\}$ and $\{\hat{d}^\dagger_s(k), \hat{d}_{s'}(k')\}$, which give

$$\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega}} \int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega'}} \sum_{s=1,2} \sum_{s'=1,2}$$

$$[u_\alpha(k,s)u^\dagger_\beta(k',s')\mathrm{e}^{-\mathrm{i}(\omega-\omega')t}\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\ \mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{y}}(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')\delta_{ss'} + \text{similar term in } \ v_\alpha v^\dagger_\beta]$$

$$= \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega}[\sum_{s=1,2} u_\alpha(k,s)u^\dagger_\beta(k,s)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} + \sum_{s=1,2} v_\alpha(k,s)v^\dagger_\beta(k,s)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}]. \quad (C)$$

We need to evaluate $\sum_s u_\alpha(k,s)u^\dagger_\beta(k,s)$ and $\sum_s v_\alpha(k,s)v^\dagger_\beta(k,s)$. Problem 7.8 below deals with similar expressions and gives some hints. First, note that '$uu^\dagger$' is a matrix ($u^\dagger u$ is single number), and $u_\alpha u^\dagger_\beta$ is its (α,β) element. We can calculate the matrix $u(k,s)u^\dagger(k,s)$ straightforwardly using (4.105):

$$\begin{aligned} u(k,s)u^\dagger(k,s) &= (\omega+m)\begin{pmatrix} \phi^s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{\omega+m}\phi^s \end{pmatrix} \begin{pmatrix} \phi^{s\dagger} & \phi^{s\dagger}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{\omega+m} \end{pmatrix} \\ &= (\omega+m)\begin{pmatrix} \phi^s\phi^{s\dagger} & \phi^s\phi^{s\dagger}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{\omega+m} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{\omega+m}\phi^s\phi^{s\dagger} & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{\omega+m}\phi^s\phi^{s\dagger}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{\omega+m} \end{pmatrix}. \end{aligned}$$

We now use $\sum_s \phi^s\phi^{s\dagger} = \mathbf{1}$ (see problem 7.8) to obtain

$$\sum_s u(k,s)u^\dagger(k,s) = (\omega+m)\begin{pmatrix} \mathbf{1} & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{\omega+m} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{\omega+m} & \frac{\boldsymbol{k}^2}{(\omega+m)^2} \end{pmatrix}.$$

Since $\boldsymbol{k}^2 = \omega^2 - m^2$ this becomes

$$\begin{pmatrix} \omega+m & \boldsymbol{\sigma}\cdot\boldsymbol{k} \\ \boldsymbol{\sigma}\cdot\boldsymbol{k} & \omega-m \end{pmatrix} \qquad (u)$$

which is in fact the matrix $\beta m + \boldsymbol{\alpha}\cdot\boldsymbol{k} + \omega$ using (4.31). For the '$\sum vv^\dagger$' term, we first change the integration variable $\boldsymbol{k}$ to $-\boldsymbol{k}$ so as to have the same exponent as the '$\sum uu^\dagger$' term. This means that in evaluating $\sum_s v(k,s)v^\dagger(k,s)$ we use (4.114) with $\boldsymbol{p}$ replaced by $-\boldsymbol{k}$ (and E by ω), which gives

$$\begin{pmatrix} \omega-m & -\boldsymbol{\sigma}\cdot\boldsymbol{k} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{k} & \omega-m \end{pmatrix}. \qquad (v)$$

The sum of (u) and (v) is just 2ω times the unit 4×4 matrix, whose (α, β) element is $\delta_{\alpha\beta}$. Inserting this into (C) gives (7.44).

7.6

$$\begin{aligned}
\hat{N}_\psi &= \int \mathrm{d}^3\boldsymbol{x} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega}} \sum_s [\hat{c}_s^\dagger(k)u^\dagger(k,s)\mathrm{e}^{\mathrm{i}k\cdot x} + \hat{d}_s(k)v^\dagger(k,s)\mathrm{e}^{-\mathrm{i}k\cdot x}] \\
&\times \int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega'}} \sum_{s'} [\hat{c}_{s'}(k')u^\dagger(k',s')\mathrm{e}^{-\mathrm{i}k'\cdot x} + \hat{d}_{s'}^\dagger(k')v^\dagger(k',s')\mathrm{e}^{\mathrm{i}k'\cdot x}].
\end{aligned}$$

Consider the term '$\hat{c}_s^\dagger \hat{c}_{s'}$'. The integral over $\boldsymbol{x}$ gives $(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')$ and the $\boldsymbol{k}'$-integral can then be done giving the term

$$\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega} \sum_{s,s'} \hat{c}_s^\dagger(k)\hat{c}_{s'}(k)u^\dagger(k,s)u(k,s).$$

Taking ϕ^1, ϕ^2 to be orthogonal (as is conventional), problem 4.12 implies that $u^\dagger(k,s)u(k,s') = 2\omega\delta_{ss'}$, so that this term is

$$\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \sum_s \hat{c}_s^\dagger(k)\hat{c}_s(k).$$

The '$\hat{d}_s\hat{d}_{s'}^\dagger$' term is handled similarly, using $v^\dagger(k,s)v(k,s') = 2\omega\delta_{ss'}$. Now consider the '$\hat{c}_s^\dagger\hat{d}_{s'}^\dagger$' term: the integral over $\boldsymbol{x}$ yields $(2\pi)^3\delta^3(\boldsymbol{k} + \boldsymbol{k}')$ so that we need to evaluate $u^\dagger((\omega,\boldsymbol{k}),s)v((\omega,-\boldsymbol{k}),s')$. It is easily verified from (4.105) and (4.114) that this vanishes; similarly for the '$\hat{d}_s\hat{c}_{s'}$' term.

$$\begin{aligned}
\hat{H}_{\mathrm{D}} &= \int \mathrm{d}^3\boldsymbol{x}\{\hat{\pi}\dot{\hat{\psi}} - \hat{\mathcal{L}}_{\mathrm{D}}\} = \int \mathrm{d}^3\boldsymbol{x}\hat{\psi}^\dagger(-\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m)\hat{\psi} \\
&= \int \mathrm{d}^3\boldsymbol{x} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega}} [\sum_s \hat{c}_s^\dagger(k)u^\dagger(k,s)\mathrm{e}^{\mathrm{i}k\cdot x} + \hat{d}_s(k)v^\dagger(k,s)\mathrm{e}^{-\mathrm{i}k\cdot x}] \\
&\times (-\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m) \int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega'}} [\sum_{s'} \hat{c}_{s'}(k')u(k',s')\mathrm{e}^{-\mathrm{i}k'\cdot x} + \hat{d}_{s'}^\dagger(k')v(k',s')\mathrm{e}^{\mathrm{i}k'\cdot x}].
\end{aligned}$$

Now note that

$$\begin{aligned}
(-\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m)u(k',s')\mathrm{e}^{-\mathrm{i}k'\cdot x} &= (\boldsymbol{\alpha}\cdot\boldsymbol{k}' + \beta m)u(k',s')\mathrm{e}^{-\mathrm{i}k'\cdot x} \\
&= \omega' u(k',s')\mathrm{e}^{-\mathrm{i}k'\cdot x}.
\end{aligned}$$

In the '$\hat{c}_s^\dagger \hat{c}_{s'}$' term, the $\boldsymbol{x}$-integral can be done, and then the $\boldsymbol{k}'$-integral, leading to

$$\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega} \sum_{s,s'} \hat{c}_s^\dagger(k)\hat{c}_{s'}(k)u^\dagger(k,s)u(k,s')\omega.$$

Use of $u^\dagger(k,s)u(k,s') = 2\omega\delta_{ss'}$ then gives the first term in (7.50). For the '$\hat{d}_s \hat{d}_{s'}^\dagger$' term note that

$$\begin{aligned}(-\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m)v(k',s')\mathrm{e}^{\mathrm{i}k'\cdot x} &= (-\boldsymbol{\alpha}\cdot\boldsymbol{k}' + \beta m)v(k',s')\mathrm{e}^{\mathrm{i}k'\cdot x} \\ &= -\omega' v(k',s')\mathrm{e}^{\mathrm{i}k'\cdot x}.\end{aligned}$$

Performing the $\boldsymbol{x}$-integral and then the $\boldsymbol{k}'$-integral leads to the second term of (7.50). The '$\hat{c}_s^\dagger \hat{d}_s^\dagger$' and '$\hat{d}_s \hat{c}_s$' terms vanish as in the calculation of $\hat{N}_\psi$.

7.8 The solution to problem 7.5 showed that

$$\sum_s u(k,s)u^\dagger(k,s) = (\beta m + \boldsymbol{\alpha}\cdot\boldsymbol{k} + \omega).$$

Hence

$$\begin{aligned}\sum_s u(k,s)\bar{u}(k,s) &= (\beta^2 m + \boldsymbol{\alpha}\beta\cdot\boldsymbol{k} + \beta\omega) = (m - \boldsymbol{\gamma}\cdot\boldsymbol{k} + \beta\omega) \\ &= (\not{k} + m).\end{aligned}$$

7.9

$$\langle 0|T(\hat{\psi}_\alpha(x_1)\hat{\bar{\psi}}(x_2))|0\rangle =$$

$$\langle 0|\theta(t_1 - t_2)\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega}}\,[\sum_s \hat{c}_s(k)u_\alpha(k,s)\mathrm{e}^{-\mathrm{i}k\cdot x_1} + \hat{d}_s^\dagger(k)v_\alpha(k,s)\mathrm{e}^{\mathrm{i}k\cdot x_1}]$$
$$\times \int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega'}}\,[\sum_{s'} \hat{c}_{s'}^\dagger(k)\bar{u}_\beta(k',s')\mathrm{e}^{\mathrm{i}k'\cdot x_2} + \hat{d}_{s'}(k')\bar{v}_\beta(k',s')\mathrm{e}^{-\mathrm{i}k'\cdot x_2}]|0\rangle$$

$$-\langle 0|\theta(t_2 - t_1)\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega}}\,[\sum_s \hat{c}_s^\dagger(k)\bar{u}_\beta(k,s)\mathrm{e}^{\mathrm{i}k\cdot x_2} + \hat{d}_s(k)\bar{v}_\beta(k,s)\mathrm{e}^{-\mathrm{i}k\cdot x_2}]$$
$$\times \int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega'}}\,[\sum_{s'} \hat{c}_{s'}(k)u_\alpha(k',s')\mathrm{e}^{-\mathrm{i}k'\cdot x_1} + \hat{d}_{s'}^\dagger(k')v_\alpha(k',s')\mathrm{e}^{\mathrm{i}k'\cdot x_1}]|0\rangle$$

where in the exponents $k^0 = (\boldsymbol{k}^2 + m^2)^{1/2} \equiv \omega$, and $k^{0\prime} = (\boldsymbol{k}'^2 + m^2)^{1/2} \equiv \omega'$. In the $\theta(t_1 - t_2)$ part, the terms in $\hat{d}_s^\dagger$ and $\hat{d}_{s'}$ vanish, using $\hat{d}|0\rangle = 0 = \langle 0|\hat{d}^\dagger$ from (7.36); and in the $\theta(t_2 - t_1)$ part, the terms in $\hat{c}_s^\dagger$ and $\hat{c}_{s'}$ vanish similarly. In the $\theta(t_1 - t_2)$ part we then use $\{\hat{c}_s(k), \hat{c}_{s'}^\dagger(k')\} = (2\pi)^3\delta^3(\boldsymbol{k} - \boldsymbol{k}')\delta_{ss'}$ to reduce it to

$$\theta(t_1 - t_2)\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega} \sum_s u_\alpha(k,s)\bar{u}_\beta(k,s)\mathrm{e}^{-\mathrm{i}k\cdot(x_1 - x_2)} =$$

$$\theta(t_1 - t_2)\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega} (\not{k} + m)_{\alpha\beta}\mathrm{e}^{-\mathrm{i}k\cdot(x_1 - x_2)}.$$

Similarly, the $\theta(t_2 - t_1)$ part reduces to

$$-\theta(t_2 - t_1)\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega} \sum_s v_\alpha(k,s)\bar{v}_\beta(k,s)\mathrm{e}^{-\mathrm{i}k\cdot(x_2 - x_1)} =$$

$$-\theta(t_2 - t_1)\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega} (\not{k} - m)_{\alpha\beta}\mathrm{e}^{-\mathrm{i}k\cdot(x_2 - x_1)}.$$

So altogether we get

$$\langle 0|T(\hat{\psi}_\alpha(x_1)\hat{\bar{\psi}}(x_2))|0\rangle = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega} [\theta(t_1 - t_2)(\not{k} + m)_{\alpha\beta}\mathrm{e}^{-\mathrm{i}\omega(t_1 - t_2) + \mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)}$$
$$-[\theta(t_2 - t_1)(\not{k} - m)_{\alpha\beta}\mathrm{e}^{-\mathrm{i}\omega(t_2 - t_1) + \mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_2 - \boldsymbol{x}_1)}]$$

where $\not{k} = \omega\beta - \boldsymbol{\gamma}\cdot\boldsymbol{k}$. We need to show that this is equal to

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}k\cdot(x_1 - x_2)}\frac{\mathrm{i}(\not{k} + m)_{\alpha\beta}}{k^2 - m^2 + \mathrm{i}\epsilon}. \qquad (T)$$

Consider the integral over k_0 in (T). There are poles at $k_0 = (\boldsymbol{k}^2 + m^2 - \mathrm{i}\epsilon)^{1/2}$ which is in the lower half k_0-plane, and which tends to the real value ω as $\epsilon \to 0$; and at $k_0 = -(\boldsymbol{k}^2 + m^2 - \mathrm{i}\epsilon)^{1/2}$ which is in the upper half k_0-plane, and tends to $-\omega$ as $\epsilon \to 0$. On the other hand, the k_0-dependence of the exponential is

$$\mathrm{e}^{-\mathrm{i}k_0(t_1 - t_2)}.$$

We must therefore close the k_0-contour (see Appendix F) in the upper half k_0-plane for $t_1 - t_2 < 0$, picking up the contribution from the pole at $k_0 = -\omega$; and in the lower half k_0-plane for $t_1 - t_2 > 0$, picking up the contribution of the pole at $k_0 = \omega$. This gives

$$\theta(t_1 - t_2) \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^4} \left(\frac{-2\pi\mathrm{i}}{2\omega}\right) \mathrm{i}(\omega\beta - \boldsymbol{\gamma}\cdot\boldsymbol{k} + m)_{\alpha\beta}\mathrm{e}^{-\mathrm{i}\omega(t_1-t_2)+\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}$$
$$+\theta(t_2 - t_1) \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^4} \left(\frac{2\pi\mathrm{i}}{-2\omega}\right) \mathrm{i}(-\omega\beta - \boldsymbol{\gamma}\cdot\boldsymbol{k} + m)_{\alpha\beta}\mathrm{e}^{\mathrm{i}\omega(t_1-t_2)+\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}.$$

In the second integral, change $\boldsymbol{k}$ to $-\boldsymbol{k}$ to obtain

$$\theta(t_1 - t_2) \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega} (\omega\beta - \boldsymbol{\gamma}\cdot\boldsymbol{k} + m)_{\alpha\beta}\mathrm{e}^{-\mathrm{i}\omega(t_1-t_2)+\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}$$
$$-\theta(t_2 - t_1) \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega} (\omega\beta - \boldsymbol{\gamma}\cdot\boldsymbol{k} - m)_{\alpha\beta}\mathrm{e}^{-\mathrm{i}\omega(t_2-t_1)+\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)}$$

as required.

7.10 The Euler-Lagrange equations for A^ν are (compare (5.134))

$$\partial^\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\nu)}\right) = \frac{\partial\mathcal{L}}{\partial A^\nu},$$

where in this case

$$\mathcal{L} = -\frac{1}{2}F_{\mu\nu}\partial^\mu A^\nu - j_{\mathrm{em}\ \nu}A^\nu$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We use a 'brute force' approach. Consider the E-L equations written out in full for one component of A^ν, say A^1:

$$\partial^0 \left(\frac{\partial\mathcal{L}}{\partial(\partial^0 A^1)}\right) + \partial^1 \left(\frac{\partial\mathcal{L}}{\partial(\partial^1 A^1)}\right) + \partial^2 \left(\frac{\partial\mathcal{L}}{\partial(\partial^2 A^1)}\right) + \partial^3 \left(\frac{\partial\mathcal{L}}{\partial(\partial^3 A^1)}\right) = \frac{\partial\mathcal{L}}{\partial A^1}.$$

We need to identify the terms in $\mathcal{L}$ which involve $\partial^0 A^1$, $\partial^1 A^1$, $\partial^2 A^1$, $\partial^3 A^1$. First, note that $\partial^1 A^1$ does not appear since $F_{\mu\nu} = 0$ when $\mu = \nu$. The relevant terms are contained in

$$-\frac{1}{2}(F_{01}\partial^0 A^1 + F_{10}\partial^1 A^0 + F_{12}\partial^1 A^2 + F_{21}\partial^2 A^1 + F_{13}\partial^1 A^3 + F_{31}\partial^3 A^1)$$
$$= -\frac{1}{2}\{(\partial_0 A_1 - \partial_1 A_0)\partial^0 A^1 + (\partial_1 A_0 - \partial_0 A_1)\partial^1 A^0 + \ldots\}$$

Now use $\partial_0 A_1 = -\partial^0 A^1$, $\partial_1 A_2 = \partial^1 A^2$, etc, to pick out just the terms involving A^1, which are

$$\frac{1}{2}(\partial^0 A^1)^2 - \frac{1}{2}(\partial^2 A^1)^2 - \frac{1}{2}(\partial^3 A^1)^2$$
$$-(\partial^0 A^1)(\partial^1 A^0) + (\partial^2 A^1)(\partial^1 A^2) + (\partial^3 A^1)(\partial^1 A^3). \quad (R)$$

We then find

$$\left.\begin{array}{lclcl} \frac{\partial \mathcal{L}}{\partial(\partial^0 A^1)} & = & \partial^0 A^1 - \partial^1 A^0 & = & \partial_1 A_0 - \partial_0 A_1 \\ \frac{\partial \mathcal{L}}{\partial(\partial^2 A^1)} & = & \partial^1 A^2 - \partial^2 A^1 & = & \partial_1 A_2 - \partial_2 A_1 \\ \frac{\partial \mathcal{L}}{\partial(\partial^3 A^1)} & = & \partial^1 A^3 - \partial^3 A^1 & = & \partial_1 A_3 - \partial_3 A_1. \end{array}\right\} \quad (D)$$

Substituting these expressions into the E-L equations for A^1 we obtain

$$\partial^0(\partial_1 A_0 - \partial_0 A_1) + \partial^2(\partial_1 A_2 - \partial_2 A_1) + \partial^3(\partial_1 A_3 - \partial_3 A_1) = -j_{\text{em } 1}, \quad (S)$$

or

$$(\partial^0\partial_0 + \partial^1\partial_1 + \partial^2\partial_2 + \partial^3\partial_3)A_1 - \partial_1(\partial^0 A_0 + \partial^1 A_1 + \partial^2 A_2 + \partial^3 A_3) = j_{\text{em } 1}$$

which is the $\nu = 1$ component of (7.62) (written with the index ν lowered).

Equations (D) can be written in covariant form as

$$\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} = \partial_\nu A_\mu - \partial_\mu A_\nu.$$

Those skilled in manipulation of indices in tensor calculus can derive this result directly from the covariant Lagrangian; (7.62) then follows trivially.

7.11 (b) The condition (7.90) is

$$(A(k^2)g^{\nu\mu} + B(k^2)k^\nu k^\mu)(-k^2 g_{\mu\sigma} + k_\mu k_\sigma) = g^\nu_\sigma.$$

Multiplying out the brackets,

$$-k^2 A(k^2)g^\nu_\sigma + A(k^2)k^\nu k_\sigma - k^2 B(k^2)k^\nu k_\sigma + B(k^2)k^\nu k^2 k_\sigma = g^\nu_\sigma,$$

giving (7.91).

7.12 Referring to (R) of problem 7.10, the addition of the term $-\frac{1}{2}(\partial_\mu A^\mu)^2$ to $\mathcal{L}$ means that we must add to (R) the additional terms

$$-\frac{1}{2}(\partial_1 A^1)^2 - (\partial_1 A^1)(\partial_0 A^0 + \partial_2 A^2 + \partial_3 A^3).$$

This produces a non-zero value for $\partial\mathcal{L}/\partial(\partial^1 A^1)$, namely

$$\frac{\partial\mathcal{L}}{\partial(\partial^1 A^1)} = -(\partial_1 A_1) + (\partial_0 A_0 - \partial_2 A_2 - \partial_3 A_3).$$

Including then the contribution of $\partial^1(\partial\mathcal{L}/\partial(\partial^1 A^2))$ in (S) of problem 7.10 (with $j_{\mathrm{em}\ 1}$ set to zero) leads to the required equation of motion for A_1.

7.13 The $\epsilon^\mu(k,\lambda)$'s are given in (7.102)-(7.104), from which we note that, for each value of λ, the only non-vanishing component of $\epsilon^\mu(k,\lambda)$ is that in which $\mu = \lambda$, which equals 1. Consider then the case $\mu = 0$ in (7.115): the $\lambda = 1$, 2 and 3 terms all vanish, and the only surviving ν-component is $\nu = 0$, for which the LHS is -1, verifying the result. Similarly for the other cases.

7.14 For (7.119): following the same steps as in (7.87)-(7.91), we require $(M^{-1})^{\mu\nu}$ where

$$M^{\mu\nu} = (-k^2 g^{\mu\nu} + (1 - 1/\xi)k^\mu k^\nu).$$

Let

$$(M^{-1})_{\nu\sigma} = A(k^2)g_{\nu\sigma} + B(k^2)k_\nu k_\sigma.$$

The condition $g^\mu_\sigma = M^{\mu\nu}(M^{-1})_{\nu\sigma}$ gives

$$\begin{aligned} g^\mu_\sigma &= (-k^2 g^{\mu\nu} + (1 - 1/\xi)k^\mu k^\nu)(A(k^2)g_{\nu\sigma} + B(k^2)k_\nu k_\sigma) \\ &= -k^2 A(k^2) g^\mu_\sigma - k^2 B(k^2)k^\mu k_\sigma + A(k^2)(1 - 1/\xi)k^\mu k_\sigma + B(k^2)(1 - 1/\xi)k^\mu k_\sigma k^2. \end{aligned}$$

To match coefficients of g^μ_σ on both sides we require $A = -1/k^2$, and then the vanishing of the $k^\mu k_\sigma$ term gives $B = (1 - \xi)/(k^2)^2$.

7.15 The Hamiltonian density is

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_S + \hat{\mathcal{H}}'_S$$

where $\hat{\mathcal{H}}_S$ is the Hamiltonian density for the free complex scalar field,

$$\hat{\mathcal{H}}_S = \hat{\pi}^\dagger\hat{\pi} + \boldsymbol{\nabla}\hat{\phi}^\dagger \cdot \boldsymbol{\nabla}\hat{\phi} + m^2\hat{\phi}^\dagger\hat{\phi},$$

and $\hat{\mathcal{H}}'_S$ is the interaction part. But also the general definition of the Hamiltonian is

$$\hat{\mathcal{H}} = \hat{\pi}\dot{\hat{\phi}} + \hat{\pi}^\dagger\dot{\hat{\phi}}^\dagger - \hat{\mathcal{L}}.$$

Now

$$\hat{\pi} = \frac{\partial\hat{\mathcal{L}}}{\partial\dot{\hat{\phi}}} = \dot{\hat{\phi}}^\dagger - \mathrm{i}q\hat{\phi}^\dagger\hat{A}_0$$

from (7.134) - (7.136), and

$$\hat{\pi}^\dagger = \dot{\hat{\phi}} + \mathrm{i}q\hat{\phi}\hat{A}_0.$$

Substituting for $\dot{\hat{\phi}}$ and $\dot{\hat{\phi}}^\dagger$ in favour of $\hat{\pi}$ and $\hat{\pi}^\dagger$, we obtain

$$\begin{aligned} \hat{\mathcal{H}} &= \hat{\pi}(\hat{\pi}^\dagger - \mathrm{i}q\hat{\phi}\hat{A}_0) + \hat{\pi}^\dagger(\hat{\pi} + \mathrm{i}q\hat{\phi}^\dagger\hat{A}_0) \\ &\quad -\{(\hat{\pi} + \mathrm{i}q\hat{\phi}^\dagger\hat{A}_0)(\hat{\pi}^\dagger - \mathrm{i}q\hat{\phi}\hat{A}_0) - \boldsymbol{\nabla}\hat{\phi}^\dagger \cdot \boldsymbol{\nabla}\hat{\phi} - m^2\hat{\phi}^\dagger\hat{\phi}\} - \hat{\mathcal{L}}_{\mathrm{int}} \\ &= \hat{\pi}^\dagger\hat{\pi} + \boldsymbol{\nabla}\hat{\phi}^\dagger \cdot \boldsymbol{\nabla}\hat{\phi} + m^2\hat{\phi}^\dagger\hat{\phi} - q^2\hat{\phi}^\dagger\hat{\phi}(\hat{A}_0)^2 - \hat{\mathcal{L}}_{\mathrm{int}} \end{aligned}$$

which establishes (7.139).

7.16 Consider first

$$\begin{aligned} &\partial_{2\nu} \quad \langle 0|T(\hat{\phi}(x_1)\hat{\phi}^\dagger(x_2)|0\rangle = \\ &\partial_{2\nu} \quad \{\theta(t_1 - t_2)\langle 0|\hat{\phi}(x_1)\hat{\phi}^\dagger(x_2)|0\rangle + \theta(t_2 - t_1)\langle 0|\hat{\phi}^\dagger(x_2)\hat{\phi}(x_1)|0\rangle\}, \end{aligned}$$

where $\partial_{2\nu} = \partial/\partial x_2^\nu$ (similarly for $\partial_{1\mu}$). If $\nu = i$, a spatial index, then the derivative passes through the θ-functions and we get simply $\langle 0|T(\hat{\phi}(x_1)\partial_{2i}\hat{\phi}^\dagger(x_2))|0\rangle$. If $\nu = 0$, then the differentiation produces extra terms

$$-\delta(t_1 - t_2)\langle 0|\hat{\phi}(x_1)\hat{\phi}^\dagger(x_2)|0\rangle + \delta(t_2 - t_1)\langle 0|\hat{\phi}^\dagger(x_2)\hat{\phi}(x_1)|0\rangle$$

which cancel using (7.32) (at equal times $(x_1 - x_2)^2$ is spacelike). So

$$\partial_{2\nu}\langle 0|T(\hat{\phi}(x_1)\hat{\phi}^\dagger(x_2))|0\rangle = \langle 0|T(\hat{\phi}(x_1)\partial_{2\nu}\hat{\phi}^\dagger(x_2))|0\rangle.$$

Now consider

$$\partial_{1\mu}\partial_{2\nu}\langle 0|T(\hat{\phi}(x_1)\hat{\phi}^\dagger(x_2))|0\rangle = \partial_{1\mu}\langle 0|T(\hat{\phi}(x_1)\partial_{2\nu}\hat{\phi}^\dagger(x_2))|0\rangle =$$
$$\partial_{1\mu}\{\theta(t_1-t_2)\langle 0|\hat{\phi}(x_1)\partial_{2\nu}\hat{\phi}^\dagger(x_2)|0\rangle + \theta(t_2-t_1)\langle 0|\partial_{2\nu}\hat{\phi}^\dagger(x_2)\hat{\phi}(x_1)|0\rangle\} \quad .$$

As before, if $\mu = j$ the derivative passes through the θ-functions and we get $\langle 0|T(\partial_{1j}\hat{\phi}(x_1)\partial_{2\nu}\hat{\phi}^\dagger(x_2)|0\rangle$. On the other hand, if $\mu = 0$ we get

$$\delta(t_1-t_2)\langle 0|\hat{\phi}(x_1)\partial_{2\nu}\hat{\phi}^\dagger(x_2)|0\rangle + \theta(t_1-t_2)\langle 0|\partial_{10}\hat{\phi}(x_1)\partial_{2\nu}\hat{\phi}^\dagger(x_2)|0\rangle$$
$$-\delta(t_1-t_2)\langle 0|\partial_{2\nu}\hat{\phi}^\dagger(x_2)\hat{\phi}(x_1)|0\rangle + \theta(t_2-t_1)\langle 0|\partial_{2\nu}\hat{\phi}^\dagger(x_2)\partial_{10}\hat{\phi}(x_1)|0\rangle.$$
$$= \langle 0|T(\partial_{10}\hat{\phi}(x_1)\partial_{2\nu}\hat{\phi}^\dagger(x_2))|0\rangle + \langle 0|[\hat{\phi}(x_1),\partial_{2\nu}\hat{\phi}^\dagger(x_2)]|0\rangle\delta(t_1-t_2).$$

If ν is a spatial index, the equal time commutator vanishes, as can be seen by differentiating (7.32) with respect to a spatial component of x_2. But if $\nu = 0$ then the equal time commutator is

$$[\hat{\phi}(x_1),\dot{\hat{\phi}}^\dagger(x_2)]\delta(t_1-t_2) = \mathrm{i}\delta^3(\boldsymbol{x}_1-\boldsymbol{x}_2)\delta(t_1-t_2).$$

These results establish (7.140).

Chapter 8

8.2 We use (8.25), and the mode expansion (7.16) in (8.23), to obtain

$$\langle \mathrm{s}^+,p'|\hat{j}^\mu_{\mathrm{em,s}}(x)|\mathrm{s}^+,p\rangle = \mathrm{i}e\sqrt{4EE'}\langle 0|\hat{a}(p')[\hat{\phi}^\dagger\partial^\mu\hat{\phi} - (\partial^\mu\hat{\phi}^\dagger)\hat{\phi}]\hat{a}^\dagger(p)|0\rangle$$
$$= \mathrm{i}e\sqrt{4EE'}\langle 0|\hat{a}(p')\{\int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega'}}\,[\hat{a}^\dagger(k')\mathrm{e}^{\mathrm{i}k'\cdot x} + \hat{b}(k')\mathrm{e}^{-\mathrm{i}k'\cdot x}]$$
$$\times\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega}}\,[-\mathrm{i}k^\mu\hat{a}(k)\mathrm{e}^{-\mathrm{i}k\cdot x} + \mathrm{i}k^\mu\hat{b}^\dagger(k)\mathrm{e}^{\mathrm{i}k\cdot x}]\}\hat{a}^\dagger(p)|0\rangle$$

$$-(\partial^\mu\hat{\phi}^\dagger)\hat{\phi}\ \text{term}$$

where $E = \sqrt{M^2+\boldsymbol{p}^2}$, $E' = \sqrt{M^2+\boldsymbol{p}'^2}$, $\omega = \sqrt{M^2+\boldsymbol{k}^2}$, and $\omega' = \sqrt{M^2+\boldsymbol{k}'^2}$. We evaluate the $\hat{\phi}^\dagger\partial^\mu\hat{\phi}$ term. The $\hat{a}$ and $\hat{b}$ operators commute with each other, so we can move the $\hat{b}(k)$ all the way to the right where it will give zero on $|0\rangle$; similarly the $\hat{b}^\dagger(k')$ will give zero on $\langle 0|$. This term therefore reduces to

$$\mathrm{i}e\sqrt{4EE'}\langle 0|\hat{a}(p')\int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega'}}\,\hat{a}^\dagger(k')\mathrm{e}^{\mathrm{i}k'\cdot x}\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega}}\,-\mathrm{i}k^\mu\hat{a}(k)\mathrm{e}^{-\mathrm{i}k\cdot x}\hat{a}^\dagger(p)|0\rangle.$$

We then use

$$\hat{a}(k)\hat{a}^\dagger(p) = \hat{a}^\dagger(p)\hat{a}(k) + (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{p}),$$

and similarly for the product $\hat{a}(p')\hat{a}^\dagger(k')$, together with

$$\hat{a}(k)|0\rangle = \langle 0|\hat{a}^\dagger(k') = 0,$$

to get rid of all the operators. The δ-functions allow all the momentum integrals to be performed, setting $\boldsymbol{k}=\boldsymbol{p}$ and $\boldsymbol{k}'=\boldsymbol{p}'$ so that $\omega = E,\ \omega' = E', k^\mu = p^\mu$, and $k'^\mu = p'^\mu$. This term becomes simply

$$ep^\mu\mathrm{e}^{-\mathrm{i}(p-p')\cdot x};$$

the $(\partial^\mu\hat{\phi}^\dagger)\hat{\phi}$ term supplies the p'^μ contribution to (8.27).

8.3 We use (8.32) to obtain

$$\langle \mathrm{s}^-,p'|\hat{j}^\mu_{\mathrm{em,s}}(x)|\mathrm{s}^-,p\rangle = \mathrm{i}e\sqrt{4EE'}\langle 0|\hat{b}(p')\{\text{same expression for the current as in problem 8.2}\}\hat{b}^\dagger(p)|0\rangle.$$

This time, in the $\hat{\phi}^\dagger\partial^\mu\hat{\phi}$ term of the current, we can move $\hat{a}(k')$ to the right to annihilate on $|0\rangle$, and $\hat{a}^\dagger(k')$ to the left to annihilate on $\langle 0|$. This leaves an expression of the form

$$\langle 0|\hat{b}(p').....\hat{b}(k')....\hat{b}^\dagger(k)....\hat{b}^\dagger(p)|0\rangle.$$

But we must remember to normally order the operators in the current: this means that the above expression should in fact be

$$\langle 0|\hat{b}(p').....\hat{b}^\dagger(k)....\hat{b}(k')....\hat{b}^\dagger(p)|0\rangle.$$

We now use the commutation relations for $\hat{b}(p')$ and $\hat{b}^\dagger(k)$ to get $\hat{b}^\dagger(k)$ anihilating on $|0\rangle$, and similarly for $\hat{b}(k')$ and $\hat{b}^\dagger(p)$ to get $\hat{b}(k')$ annihilating on $|0\rangle$. This leaves delta-functions $\delta(\boldsymbol{p}-\boldsymbol{k}')$ and $\delta(\boldsymbol{k}-\boldsymbol{p}')$, allowing the momentum integrals (in this first term) to be performed, setting $\boldsymbol{k} = \boldsymbol{p}', \boldsymbol{k}' = \boldsymbol{p}$, and hence $\omega = E'$, $\omega' = E, k = p'$ and $k' = p$. The result is

$$-ep'^{\mu}\mathrm{e}^{-\mathrm{i}(p-p')\cdot x}$$

for this first term. The second supplies the $-ep^{\mu}$ part, and the whole matrix element is indeed the negative of the one in problem 8.2.

8.4 We consider a Lorentz transformation along the x^1-axis. Then

$$\begin{aligned} j'_x = \psi'^{\dagger}\alpha_x\psi' &= \psi^{\dagger}\mathrm{e}^{\alpha_x\vartheta/2}\alpha_x\mathrm{e}^{\alpha_x\vartheta/2}\psi \\ &= \psi^{\dagger}\alpha_x\mathrm{e}^{\alpha_x\vartheta}\psi \\ &= \psi^{\dagger}\alpha_x(\cosh\vartheta + \alpha_x\sinh\vartheta)\psi \\ &= \cosh\vartheta\ j_x + \sinh\vartheta\ \rho \end{aligned}$$

as in (4.86), where we have used (4.90), and (4.91) with $\vartheta/2$ replaced by ϑ.

8.5 (a) We are assuming $E = E'$, as in the required application. Then

$$\begin{aligned} u^{\dagger}(k', s'=1)u(k, s=1) &= (E+m)\begin{pmatrix} \phi^{1\dagger} & \phi^{1\dagger}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}'}{(E+m)} \end{pmatrix}\begin{pmatrix} \phi^1 \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{(E+m)}\phi^1 \end{pmatrix} \\ &= (E+m)\left\{1 + \phi^{1\dagger}\left(\frac{\boldsymbol{k}\cdot\boldsymbol{k}'}{(E+m)^2} + \mathrm{i}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}'\times\boldsymbol{k}}{(E+m)^2}\right)\phi^1\right\} \end{aligned}$$

which reduces immediately to the given expression.

8.5 (b) We have

$$\phi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \phi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Also

$$\boldsymbol{\sigma}\cdot\boldsymbol{A} = \begin{pmatrix} A_z & A_x - \mathrm{i}A_y \\ A_x + \mathrm{i}A_y & -A_z \end{pmatrix}.$$

Then, by straightforward matrix multiplication we obtain, for instance,

$$\phi^{1\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{A}\phi^2 = A_x - \mathrm{i}A_y.$$

Generally,

$$\phi^{i\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{A}\phi^{j} = (\boldsymbol{\sigma}\cdot\boldsymbol{A})_{ij}.$$

(c) The expression S of (8.46) is

$$S = \frac{1}{2}\{|u'^{\dagger}_{s'=1}u_{s=1}|^2 + |u'^{\dagger}_{s'=1}u_{s=2}|^2 + |u'^{\dagger}_{s'=2}u_{s=1}|^2 + |u'^{\dagger}_{s'=2}u_{s=2}|^2\}.$$

Part (a) evaluated $u'^{\dagger}_{s'=1}u_{s=1}$, and $u'^{\dagger}_{s'=2}u_{s=2}$ is the same but with ϕ^1 replaced by ϕ^2. Since $\phi^{1\dagger}\phi^2 = 0$, we find

$$u'^{\dagger}_{s'=1}u_{s=2} = (E+m)\frac{\mathrm{i}\phi^{1\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{k}'\times\boldsymbol{k}\phi^2}{(E+m)^2}$$

and similarly for $u'^{\dagger}_{s'=2}u_{s=1}$. The result follows after noting that

$$|\phi^{1\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{A}\phi^1|^2 + |\phi^{1\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{A}\phi^2|^2 + |\phi^{2\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{A}\phi^1|^2 + |\phi^{2\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{A}\phi^2|^2 = 2\boldsymbol{A}^2.$$

(d)

$$\begin{aligned}
S &= (E=m)^2\left\{\left[1+\frac{\boldsymbol{k}^2\cos\theta}{(E+m)^2}\right]^2 + \frac{(\boldsymbol{k}^2)^2\sin^2\theta}{(E+m)^4}\right\} \\
&= (E+m)^2\left\{\left[1+\left(\frac{E-m}{E+m}\right)\cos\theta\right]^2 + \left(\frac{E-m}{E+m}\right)^2\sin^2\theta\right\} \\
&= \{[(E+m)+(E-m)\cos\theta]^2 + (E-m)^2\sin^2\theta\} \\
&= (E+m)^2 + 2\boldsymbol{k}^2\cos\theta + (E-m)^2 \\
&= 2(E^2+m^2+\boldsymbol{k}^2\cos\theta) \\
&= 2[E^2+m^2+\boldsymbol{k}^2(1-2\sin^2\theta/2)] \\
&= 4E^2(1-\frac{\boldsymbol{k}^2}{E^2}\sin^2\theta/2)
\end{aligned}$$

and the result follows using $v = |\boldsymbol{k}|/E$.

8.6 The manipulations are similar to those in problem 8.2 except that anticommutation relations are used.

8.7 $\gamma^\mu = (\gamma^0, \gamma^0\boldsymbol{\alpha})$. So

$$\gamma^{\mu\dagger} = (\gamma^{0\dagger}, \boldsymbol{\alpha}^\dagger\gamma^{0\dagger}) = (\gamma^0, \boldsymbol{\alpha}\gamma^0).$$

Hence
$$\gamma^0\gamma^{\mu\dagger}\gamma^0 = ((\gamma^0)^3, \gamma^0\boldsymbol{\alpha}(\gamma^0)^2) = (\gamma^0, \gamma^0\boldsymbol{\alpha}) = \gamma^\mu.$$

8.8

$$\begin{aligned}\mathrm{Tr}[(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma^\nu] = {} & \mathrm{Tr}(\not{k}'\gamma^\mu \not{k}\gamma^\nu) + m\mathrm{Tr}(\gamma^\mu \not{k}\gamma^\nu) \\ & + m\mathrm{Tr}(\not{k}'\gamma^\mu\gamma^\nu) + m^2\mathrm{Tr}(\gamma^\mu\gamma^\nu).\end{aligned}$$

The terms linear in m vanish by (8.73); (8.74) gives $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$, and (8.75) implies that

$$\mathrm{Tr}[\not{k}'\gamma^\mu \not{k}\gamma^\nu] = 4[k'^\mu k^\nu + k'^\nu k^\mu - k'\cdot k g^{\mu\nu}].$$

These results establish (8.78).

8.9
$$L^{00} = 2[2k'^0k^0 - (k'\cdot k)] + 2m^2.$$
Here, $k^0 = (m^2 + \boldsymbol{k}^2)^{1/2} = k'^0 \equiv E; k'\cdot k = k^0k'^0 - \boldsymbol{k}\cdot\boldsymbol{k}' = E^2 - \boldsymbol{k}^2\cos\theta$. So

$$\begin{aligned} L^{00} &= 2[E^2 + \boldsymbol{k}^2\cos\theta] + 2m^2 \\ &= 2[E^2 + m^2 + \boldsymbol{k}^2\cos\theta]\end{aligned}$$

and the result follows as in problem 8.5(d).

8.11 Using (8.27), (8.55), and the Fourier transform of (7.119) (compare (6.98) and (7.58)), the term written explicitly in (8.93) becomes

$$\frac{(-\mathrm{i})^2}{2}\int\int \mathrm{d}^4x_1\ \mathrm{d}^4x_2\{e(p+p')^\mu \mathrm{e}^{-\mathrm{i}(p-p')\cdot x_1}$$

$$\times\int\frac{\mathrm{d}^4l}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}l\cdot(x_1-x_2)}\,.\frac{\mathrm{i}[-g_{\mu\nu} + (1-\xi)l_\mu l_\nu/l^2]}{l^2}\,.-e\bar{u}(k',s')\gamma^\nu u(k,s)\mathrm{e}^{-\mathrm{i}(k-k')\cdot x_2}\}.$$

The integral over x_1 gives $(2\pi)^4\delta^4(p+l-p')$, and that over x_2 gives $(2\pi)^4\delta^4(l+k'-k)$. The integral over l can now be done using one of these delta functions, which sets $l = k - k' = q$ everywhere, including in the other delta function. This gives one half times (8.95) - the other half is supplied by the '$(x_1 \leftrightarrow x_2)$' part of (8.93).

8.12 Using

$$\partial_\mu \mathrm{e}^{-\mathrm{i}(p-p')\cdot x} = -\mathrm{i}(p-p')_\mu \mathrm{e}^{-\mathrm{i}(p-p')\cdot x},$$

we find from (8.27)

$$\partial_\mu \langle \mathrm{s}^+, p' | \hat{j}^\mu_{\mathrm{em,s}}(x) | \mathrm{s}^+, p\rangle = -\mathrm{i}e(p^2 - p'^2)\mathrm{e}^{-\mathrm{i}(p-p')\cdot x}$$

which vanishes since $p^2 = p'^2 = M^2$. Similarly,

$$\partial_\mu \langle \mathrm{e}^-, k', s' | \hat{j}^\mu_{\mathrm{em,e}}(x) | \mathrm{e}^-, k, s\rangle = \mathrm{i}e\bar{u}'(\not{k} - \not{k}')u\mathrm{e}^{-\mathrm{i}(k-k')\cdot x}$$

which vanishes since $\not{k}u = mu$ and $\bar{u}'\not{k}' = m\bar{u}'$. (The particles in the external states are 'on-shell' - i.e. their wavefunctions satisfy the free-particle equations of motion, or equivalently their 4-momenta satisfy the energy-momentum condition for a free particle.]

8.13 We have

$$L^{\mu\nu}T_{\mu\nu} = 2[k'^\mu k^\nu + k'^\nu k^\mu + (q^2/2)g^{\mu\nu}](p+p')_\mu (p+p')_\nu.$$

It is advantageous to write $p' = p + q$, so that $(p+p')_\mu = (2p+q)_\mu$, and similarly for $(p+p')_\nu$, and then drop the $q_\mu q_\nu$ part using (8.188). Then

$$\begin{aligned} L^{\mu\nu}T_{\mu\nu} &= 8[2p\cdot k'\ p\cdot k + (q^2/2)p^2] \\ &= 8[2p\cdot k'\ p\cdot k + (q^2/2)M^2]. \end{aligned}$$

8.14 We have (correcting the misprint in (8.134))

$$\begin{aligned} F(\boldsymbol{q}^2) &= \int \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}}\rho(\boldsymbol{x})\mathrm{d}^3\boldsymbol{x} = \int \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}}\ \frac{\mathrm{e}^{-|\boldsymbol{x}|/a}}{8\pi a^3}\ \mathrm{d}^3\boldsymbol{x} \\ &= \frac{1}{8\pi a^3}\int \mathrm{e}^{-\mathrm{i}|\boldsymbol{q}||\boldsymbol{x}|\cos\theta}\ \mathrm{e}^{-|\boldsymbol{x}|/a}\ |\boldsymbol{x}|\ \mathrm{d}|\boldsymbol{x}|\ 2\pi\mathrm{d}(\cos\theta). \end{aligned}$$

The integral over $\cos\theta$ can be done with the result

$$\frac{1}{\mathrm{i}|\boldsymbol{q}||\boldsymbol{x}|}\left(\mathrm{e}^{\mathrm{i}|\boldsymbol{q}||\boldsymbol{x}|} - \mathrm{e}^{-\mathrm{i}|\boldsymbol{q}||\boldsymbol{x}|}\right)$$

so that $F(\boldsymbol{q}^2)$ becomes

$$F(\boldsymbol{q}^2) = \frac{1}{4a^3\mathrm{i}|\boldsymbol{q}|}\int_0^\infty |\boldsymbol{x}|\mathrm{e}^{-|\boldsymbol{x}|/a}\ (\mathrm{e}^{\mathrm{i}|\boldsymbol{q}||\boldsymbol{x}|} - \mathrm{e}^{-\mathrm{i}|\boldsymbol{q}||\boldsymbol{x}|})\ \mathrm{d}|\boldsymbol{x}|.$$

One way of doing the $|\boldsymbol{x}|$-integral is to write it as

$$\frac{\partial}{\partial(-1/a)}\int_0^\infty \mathrm{e}^{-|\boldsymbol{x}|/a}\left(\mathrm{e}^{\mathrm{i}|\boldsymbol{q}||\boldsymbol{x}|}-\mathrm{e}^{-\mathrm{i}|\boldsymbol{q}||\boldsymbol{x}|}\right)\mathrm{d}|\boldsymbol{x}|,$$

where now the simple exponential integrals can be done, and the differentiation with respect to a performed, after using

$$\frac{\partial}{\partial(-1/a)} = a^2\frac{\partial}{\partial a}.$$

This leads to (8.135).

8.15 After replacing ϵ_μ by k_μ, (8.161) becomes (after correcting the misprints in it)

$$-e^2\epsilon_\nu^*(k',\lambda')\bar{u}(p',s')\gamma^\nu\frac{(\not p+\not k+m)}{(p+k)^2-m^2}\not k u(p,s)$$
$$-e^2\epsilon_\nu^*(k',\lambda')\bar{u}(p',s')\not k\frac{(\not p-\not k'+m)}{(p-k')^2+m^2}\gamma^\nu u(p,s).$$

In the first term, replace $\not k u(p,s)$ by

$$(\not k'+\not p'-\not p)u(p,s) = (\not k'+\not p'-m)u(p,s) = (\not p+\not k-m)u(p,s)$$

using the Dirac equation and 4-momentum conservation. Also note that

$$(\not p+\not k+m)(\not p+\not k-m) = (p+k)^2-m^2.$$

The first term is then

$$-e^2\epsilon_\nu^*(k',\lambda')\bar{u}\gamma^\nu u.$$

In the second term, replace $\bar{u}(p',s')\not k$ by

$$\bar{u}(p',s')(\not k'+\not p'-\not p) = \bar{u}(p',s')(\not k'-\not p+m) = -\bar{u}(p',s')(\not p-\not k'-m).$$

The second term becomes

$$e^2\epsilon_\nu^*(k',\lambda')\bar{u}\gamma^\nu u$$

which cancels the first. Similarly when $\epsilon_\nu^*(k',\lambda')$ is replaced by k'_ν.

8.16 (a) We work in the massless limit, as in section 8.6.3. We have

$$\sum_{\lambda,\lambda',s,s'} \mathcal{M}^{(s)}_{\gamma e^-} \mathcal{M}^{(u)*}_{\gamma e^-} = \frac{e^4}{su} \sum_{\lambda,\lambda',s,s'}$$

$$\epsilon^*_\nu(k',\lambda')\epsilon_\rho(k',\lambda')\epsilon_\mu(k,\lambda)\epsilon^*_\sigma(k,\lambda)\bar{u}(p',s')\gamma^\nu(\not{p}+\not{k})\gamma^\mu u(p,s)\bar{u}(p,s)\gamma^\rho(\not{p}-\not{k}')\gamma^\sigma u(p',s')$$

$$\begin{aligned} &= \frac{e^4}{su} g_{\nu\rho} g_{\mu\sigma} \mathrm{Tr}\{\gamma^\nu(\not{p}+\not{k})\gamma^\mu \not{p}\gamma^\rho(\not{p}-\not{k}')\gamma^\sigma \not{p}'\} \\ &= \frac{e^4}{su} \mathrm{Tr}\{\gamma^\nu(\not{p}+\not{k})\gamma^\mu \not{p}\gamma_\nu(\not{p}-\not{k}')\gamma_\mu \not{p}'\}. \end{aligned}$$

We now use

$$\gamma^\mu \not{p}\gamma_\nu(\not{p}-\not{k}')\gamma_\mu = -2(\not{p}-\not{k}')\gamma_\nu \not{p}$$

from (J.5), and then

$$\gamma^\nu(\not{p}+\not{k})(\not{p}-\not{k}')\gamma_\nu = 4(p+k)\cdot(p-k')$$

from (J.4), and finally

$$\mathrm{Tr}(\not{p}\not{p}') = 4p\cdot p'$$

from (J.30) to obtain

$$\sum_{\lambda,\lambda',s,s'} \mathcal{M}^{(s)}_{\gamma e^-} \mathcal{M}^{(u)*}_{\gamma e^-} = -32\frac{e^4}{su}(p+k)\cdot(p-k')\; p'\cdot p.$$

Now

$$\begin{aligned} (p+k)\cdot(p-k') &= p^2 - p\cdot k' + k\cdot p - k\cdot k' \\ &= -p'\cdot k + k\cdot p - k\cdot k' \end{aligned}$$

since $p\cdot k' = p'\cdot k$ in the massless limit for all external lines (as follows from squaring $p-k' = p'-k$). Hence

$$(p+k)\cdot(p-k') = k\cdot(p-p'-k') = -k^2.$$

This therefore vanishes when the photon is on-shell ($k^2=0$).

8.17 The interference term calculated in problem 8.16 (a) had the value

$$\sum_{\lambda,\lambda',s,s'} \mathcal{M}^{(s)}_{\gamma e^-}\,\mathcal{M}^{(u)*}_{\gamma e^-} = -32\frac{e^4}{su}(p+k)\cdot(p-k')\;p'\cdot p.$$

This time (with $k^2 = -Q^2$) we have

$$\begin{aligned}(p+k)\cdot(p-k') &= p^2 - p\cdot k' + k\cdot p - k\cdot k' \\ &= -p\cdot k' + k'\cdot p' + Q^2/2 - k\cdot k'\end{aligned}$$

using $k\cdot p = k'\cdot p' + Q^2/2$ (from squaring $k+p = k'+p'$). So

$$(p+k)\cdot(p-k') = Q^2/2 + k'\cdot(p'-p-k) = Q^2/2 - (k')^2 = Q^2/2.$$

Hence using $p\cdot p' = -t/2$, we obtain

$$\frac{1}{4}\sum_{\lambda,\lambda',s,s'} \mathcal{M}^{(s)}_{\gamma e^-}\,\mathcal{M}^{(u)*}_{\gamma e^-} = e^4\frac{2tQ^2}{su}.$$

There are two such terms in the product

$$\left(\mathcal{M}^{(s)}_{\gamma e^-} + \mathcal{M}^{(u)}_{\gamma e^-}\right)\left(\mathcal{M}^{(s)}_{\gamma e^-} + \mathcal{M}^{(u)}_{\gamma e^-}\right)^*,$$

which are equal.

8.18 (a) We calculate $L_{\mu\nu}M^{\mu\nu}_{\text{eff}}$ using (8.185) and (8.190).

$$\begin{aligned}L_{\mu\nu}M^{\mu\nu}_{\text{eff}} &= 4[k'_\mu k_\nu + k'_\nu k_\mu + (q^2/2)g_{\mu\nu}][2p^\mu p^\nu + (q^2/2)g^{\mu\nu}] \\ &= 4[4k'\cdot p\;k\cdot p + q^2 k\cdot k' + q^2p^2 + 4(q^2/2)^2] \\ &= 4[4k'\cdot p\;k\cdot p + q^2(-q^2/2) + q^2M^2 + (q^2)^2] \\ &= 8[2k'\cdot p\;k\cdot p + (q^2)^2/4 + q^2M^2/2].\end{aligned}$$

In 'laboratory' frame with $p^\mu = (M,\mathbf{0})$, neglecting the electron mass so that $\omega = |\boldsymbol{k}| \equiv k, \omega' = |\boldsymbol{k}'| \equiv k'$, and using (8.219), the preceding expression becomes

$$\begin{aligned}L_{\mu\nu}M^{\mu\nu}_{\text{eff}} &= 8[2kk'M^2 + 4k^2k'^2\sin^4\theta/2 - 2kk'M^2\sin^2\theta/2] \\ &= 8[2kk'M^2(1-\sin^2\theta/2) + 4k^2k'^2\sin^4\theta/2] \\ &= [16kk'M^2\cos^2\theta/2][1 + 2\frac{kk'}{M^2}\sin^2\theta/2\tan^2\theta/2] \\ &= [16kk'M^2\cos^2\theta/2][1 - (q^2/2M^2)\tan^2\theta/2]\end{aligned}$$

The first factor is exactly that in (K.24), leading to the 'no-structure' cross section.

(b) Returning to line 3 of part (a):

$$L_{\mu\nu}M_{\text{eff}}^{\mu\nu} = 4[4k' \cdot p \; k \cdot p + q^2 k \cdot k' + q^2 p^2 + 4(q^2/2)^2].$$

With all particles massless, $s = 2k \cdot p$, $t = -2k \cdot k'$, and $u = -2p \cdot k'$. Hence our expression becomes

$$4[-us - t^2/2 + t^2] = 4[-us + t^2/2].$$

$\mathrm{d}\sigma/\mathrm{d}t$ is then given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi}\frac{(4\pi\alpha)^2}{t^2}4[-\frac{u}{s} + \frac{t^2}{2s^2}].$$

In the massless case we have $s+t+u=0$ so that $t^2 = (u+s)^2$, leading to the required expression for $\mathrm{d}\sigma/\mathrm{d}t$.

From (K.41), (K.46) and (K.50) we find

$$\mathrm{d}y = \frac{k'^2}{2\pi kM}2\pi\frac{\mathrm{d}t}{2k'^2}.$$

But in this frame $s = (k+p)^2 = 2kM$. Hence $\mathrm{d}t = s\mathrm{d}y$ and the result is established.

8.19 (a) We shall label the 4-momentum and spin of the ingoing e^- by k, s, of the ingoing e^+ by k_1, s_1, of the outgoing μ^- by p', r', and of the outgoing μ^+ by p_1, r_1. The Mandelstam variables are

$$Q^2 = (k+k_1)^2 = (p'+p_1)^2,\; t = (k_1-p_1)^2 = (k-p')^2,\; u = (k-p_1)^2 = (k_1-p')^2.$$

The amplitude is

$$\mathrm{i}e\bar{v}(k_1,s_1)\gamma^\mu u(k,s)\;\frac{-\mathrm{i}g_{\mu\nu}}{Q^2}\;\mathrm{i}e\bar{u}(p',r')\gamma_\nu v(p_1,r_1).$$

Note that the $\bar{v}\gamma^\mu u$ factor depends only on the e^- and e^+ momenta, while the $\bar{u}\gamma_\nu v$ factor depends only on the momenta of the μ^- and μ^+.

(b) The spin-averaged squared cross matrix element is then

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\left(\frac{e^2}{Q^2}\right)\sum_{s,s_1}\bar{v}(k_1,s_1)\gamma_\mu u(k,s)\bar{u}(k,s)\gamma_\nu v(k_1,s_1)$$

$$\times\sum_{r',r_1}\bar{u}(p',r')\gamma^\mu v(p_1,r_1)\bar{v}(p_1,r_1)\gamma^\nu u(p',r')$$

$$= \frac{(4\pi\alpha)^2}{Q^4}L(\mathrm{e})_{\mu\nu}L(\mu)^{\mu\nu}$$

where

$$L(\mathrm{e})_{\mu\nu} = \frac{1}{2}\mathrm{Tr}[(\not{k}_1 - m)\gamma_\mu(\not{k} + m)\gamma_\nu]$$

and

$$L(\mu)^{\mu\nu} = \frac{1}{2}\mathrm{Tr}[(\not{p}' + M)\gamma^\mu(\not{p}_1 - M)\gamma^\nu],$$

using (7.61).

(c) Using the trace theorems as in (8.78), the lepton tensors are

$$L(\mathrm{e})_{\mu\nu} = 2[k_{1\mu}k_\nu + k_{1\nu}k_\mu - (k_1\cdot k)g_{\mu\nu}]$$

and

$$L(\mu)^{\mu\nu} = 2[p'^\mu p_1^\nu + p'^\nu p_1^\mu - (p'\cdot p_1)g^{\mu\nu}]$$

where now (in the massless limit) $Q^2/2 = p'\cdot p_1 = k_1\cdot k$. Hence

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \left(\frac{e^2}{Q^2}\right)4[2p'\cdot k_1\ p_1\cdot k + 2p'\cdot k\ p_1\cdot k_1 - Q^2p'\cdot p_1 - Q^2k_1\cdot k + (Q^2)^2] \\
&= \left(\frac{e^2}{Q^2}\right)4[2\frac{u^2}{4} + 2\frac{t^2}{4} - Q^2Q^2/2 - Q^2Q^2/2 + (Q^2)^2] \\
&= \left(\frac{e^2}{Q^2}\right)2[t^2 + u^2].
\end{aligned}$$

In the massless limit, all 3-momenta have equal modulus which (slightly confusingly) we denote by k. Then

$$t = -2k^2(1-\cos\theta),\ \ u = -2k^2(1+\cos\theta)$$

so that

$$t^2 + u^2 = 8k^4(1 + \cos^2\theta).$$

Since $Q^2 = 4k^2$ the result follows.

Crossing symmetry implies that the amplitude for

$$\mathrm{e}^-(k, s) + \mathrm{e}^+(k_1, s_1) \to \mu^-(p', r') + \mu^+(p_1, r_1)$$

is equal to (minus) the amplitude for

$$\mathrm{e}^-(k, s) + \mu^-(-p_1, -r_1) \to \mu^-(p', r') + \mathrm{e}^-(-k_1, -s_1).$$

We can therefore obtain this amplitude from the one calculated in section 8.7 by making the replacements

$$p, r \to -p_1, -r_1 \text{ and } k', s' \to -k_1, -s_1.$$

The spin labels disappear at the Trace stage, of course, and

$$q^2(\text{section 8.7}) = (k - k')^2 \to (k + k_1)^2 = Q^2.$$

So (8.185) and (8.186) become

$$(8.185) \to 2[-k_{1\mu}k_\nu - k_{1\nu}k_\mu + (Q^2/2)g_{\mu\nu}]$$

and

$$(8.186) \to 2[-p'^\mu p^\nu - p'^\nu p^\mu + (Q^2/2)g^{\mu\nu}]$$

leading to exactly the same result for $\overline{|\mathcal{M}|^2}$.

(d) Using the formula (6.129) for the differential cross section for elastic scattering in the centre of mass, and replacing $|\mathcal{M}_{\mathrm{fi}}|^2$ by $\overline{|\mathcal{M}|^2}$, we obtain

$$\begin{aligned} \sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \int \frac{1}{64\pi^2 Q^2} 16\pi^2\alpha^2(1 + \cos^2\theta)\ 2\pi\ \mathrm{d}\cos\theta \\ &= \frac{\alpha^2}{4Q^2} 2\pi \left(\cos\theta + \frac{\cos^3\theta}{3}\right)_{-1}^{1} \\ &= 4\pi\alpha^2/3Q^2 \end{aligned}$$

as required.

8.20 We have

$$\begin{aligned}
\bar{u}(p')\mathrm{i}\frac{\sigma^{\mu\nu}}{2M}q_\nu u(p) &= \mathrm{i}\bar{u}(p')\frac{1}{2}\mathrm{i}\left(\frac{\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu}{2M}\right)(p'_\nu-p_\nu)u(p) \\
&= -\frac{1}{4M}\bar{u}(p')(\gamma^\mu \not{p}'-\gamma^\mu\not{p}-\not{p}'\gamma^\mu+\not{p}\gamma^\mu)u(p) \\
&= -\frac{1}{4M}\bar{u}(p')[\gamma^\mu\not{p}'+\not{p}\gamma^\mu]u(p)+\frac{1}{2}\bar{u}(p')\gamma^\mu u(p).
\end{aligned}$$

Now

$$\begin{aligned}
\bar{u}(p')\gamma^\mu\not{p}' u(p) &= \bar{u}(p')\gamma^\mu\gamma^\rho p'_\rho u(p) \\
&= \bar{u}(p')(2g^{\mu\rho}-\gamma^\rho\gamma^\mu)p'_\rho u(p) \\
&= \bar{u}(p')2p'^\mu u(p)-\bar{u}(p')\not{p}'\gamma^\mu u(p) \\
&= \bar{u}(p')2p'^\mu u(p)-M\bar{u}(p')\gamma^\mu u(p),
\end{aligned}$$

and similarly

$$\bar{u}(p')\not{p}\gamma^\mu u(p)=\bar{u}(p')2p^\mu u(p)-M\bar{u}(p')\gamma^\mu u(p).$$

These formulae establish the required result.

Chapter 9

9.1 Using (8.206) with $\mathcal{F}_1=1$ and $\kappa=0$ for the current matrix elements, and cancelling a factor of e^2, we obtain

$$\begin{aligned}
W_{\mathrm{el}}^{\mu\nu} &= \frac{1}{8\pi M}\sum_{s,s'}\bar{u}(p,s)\gamma^\mu u(p',s')\bar{u}(p',s')\gamma^\nu u(p,s)(2\pi)^4\delta^4(p+q-p')\frac{1}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{p}'}{2E'} \\
&= \frac{1}{8\pi M}\mathrm{Tr}\{\gamma^\mu(\not{p}'+M)\gamma^\nu(\not{p}+M)\}(2\pi)^4\delta^4(p+q-p')\frac{1}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{p}'}{2E'}
\end{aligned}$$

where the Trace is $2M^{\mu\nu}$ (see (8.186)). Equation (9.104) then becomes

$$\mathrm{d}\sigma=\left(\frac{4\pi\alpha}{q^2}\right)^2\frac{1}{4[(k\cdot p)^2-m^2M^2]^{1/2}}L_{\mu\nu}M^{\mu\nu}(2\pi)^4\delta^4(p+q-p')\frac{\mathrm{d}^3\boldsymbol{p}'}{2E'(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{k}'}{2\omega'(2\pi)^3}.$$

This is precisely the formula which which yields the cross section for elastic eμ scattering (see Appendix K for the flux and phase space factors).

9.2 (a) Omitting the terms involving q_μ and q_ν from the expression (9.10) for $W^{\mu\nu}$, we need to evaluate

$$\begin{aligned} L_{\mu\nu}W^{\mu\nu} &= 2[k'^\mu k^\nu + k'^\nu k^\mu + (q^2/2)g^{\mu\nu}][-g_{\mu\nu}W_1 + (p_\mu p_\nu/M^2)W_2] \\ &= 2[(-2k'\cdot k - 2q^2)W_1 + (2p\cdot k'\ p\cdot k + q^2p^2/2)W_2/M^2]. \end{aligned}$$

In the 'laboratory' system, and neglecting the electron mass (compare (8.217) and (8.218)),

$$p\cdot k' = \omega' M,\ \ p\cdot k = \omega M,\ \ q^2 = -2k\cdot k',$$

and so

$$(2p\cdot k'\ p\cdot k + q^2p^2/2)/M^2 = 2\omega\omega' + q^2/2 = 2\omega\omega' - k\cdot k'.$$

Writing as usual $k = \omega = |\boldsymbol{k}|,\ k' = \omega' = |\boldsymbol{k}'|$, we have

$$k\cdot k' = kk'(1-\cos\theta) = 2kk'\sin^2\theta/2$$

and

$$2\omega\omega' - k\cdot k' = kk'(1+\cos\theta) = 2kk'\cos^2\theta/2.$$

Hence

$$L_{\mu\nu}W^{\mu\nu} = 4kk'[2W_1\sin^2\theta/2 + W_2\cos^2\theta/2]$$

and, from (9.104),

$$\mathrm{d}\sigma = \left(\frac{4\pi\alpha}{q^2}\right)^2 \frac{1}{4[(k\cdot p)^2 - m^2M^2]^{1/2}} 4\pi M 4kk'[2W_1\sin^2\theta/2 + W_2\cos^2\theta/2]\frac{\mathrm{d}^3\boldsymbol{k}'}{2\omega'(2\pi)^3}.$$

Now $(q^2)^2 = 16k^2k'^2\sin^4\theta/2$, and (neglecting the electron mass) $[(k\cdot p)^2 - m^2M^2]^{1/2} = kM$. Also,

$$\mathrm{d}^3\boldsymbol{k}' = k'^2\mathrm{d}k'\mathrm{d}\Omega.$$

Hence

$$\mathrm{d}\sigma = \frac{\alpha^2}{4k^2\sin^4\theta/2}[2W_1\sin^2\theta/2 + W_2\cos^2\theta/2]\mathrm{d}k'\mathrm{d}\Omega$$

as required.

(b) We have

$$Q^2 = 2kk'(1-\cos\theta) \text{ and } \nu = k - k'.$$

Hence

$$\begin{aligned}
\mathrm{d}\cos\theta\ \mathrm{d}k' &= \frac{1}{\begin{vmatrix} \frac{\partial Q^2}{\partial\cos\theta} & \frac{\partial Q^2}{\partial k'} \\ \frac{\partial \nu}{\partial\cos\theta} & \frac{\partial \nu}{\partial k'} \end{vmatrix}}\ \mathrm{d}Q^2\ \mathrm{d}\nu \\
&= \frac{1}{\begin{vmatrix} 2kk' & 2k(1-\cos\theta) \\ 0 & 1 \end{vmatrix}}\ \mathrm{d}Q^2\ \mathrm{d}\nu = \frac{1}{2kk'}\ \mathrm{d}Q^2\ \mathrm{d}\nu.
\end{aligned}$$

Using $\mathrm{d}\Omega = 2\pi \mathrm{d}\cos\theta$ and the result of part (a), this leads to (9.16) from (9.12).

Similarly, since

$$x = Q^2/2M\nu \text{ and } y = \nu/k,$$

we have

$$\begin{aligned}
\mathrm{d}Q^2\mathrm{d}\nu &= \frac{1}{\begin{vmatrix} \frac{\partial x}{\partial Q^2} & \frac{\partial x}{\partial \nu} \\ \frac{\partial y}{\partial Q^2} & \frac{\partial y}{\partial \nu} \end{vmatrix}}\ \mathrm{d}x\ \mathrm{d}y \\
&= \frac{1}{\begin{vmatrix} \frac{1}{2M\nu} & -\frac{Q^2}{2M\nu^2} \\ 0 & \frac{1}{k} \end{vmatrix}} = 2M\nu k\ \mathrm{d}x\ \mathrm{d}y \\
&= 2Mk^2y\ \mathrm{d}x\ \mathrm{d}y
\end{aligned}$$

as in (9.19), leading straightforwardly to the formula for $\mathrm{d}^2\sigma/\mathrm{d}x\mathrm{d}y$.

9.3 (a) The transverse polarization vectors are given in (9.38). These satisfy $\epsilon(\lambda = \pm 1)\cdot p = 0$ in the laboratory frame, and also (9.40). Hence in the product $\epsilon_\mu \epsilon^*_\nu W^{\mu\nu}$, with $W^{\mu\nu}$ given by (9.10), only the contraction with $-g^{\mu\nu}W_1$ survives, leading to

$$\begin{aligned}
\frac{1}{2}\sum_{\lambda=\pm 1} \epsilon_\mu(\lambda)\epsilon^*_\nu(\lambda)W^{\mu\nu} &= \frac{1}{2}[-\epsilon^\mu(\lambda=1)\epsilon^*_\mu(\lambda=1) - \epsilon^\mu(\lambda=-1)\epsilon^*_\mu(\lambda=-1)]W_1 \\
&= \frac{1}{2}[1+1]W_1 = W_1
\end{aligned}$$

and (9.46) follows from (9.45).

(b) From (9.47) the longitudinal/scalar virtual photon cross section is

$$\sigma_{\mathrm{S}} = (4\pi^2\alpha/K)\epsilon^*_\mu(\lambda = 0)\epsilon_\nu(\lambda = 0)W^{\mu\nu}$$

where $W^{\mu\nu}$ given by (9.10), and where $\epsilon^\mu(\lambda = 0)$ is real and given by (9.41), and satisfies $q \cdot \epsilon = 0$ (see (9.49)). Thus in the contractions with $W^{\mu\nu}$, terms involving q^μ and q^ν can be dropped. The W_1 term in '$\epsilon \cdot \epsilon\ W$' is then simply $(-\epsilon \cdot \epsilon)W_1 = -W_1$ (note (9.42)), while the W_2 term is

$$\begin{aligned}\frac{1}{M^2}\epsilon(\lambda = 0) \cdot p\ \epsilon(\lambda = 0) \cdot p\ W_2 &= \frac{1}{M^2Q^2}(q^3M)(q^3M)W_2 \\ &= \frac{(q^3)^2}{Q^2}W_2 = \frac{Q^2 + (q^0)^2}{Q^2}W_2 = \left(1 + \frac{\nu^2}{Q^2}\right) W_2,\end{aligned}$$

and (9.48) follows from these results.

From (9.46) we obtain

$$W_1 = \frac{K}{4\pi^2\alpha}\sigma_{\mathrm{T}},$$

and substituting this into (9.48) gives the required result for W_2.

9.4 In the limit $m \to 0$ the spinors ϕ and χ satisfy $\boldsymbol{\sigma} \cdot \boldsymbol{p}\phi = E\phi$ and $\boldsymbol{\sigma} \cdot \boldsymbol{p}\chi = -E\chi$ respectively, where in both cases $E = |\boldsymbol{p}|$. Hence ϕ satisfies

$$\frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{|\boldsymbol{p}|}\phi = \phi$$

which shows it has positive helicity (compare (4.67)); similarly χ has negative helicity.

(b) For example,

$$P_{\mathrm{R}}P_{\mathrm{L}} = \left(\frac{1 + \gamma_5}{2}\right) \left(\frac{1 - \gamma_5}{2}\right) = \frac{1}{4}[1 - \gamma_5^2] = 0.$$

When $m \neq 0$, the operators P_{R} and P_{L} still project out the ϕ and χ components of the 4-component spinor, but these 2-component objects are no longer (with $m \neq 0$) helicity eigenstates (since, for example, $(\boldsymbol{\sigma} \cdot \boldsymbol{p}/|\boldsymbol{p}|)\phi$ is no longer equal to ϕ or $-\phi$).

(c) We may write

$$\bar{u}\gamma^\mu u = u^\dagger(P_{\rm R} + P_{\rm L})\gamma^0\gamma^\mu(P_{\rm R} + P_{\rm L})u.$$

We exploit the fundamental relation $\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu$ (see (J.11)). Consider one 'cross' term:

$$\begin{aligned} u^\dagger P_{\rm R}\gamma^0\gamma^\mu P_{\rm L}u &= u^\dagger\gamma^0 P_{\rm L}\gamma^\mu P_{\rm L}u \\ &= u^\dagger\gamma^0\gamma^\mu P_{\rm R}P_{\rm L}u \\ &= 0. \end{aligned}$$

Similarly for the term $u^\dagger P_{\rm L}\gamma^0\gamma^\mu P_{\rm R}u$. The only surviving terms are

$$u^\dagger P_{\rm R}\gamma^0\gamma^\mu P_{\rm R}u + u^\dagger P_{\rm L}\gamma^0\gamma^\mu P_{\rm L}u$$

which is just $\bar{u}_{\rm R}\gamma^\mu u + \bar{u}_{\rm L}\gamma^\mu u$. Hence 'R' states connect only to 'R' states, and similarly for 'L' states, and so helicity (in the massless limit) is conserved.

Note that '$\bar{u}_{\rm R}$' could perhaps more clearly be written as

$$\overline{u_{\rm R}}$$

since we form it by taking the dagger of $u_{\rm R}$ and then multiplying by γ^0 - i.e. we take the Dirac 'bar' of $u_{\rm R}$. $\bar{u}_{\rm R}$ is however the conventional notation.

(d) In this case a typical cross term is

$$\begin{aligned} u^\dagger P_{\rm R}\gamma^0\gamma^\mu\gamma_5 P_{\rm L}u &= u^\dagger\gamma^0 P_{\rm L}\gamma^\mu\gamma_5 P_{\rm L}u \\ &= u^\dagger\gamma^0\gamma^\mu P_{\rm R}\gamma_5 P_{\rm L}u \\ &= u^\dagger\gamma^0\gamma^\mu\gamma_5 P_{\rm R}P_{\rm L}u \\ &= 0, \end{aligned}$$

and again helicity is conserved.

(e) The Dirac mass term is

$$\bar{\hat{\psi}}\hat{\psi} = \hat{\psi}^\dagger(P_{\rm R} + P_{\rm L})\gamma^0(P_{\rm R} + P_{\rm L})\hat{\psi}.$$

Consider a 'diagonal' term:

$$\hat{\psi}^\dagger P_{\rm R}\gamma^0 P_{\rm R}\hat{\psi} = \hat{\psi}^\dagger\gamma^0 P_{\rm L} P_{\rm R}\hat{\psi} = 0,$$

and similarly for the other diagonal term. Only the 'L-R' and 'R-L' terms survive (the daggers in the printed answer are a misprint).

9.5 Neglecting the positron and proton masses, their 4-momenta are $p_{\rm e^+} = (k, 0, 0, -k)$, say, and $p_{\rm p} = (p, 0, 0, p)$. Then

$$W_{\rm CM}^2 = (p_{\rm e^+} + p_{\rm p})^2 = (k+p)^2 - (k-p)^2 = 4kp.$$

So $W_{\rm CM} = 2\sqrt{kp} = 300.3$ GeV.

A leptoquark of mass $M_{\rm lq}$ formed as a resonance state of the $\rm e^+$ and the struck quark would appear as a peak in the effective mass of the $\rm e^+$ and the quark, at an effective mass equal to $M_{\rm lq}$. In a simple parton model picture, this efective mass is $\sqrt{(p_{\rm e^+} + xp_{\rm p})^2}$. So we expect a peak when

$$p_{\rm e^+}^2 + 2x\ p_{\rm e^+}\cdot p_{\rm p} + x^2 p_{\rm p}^2 = M_{\rm lq}^2,$$

or, neglecting the positron and proton masses, at $x = M_{\rm lq}^2/W_{\rm CM}^2$.

9.6 (a) Using (9.92) for $\mathrm{d}^2\sigma/\mathrm{d}x_1\,\mathrm{d}x_2$, we have

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} &= \int \mathrm{d}x_1\,\mathrm{d}x_2 \frac{4\pi\alpha^2}{9q^2}\sum_a e_a^2[q_a(x_1)\bar{q}_a(x_2) + \bar{q}_a(x_1)q_a(x_2)]\delta(q^2 - sx_1x_2)\\ &= \frac{4\pi\alpha^2}{9q^2}\int \mathrm{d}x_1\,\mathrm{d}x_2\sum_a e_a^2[q_a(x_1)\bar{q}_a(x_2) + \bar{q}_a(x_1)q_a(x_2)]\frac{1}{s}\delta(q^2/s - x_1x_2)\end{aligned}$$

with the help of (E.29), and then writing $1/s = x_1x_2/q^2$ and $\tau = q^2/s$ we obtain the desired formula.

(b)

$$\begin{aligned}\mathrm{d}q^2\,\mathrm{d}x_{\rm F} &= \begin{vmatrix} \frac{\partial q^2}{\partial x_1} & \frac{\partial q^2}{\partial x_2} \\ \frac{\partial x_{\rm F}}{\partial x_1} & \frac{\partial x_{\rm F}}{\partial x_2}\end{vmatrix}\ \mathrm{d}x_1\,\mathrm{d}x_2\\ &= \begin{vmatrix} sx_2 & sx_1 \\ 1 & -1\end{vmatrix}\ \mathrm{d}x_1\,\mathrm{d}x_2\\ &= -s(x_1+x_2)\ \mathrm{d}x_1\,\mathrm{d}x_2.\end{aligned}$$

The minus sign can be absorbed by appropriate choice of limits in the $q^2 - x_{\rm F}$ integration. In the variables (x_1, x_2), the integration is over the square $0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1$. Consider performing the integration holding x_1 fixed and integrating over x_2, and then integrating over x_1. Take $x_1 = 1/2$ as an example, with x_2 running from $x_2 = 0$ to $x_2 = 1$. In the $q^2 - x_{\rm F}$ plane, this line maps into the line $2q^2/s + x_{\rm F} = 1/2$, and it is traversed in the sense of q^2/s increasing (from 0 to 1/2) but $x_{\rm F}$ *decreasing* (from 1/2 to -1/2). We can reverse the sense in which $x_{\rm F}$ is covered by invoking the minus sign from the determinant.

The variables x_1 and x_2 are given in terms of $x_{\rm F}$ and τ by $x_1 - x_2 = x_{\rm F}$ and $x_2 = \tau/x_1$. So we have

$$x_1 - \tau/x_1 = x_{\rm F}.$$

Solving for x_1 (which is greater than 0) we find

$$x_1 = \frac{1}{2}[x_{\rm F} + (x_{\rm F}^2 + 4\tau)^{1/2}],$$

and hence

$$x_2 = \frac{1}{2}[-x_{\rm F} + (x_{\rm F}^2 + 4\tau)^{1/2}],$$

so that $x_1 + x_2 = (x_{\rm F}^2 + \tau)^{1/2}$. Hence

$$\begin{aligned}
{\rm d}^2\sigma &= \frac{4\pi\alpha^2}{9q^2}\sum_a e_a^2[q_a(x_1)\bar{q}_a(x_2) + \bar{q}_a(x_1)q_a(x_2)]\ {\rm d}x_1\ {\rm d}x_2 \\
&= \frac{1}{(x_1+x_2)s}\{\ldots\}\ {\rm d}q^2\ {\rm d}x_{\rm F} \\
&= \frac{1}{(x_{\rm F}^2 + 4\tau)^{1/2}}\ \frac{4\pi\alpha^2\tau}{9q^4}\sum_a e_a^2[q_a(x_1)\bar{q}_a(x_2) + \bar{q}_a(x_1)q_a(x_2)]\ {\rm d}q^2\ {\rm d}x_{\rm F}
\end{aligned}$$

which leads to the desired expression.

9.7 Let the 4-momenta of the incoming q and $\bar{\rm q}$ be k and k_1 respectively, and let those of the outgoing μ^- and μ^+ be p' and p_1. Then the $\rm q - \bar{q} - \gamma$ vertex, for scalar quarks, is proportional to $(k - k_1)^\mu$, while the $\gamma - \mu^- - \mu^+$ vertex is same as in problem 8.19(b). Thus in evaluating the unpolarized cross section we need the contraction

$$\begin{aligned}
T &= (k-k_1)^\mu(k-k_1)^\nu(p'_\mu p_{1\nu} + p'_\nu p_{1\mu} - (Q^2/2)g_{\mu\nu}) \\
&= 2p'\cdot(k-k_1)\ p_1\cdot(k-k_1) - (Q^2/2)(k-k_1)^2.
\end{aligned}$$

Introduce the Mandelstam variables

$s = (k+k_1)^2 = Q^2$, $t = (k-p')^2 = (k_1-p_1)^2$, and $u = (k-p_1)^2 = (k_1-p')^2$.

Then, negelecting lepton masses,

$$\begin{aligned} T &= 2[-\frac{t}{2}+\frac{u}{2}][-\frac{u}{2}+\frac{t}{2}] - (Q^2/2)(-Q^2) \\ &= -\frac{1}{2}(t-u)^2 + \frac{1}{2}(Q^2)^2 \\ &= -\frac{1}{2}(4k^2\cos\theta)^2 + \frac{1}{2}(4k^2)^2 \\ &\propto (1-\cos^2\theta) \end{aligned}$$

where k is the CM momentum.

Chapter 10

10.1 We need to calculate the Jacobian for the change of variables

$$\{x_1, x_2, x_3, x_4\} \to \{x, y, z, X\}. \quad \text{(A)}$$

Each of these variables is in fact a 4-vector, with (therefore) four components, as is explicit in the notation d^4x_1 etc. Thus there are a total of 16 variables altogether, and we certainly don't want to deal with a 16×16 determinant. We can however write the 16-dimensional integration element as

$$[\mathrm{d}x_1^0\,\mathrm{d}x_2^0\,\mathrm{d}x_3^0\,\mathrm{d}x_4^0]\,[\mathrm{d}x_1^1\,\mathrm{d}x_2^1\,\mathrm{d}x_3^1\,\mathrm{d}x_4^1]\,[\mathrm{d}x_1^2\ \mathrm{etc}]$$

and imagine doing the transformation first for the '^{0}' components of the two sets in (A):

$$\{x_1^0, x_2^0, x_3^0, x_4^0\} \to \{x^0, y^0, z^0, X^0\},$$

then for the '^{1}' components, etc. The total Jacobian will then be the product of four 4×4 Jacobians, one for each of the components. But since they all have exactly the same form, we shall suppress the explicit 'component' label, and write simply

$$J = \begin{vmatrix} \frac{\partial x}{\partial x_1} & \frac{\partial x}{\partial x_2} & \frac{\partial x}{\partial x_3} & \frac{\partial x}{\partial x_4} \\ \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} & \frac{\partial y}{\partial x_3} & \frac{\partial y}{\partial x_4} \\ \frac{\partial z}{\partial x_1} & \frac{\partial z}{\partial x_2} & \frac{\partial z}{\partial x_3} & \frac{\partial z}{\partial x_4} \\ \frac{\partial X}{\partial x_1} & \frac{\partial X}{\partial x_2} & \frac{\partial X}{\partial x_3} & \frac{\partial X}{\partial x_4} \end{vmatrix}$$

for one of the determinants. Evaluating the partial derivatives, we obtain

$$\begin{aligned} J &= \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{vmatrix} \\ &= \frac{1}{4}\left\{ \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{vmatrix} \right\} \\ &= 1. \end{aligned}$$

Hence the full Jacobian is unity.

Inverting the relations giving $\{x, y, z, X\}$ in terms of $\{x_1, x_2, x_3, x_4\}$, we find

$$\begin{aligned}
x_1 &= \frac{1}{4}(4X + 3x - y + 2z) \\
x_2 &= \frac{1}{4}(4X - x + 3y - 2z) \\
x_3 &= \frac{1}{4}(4X - x - y + 2z) \\
x_4 &= \frac{1}{4}(4X - x - y - 2z).
\end{aligned}$$

The exponentials in (10.3) then become

$$\mathrm{e}^{\mathrm{i}(\frac{p'_{\mathrm{A}} - p_{\mathrm{B}}}{4})\cdot(4X+3x-y+2z)}\ \mathrm{e}^{\mathrm{i}(\frac{p'_{\mathrm{B}} - p_{\mathrm{A}}}{4})\cdot(4X-x+3y-2z)}$$

which reduces to the exponentials given in (10.4). The propagator factors only depend on the difference in the arguments of the two fields (see (6.98)).

10.2 The integral is

$$\begin{aligned}
\int_0^1 \frac{1}{[A + x(B-A)]^2} &= -\frac{1}{(B-A)}\left[\frac{1}{A + x(B-A)}\right]_0^1 \\
&= \frac{1}{A-B}\left[\frac{1}{B} - \frac{1}{A}\right] = \frac{1}{AB}.
\end{aligned}$$

10.3 The $[\ldots]$ bracket is

$$\begin{aligned}
&[k^2 - xk^2 - (1-x)m_{\mathrm{A}}^2 + xq^2 - 2x\ q\cdot k + xk^2 - xm_{\mathrm{B}}^2 + \mathrm{i}\epsilon] \\
&\qquad = [k^2 - 2x\ q\cdot k + xq^2 - xm_{\mathrm{B}}^2 - (1-x)m_{\mathrm{A}}^2 + \mathrm{i}\epsilon] \\
&\qquad = [(k - xq)^2 - x^2q^2 + xq^2 - xm_{\mathrm{B}}^2 - (1-x)m_{\mathrm{A}}^2 + \mathrm{i}\epsilon] \\
&\qquad = [k'^2 - \{-x(1-x)q^2 + xm_{\mathrm{B}}^2 + (1-x)m_{\mathrm{A}}^2\} + \mathrm{i}\epsilon]
\end{aligned}$$

which is just $[k'^2 - \Delta + \mathrm{i}\epsilon]$. Note that we have replaced '$(1-x)\mathrm{i}\epsilon + \mathrm{i}\epsilon$' by '$\mathrm{i}\epsilon$' since all that matters is the sign of this infinitesimal imaginary part, and $(1-x)$ can never be negative.

10.4 We have

$$\Pi_{\mathrm{C}}^{[2]}(q^2) = \frac{-g^2}{8\pi^2}\int_0^1\int_0^\Lambda \frac{u^2\mathrm{d}u}{(u^2+\Delta)^{3/2}}.$$

One way of evaluating the integral over u is to substitute $u = \Delta^{1/2}\tan\theta$. Then

$$\mathrm{d}u = \Delta^{1/2}\sec^2\theta\ \mathrm{d}\theta$$

and

$$(u^2+\Delta)^{3/2} = \Delta^{3/2}\sec^3\theta.$$

The integral over u becomes

$$\begin{aligned}
\int \frac{\tan^2\theta}{\sec^3\theta}\ \sec^2\theta\ \mathrm{d}\theta &= \int\frac{\sin^2\theta}{\cos\theta}\ \mathrm{d}\theta \\
&= \int\left(\frac{1}{\cos\theta} - \cos\theta\right)\ \mathrm{d}\theta \\
&= \ln(\sec\theta + \tan\theta) - \sin\theta \\
&= \ln\left(\frac{(u^2+\Delta)^{1/2}}{\Delta^{1/2}} + \frac{u}{\Delta^{1/2}}\right) - \frac{\Lambda}{(u^2+\Delta)^{1/2}}.
\end{aligned}$$

This expression vanishes at $u = 0$ and has the value

$$\left\{\ln\left(\frac{\Lambda + (\Lambda^2+\Delta)^{1/2}}{\Delta^{1/2}}\right) - \frac{\Lambda}{(\Lambda+\Delta)^{1/2}}\right\}$$

at $u = \Lambda$ (correcting the formula given in (10.51)).

10.5 $\Pi_{\mathrm{C}}^{[2]}(q^2)$ is given by (10.42) and its dependence on q^2 is contained in the quantity Δ of (10.43). Hence

$$\begin{aligned}
\frac{\mathrm{d}\Pi_{\mathrm{C}}^{[2]}(q^2)}{\mathrm{d}q^2} &= \mathrm{i}g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\ \frac{-2}{(k'^2-\Delta+\mathrm{i}\epsilon)^3}\ \frac{\mathrm{d}\Delta}{\mathrm{d}q^2} \\
&= \mathrm{i}g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\ \frac{2}{(k'^2-\Delta+\mathrm{i}\epsilon)^3}\ x(1-x).
\end{aligned}$$

The denominator of the k' integral now behaves like $(k')^6$ at large values of k', which will ensure convergence. In more detail, we now have, in place of (10.44), a k'^0 integral of the form

$$\int\frac{\mathrm{d}k'^0}{[(k'^0)^2-A]^3} = \frac{1}{2}\frac{\partial^2}{\partial A^2}\int\frac{\mathrm{d}k'^0}{[(k'^0)^2-A]}$$

which will result in a denominator $(u^2+\Delta)^{5/2}$ in (10.50), so that the u-integral is manifestly convergent, by counting powers of u at large values of u.

10.6 Consider the counter term

$$\mathcal{L}_{\rm ct} = \frac{1}{2}\delta Z_{\rm C}\partial_\mu\hat{\phi}_{\rm ph,C}\partial^\mu\hat{\phi}_{\rm ph,C}$$

where normal-ordering is understood. Remembering that '$L=T-V$', the corresponding interaction potential is $-\mathcal{L}_{\rm ct}$, and in the Dyson-Wick expansion it is '$-$i' times the interaction that appears. Hence the one-C matrix element required is

$$\frac{\rm i}{2}\delta Z_{\rm C}\langle {\rm C},k|\partial_\mu\hat{\phi}_{\rm ph,C}\partial^\mu\hat{\phi}_{\rm ph,C}|{\rm C},k\rangle$$

$$= \frac{\rm i}{2}2E\delta Z_{\rm C}\langle 0|\hat{a}_{\rm C}(k)\int\frac{{\rm d}^3\boldsymbol{k}_1}{(2\pi)^3\sqrt{2E_1}}\,[\hat{a}_{\rm C}(k_1)(-{\rm i}k_{1\mu}){\rm e}^{-{\rm i}k_1\cdot x}+\hat{a}^\dagger_{\rm C}(k_1)({\rm i}k_{1\mu}){\rm e}^{{\rm i}k_1\cdot x}]$$

$$\times\int\frac{{\rm d}^3\boldsymbol{k}_2}{(2\pi)^3\sqrt{2E_2}}\,[\hat{a}_{\rm C}(k_2)(-{\rm i}k^{2\mu}){\rm e}^{-{\rm i}k_2\cdot x}+\hat{a}^\dagger_{\rm C}(k_2)({\rm i}k^{2\mu}){\rm e}^{{\rm i}k_2\cdot x}]\ \hat{a}^\dagger_{\rm C}(k)|0\rangle$$

where normal-ordering is understood but not shown explicitly. The only terms which give a non-zero vev are those in which the number of $\hat{a}$ operators is equal to the number of $\hat{a}^\dagger$ operators. One of these is

$$\langle 0|\hat{a}_{\rm C}(k)({\rm i}k_{1\mu})\hat{a}^\dagger_{\rm C}(k_1)(-{\rm i}k^{2\mu})\hat{a}_{\rm C}(k_2)\hat{a}^\dagger_{\rm C}(k)|0\rangle.$$

As usual, we write

$$\hat{a}_{\rm C}(k)\hat{a}^\dagger_{\rm C}(k_1) = \hat{a}^\dagger_{\rm C}(k_1)\hat{a}_{\rm C}(k)+(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}_1),$$

the $\hat{a}^\dagger_{\rm C}(k_1)$ then annihilating on $\langle 0|$, and similarly with $\hat{a}_{\rm C}(k_2)\hat{a}^\dagger_{\rm C}|0\rangle$ which yields a factor $(2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}_2)$. The $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ integrations can then be done, setting $\boldsymbol{k}_1=\boldsymbol{k}$ and $\boldsymbol{k}_2=\boldsymbol{k}$, and hence $E_1=E$, and $E_2=E$, and $k_1^\mu=k^\mu, k_2^\mu=k^\mu$. This term therefore finally yields

$$\frac{\rm i}{2}\delta Z_{\rm C}k^2.$$

The other surviving term is (after normal-ordering)

$$\langle 0|\hat{a}_{\mathrm{C}}(k)(\mathrm{i}k^{2\mu})\hat{a}^{\dagger}_{\mathrm{C}}(k_2)(-\mathrm{i}k_{1\mu})\hat{a}_{\mathrm{C}}(k_1)\hat{a}^{\dagger}_{\mathrm{C}}(k)|0\rangle.$$

This contributes exactly the same result, removing the factor $\frac{1}{2}$ as required. The 1-C matrix element of the counter term involving $\hat{\phi}^2_{\mathrm{ph,C}}$ is very similar, lacking the k^2 factor coming from the gradients.

Chapter 11

11.1 The Feynman rule for the counter term in the fermion propagator is given by (a) of (11.7). The fermion analogue of (10.63) is then

$$S = \frac{\mathrm{i}}{\not{p} - m + (Z_2 - 1)\not{p} - \delta m - \Sigma^{[2]}(p)}.$$

For $\not{p} \approx m$ we expand $\Sigma^{[2]}(p)$ as

$$\Sigma^{[2]}(p) \approx \Sigma^{[2]}(\not{p} = m) + (\not{p} - m)\left.\frac{\mathrm{d}\Sigma^{[2]}}{\mathrm{d}\not{p}}\right|_{\not{p}=m},$$

so that

$$\begin{aligned} S &\approx \frac{\mathrm{i}}{\not{p} - m + (Z_2 - 1)\not{p} - \delta m - \Sigma^{[2]}(\not{p} = m) - (\not{p} - m)\left.\frac{\mathrm{d}\Sigma^{[2]}}{\mathrm{d}\not{p}}\right|_{\not{p}=m}} \\ &= \frac{\mathrm{i}}{\not{p}[Z_2 - \left.\frac{\mathrm{d}\Sigma^{[2]}}{\mathrm{d}\not{p}}\right|_{\not{p}=m}] + m\left.\frac{\mathrm{d}\Sigma^{[2]}}{\mathrm{d}\not{p}}\right|_{\not{p}=m} - \Sigma^{[2]}(\not{p} = m) - \delta m - m}, \end{aligned}$$

where Z_2 and δm must be chosen so that this expression is equal to $\mathrm{i}/(\not{p} - m)$. The coefficient of $\not{p}$ therefore yields

$$Z_2 = 1 + \left.\frac{\mathrm{d}\Sigma^{[2]}}{\mathrm{d}\not{p}}\right|_{\not{p}=m},$$

while the mass terms may be written as

$$\begin{aligned} m\left.\frac{\mathrm{d}\Sigma^{[2]}}{\mathrm{d}\not{p}}\right|_{\not{p}=m} - \Sigma^{[2]}(\not{p} = m) - \delta m - m &= m(Z_2 - 1) - \Sigma^{[2]}(\not{p} = m) - (m_0 Z_2 - m) - m \\ &= -m + (m - m_0)Z_2 - \Sigma^{[2]}(\not{p} = m), \end{aligned}$$

whence we require

$$m_0 - m = -Z_2^{-1}\Sigma^{[2]}(\not{p} = m).$$

When these values are substituted back into the first expression for S, we obtain (11.10).

11.2 Consider QED for definiteness, with interaction Lagrangian

$$\hat{\mathcal{L}}_{\text{int}}(x) = -q\bar{\hat{\psi}}(x)\gamma^\mu\hat{\psi}(x).$$

The amplitude for a typical closed fermion loop will involve (compare (10.3)) a sequence of fermion propagators starting from a certain space-time point and ending at the same point:

$$\langle 0|T(\hat{\psi}(x_1)\bar{\hat{\psi}}(x_2))|0\rangle\langle 0|T(\hat{\psi}(x_2)\bar{\hat{\psi}}(x_3))|0\rangle\ldots\langle 0|T(\hat{\psi}(x_N)\bar{\hat{\psi}}(x_1))|0\rangle. \quad (A)$$

Note that each propagator involves fields from different interaction Lagrangians in a term of given order in the Dyson expansion analogous to (6.42), since each interaction involves two fields at the same point. In a given 'Dyson' term, however, the interactions may be written in any order within the overall time-ordering symbol T, for example (referring to (A)) as

$$\langle 0|\ldots T\{\ldots[\bar{\hat{\psi}}(x_1)\not{A}(x_1)\hat{\psi}(x_1)][\bar{\hat{\psi}}(x_2)\not{A}(x_2)\hat{\psi}(x_2)]\ldots[\bar{\hat{\psi}}(x_N)\not{A}(x_1)\hat{\psi}(x_N)]\}\ldots|0\rangle,$$

without introducing any sign changes. The product of contractions shown in (A) is now obtained after permuting $\bar{\hat{\psi}}(x_1)$ through an odd number of fermion fields (namely the single field $\hat{\psi}(x_1)$ and the appropriate number of pairs $\bar{\hat{\psi}}(x_2)\hat{\psi}(x_2),\ldots,\bar{\hat{\psi}}(x_N)\hat{\psi}(x_N)$), which produces an overall minus sign.

11.3 The amplitude arises from the second-order term in the Dyson expansion, which involves the T-product

$$T(\bar{\hat{\psi}}_\alpha(x_1)\gamma^\mu_{\alpha\beta}\hat{\psi}_\beta(x_1)\ \bar{\hat{\psi}}_\rho(x_2)\gamma^\nu_{\rho\sigma}\hat{\psi}_\sigma(x_2)),$$

where we have indicated the (summed over) Dirac matrix indices $\alpha, \beta, \rho, \sigma$ explicitly. Figure 11.2(b) corresponds to the contraction

$$\ldots\gamma^\mu_{\alpha\beta}\langle 0|T(\hat{\psi}_\beta(x_1)\bar{\hat{\psi}}_\rho(x_2))|0\rangle\gamma^\nu_{\rho\sigma}\langle 0|T(\hat{\psi}_\sigma(x_2)\bar{\hat{\psi}}_\alpha(x_1))|0\rangle\ldots$$

which has the structure

$$\gamma^\mu_{\alpha\beta} S_{\beta\rho}(x_1 - x_2)\gamma^\nu_{\rho\sigma} S_{\sigma\alpha}(x_2 - x_1)$$
$$= \sum_\alpha [\gamma^\mu S(x_1 - x_2)\gamma^\nu S(x_2 - x_1)]_{\alpha\alpha}$$

which exhibits the required Trace.

11.4 The result is (up to a factor of 2) the same as (8.78) with $k' = k + q$.

11.5 We have, from (11.32),

$$\bar{\Pi}^{[2]}_\gamma(q^2) = \Pi^{[2]}_\gamma(q^2) - \Pi^{[2]}_\gamma(0),$$

where from (11.23)

$$\Pi^{[2]}_\gamma(q^2) = 8\mathrm{i}e^2 \int_0^1 \mathrm{d}x\; x(1-x) I(x, q^2)$$

with

$$I(x, q^2) = \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{1}{(k'^2 - \Delta_\gamma(x, q^2) + \mathrm{i}\epsilon)^2},$$

where

$$\Delta_\gamma(x, q^2) = -x(1-x)q^2 + m^2.$$

As noted in the text, the k'-integral in $I(x, q^2)$ has been evaluated in section 10.3.1 and 10.3.2, with the result (correcting (10.51))

$$I(x, q^2) = \frac{\mathrm{i}}{8\pi^2}\left\{\ln\left(\frac{\Lambda + (\Lambda^2 + \Delta_\gamma(x, q^2))^{1/2}}{\Delta_\gamma^{1/2}(x, q^2)}\right) - \frac{\Lambda}{(\Lambda^2 + \Delta_\gamma(x, q^2))^{1/2}}\right\},$$

where Λ is the ultraviolet cut off. The quantity $\bar{\Pi}^{[2]}_\gamma(q^2)$ involves the x-integral of the subtracted quantity

$$I(x, q^2) - I(x, q^2 = 0),$$

and we shall see that this has a well-defined finite limit as $\Lambda \to \infty$. We have

$$I(x, q^2) - I(x, q^2 = 0) = \frac{\mathrm{i}}{8\pi^2}\left\{\ln\left(\frac{\Lambda + (\Lambda^2 + \Delta_\gamma(x, q^2))^{1/2}}{\Lambda + (\Lambda^2 + \Delta_\gamma(x, q^2 = 0))^{1/2}}\; \frac{\Delta_\gamma^{1/2}(x, q^2 = 0)}{\Delta_\gamma^{1/2}(x, q^2)}\right)\right.$$

$$- \frac{\Lambda}{(\Lambda^2 + \Delta_\gamma(x, q^2))^{1/2}} + \frac{\Lambda}{(\Lambda^2 + \Delta_\gamma(x, q^2 = 0))^{1/2}} \Bigg\} .$$

Now

$$\lim_{\Lambda \to \infty} \frac{\Lambda + (\Lambda^2 + \Delta_\gamma(x, q^2))^{1/2}}{\Lambda + (\Lambda^2 + \Delta_\gamma(x, q^2 = 0))^{1/2}} = 1$$

and

$$\lim_{\Lambda \to \infty} - \frac{\Lambda}{(\Lambda^2 + \Delta_\gamma(x, q^2))^{1/2}} + \frac{\Lambda}{(\Lambda^2 + \Delta_\gamma(x, q^2 = 0))^{1/2}} = 0.$$

Noting also that $\Delta_\gamma(x, q^2 = 0) = m^2$, we obtain

$$\bar{\Pi}^{[2]}\gamma(q^2) = -\frac{e^2}{2\pi^2} \int_0^1 \mathrm{d}x \; x(1-x) \ln \left[\frac{m^2}{\Delta_\gamma(x, q^2)} \right]$$

which is equivalent to (11.36).

11.7 For $q^2 = -\boldsymbol{q}^2 \ll m^2$ we write

$$\begin{aligned} \bar{\Pi}^{[2]}_\gamma(\boldsymbol{q}^2) &= \frac{2\alpha}{\pi} \int_0^1 \mathrm{d}x \; x(1-x) \ln[1 + \boldsymbol{q}^2 x(1-x)/m^2] \\ &\approx \frac{2\alpha}{\pi} \int_0^1 \mathrm{d}x \; x(1-x) \frac{\boldsymbol{q}^2}{m^2} x(1-x) \\ &= \frac{2\alpha \boldsymbol{q}^2}{\pi m^2} \int_0^1 \mathrm{d}x \; x^2(1-x)^2 \end{aligned}$$

which leads to (11.38).

11.8 In this case we are interested in $q^2 = -|q^2| \gg m^2$. We write

$$\begin{aligned} \bar{\Pi}^{[2]}_\gamma(|q^2|) &= \frac{2\alpha}{\pi} \int_0^1 \mathrm{d}x \; x(1-x) \; \ln \left(\frac{|q^2|}{m^2} x(1-x) + 1 \right) \\ &= \frac{2\alpha}{\pi} \int_0^1 \mathrm{d}x \; x(1-x) \; \ln \left\{ \frac{|q^2|}{m^2} x(1-x)[1 + \frac{m^2}{|q^2| x(1-x)}] \right\} \\ &= \frac{2\alpha}{\pi} \int_0^1 \mathrm{d}x \; x(1-x) \left\{ \ln \left(\frac{|q^2|}{m^2} \right) + \ln[x(1-x)] + O\left(\frac{m^2}{|q^2|} \right) \right\} . \end{aligned}$$

Then using

$$\int_0^1 x \; \ln x \; \mathrm{d}x = \left(\frac{x^2}{2} \ln x - \frac{x^2}{4} \right)_0^1 = -\frac{1}{4}$$

and

$$\int_0^1 x^2 \ \ln x \ \mathrm{d}x = \left(\frac{x^3}{3}\ln x - \frac{x^3}{3}\right)_0^1 = -\frac{1}{9},$$

together with

$$\int_0^1 x(1-x)\ln(1-x) \ \mathrm{d}x = \int_0^1 x(1-x)\ln x \mathrm{d}x,$$

we arrive at (11.54).

11.9 The e contributes 0.01639, the μ 0.00825 and the τ 0.00388 for a total of 0.0285.

11.10 In the system $\hbar = c = 1$, with one independent dimension which we take to be mass (see Appendix B), the electromagnetic field strength tensor $\hat{F}^{\mu\nu}$ has dimension M^2, so that the Maxwell action

$$-\frac{1}{4}\int \mathrm{d}^4x \hat{F}^{\mu\nu}\hat{F}_{\mu\nu}$$

is dimensionless. Hence $(\hat{F}^{\mu\nu}\hat{F}_{\mu\nu})^2$ has dimension M^8, and therefore the coupling constant 'E' must have dimension M^{-4} so that

$$E\int \mathrm{d}^4x \ (\hat{F}^{\mu\nu}\hat{F}_{\mu\nu})^2$$

is dimensionless.

In terms of the fields $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{B}}$,

$$\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} = 2(\hat{\boldsymbol{B}}^2 - \hat{\boldsymbol{E}}^2).$$

Hence

$$E(\hat{F}^{\mu\nu}\hat{F}_{\mu\nu})^2 = 4E(\hat{\boldsymbol{B}}^2 - \hat{\boldsymbol{E}}^2)^2.$$

This fourth-order (in the fields) interaction will contribute to $\gamma - \gamma$ scattering, which does not occur in the classical (Maxwell) theory. See for example *The Quantum Theory of Fields*, volume 1, by S. Weinberg (CUP), pages 533-4.