Large N : confining flux tubes and string theory

Michael Teper (Oxford) - Humboldt/Zeuthen '09

- Large N :

is $N = \infty$ physically relevant: i.e. is large-N confining and is $N = 3$ close to $N = \infty$?

Monte Carlo algorithm and its cost as $N \uparrow$

- Flux tubes and String Theory :

effective string theories - recent progress

lattice results : fundamental and $k = 2$ flux tubes in D=2+1

lattice results : fundamental flux in D=3+1

Large N – a brief overview

• QCD : value of $g^2 \leftrightarrow$ scale of physics,

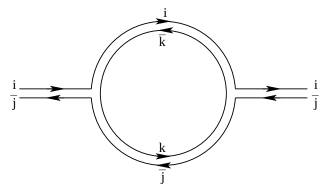
$\Rightarrow$ there is no obvious expansion parameter

$\Rightarrow$ try something much less obvious 't Hooft 1974:

expand SU(N) as a power series in $1/N$ around SU(∞)

$$SU(N) \simeq SU(\infty) + O(1/N^2)$$

• now, in perturbation theory gluon loop is $O(g^2N)$



so to have smooth large-N limit for perturbative physics

$\Rightarrow \quad g^2N = \text{const} \quad$ at large N

• $N \to \infty$ colour singlet phenomenology

't Hooft, Witten-Veneziano, Dashen-Manohar, ...

zero decay widths; no mixing; exact OZI, η'; SU(4) spin-flavour symmetry for baryons; chiral effective lagrangians, ...

• no scattering of colour singlets – integrability?

but strongly interacting bound states

• factorisation colour singlet operators: e.g.

$$\langle \Phi_1(x_1)\Phi_2(x_2)\rangle = \langle \Phi_1\rangle\langle \Phi_2\rangle \left\{1 + O\left(\frac{1}{N^2}\right)\right\}$$

$\Rightarrow$ Witten's Master Field $\to$ translation invariant $\to$ Eguchi-Kawai single point reduction

$\to$ large-N lattice calculations (rough!) in mid-80's

- the large N counting for hadrons follows from:

– coupling variation with N determined so as to control the 'non-confined'perturbative dynamics

– in the confined phase the probability for colour singlets from products of adjoints $\rightarrow 0$ as $N \rightarrow \infty$

- Feynman diagrams on 2D surfaces :

$g^2 N \rightarrow \infty \rightarrow$ vertices dense $\rightarrow$ stringy sheets

$\Rightarrow$

$N = \infty$ gauge theory $\sim$ a string theory 't Hooft, 1974

$N = \infty$ gauge theory $\sim$ dual to a string theory Maldacena, 1997

• Since 1997 Maldacena new hope of a solution at $N = \infty$ has been provided by the strong-weak coupling gauge-gravity dualities and this has provided new motivation for numerical calculations at large N

• A trivial but effective strategy is to repeat the calculations for larger N and compare the results, i.e. SU(2), SU(3), SU(4), SU(5), SU(6), ...

• Since the leading correction in a theory with just adjoint fields is expected to be $O(1/N^2)$, going to say $N = 8$ should usually be sufficient to provide a range of N from which we can extrapolate using

$$\frac{m(N)}{\sqrt{\sigma(N)}} = \frac{m(\infty)}{\sqrt{\sigma(\infty)}} + \frac{c}{N^2} + O\left(\frac{1}{N^4}\right)$$

Lattice strategy

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Calculating masses : Wilson, Coseners House, March 1981

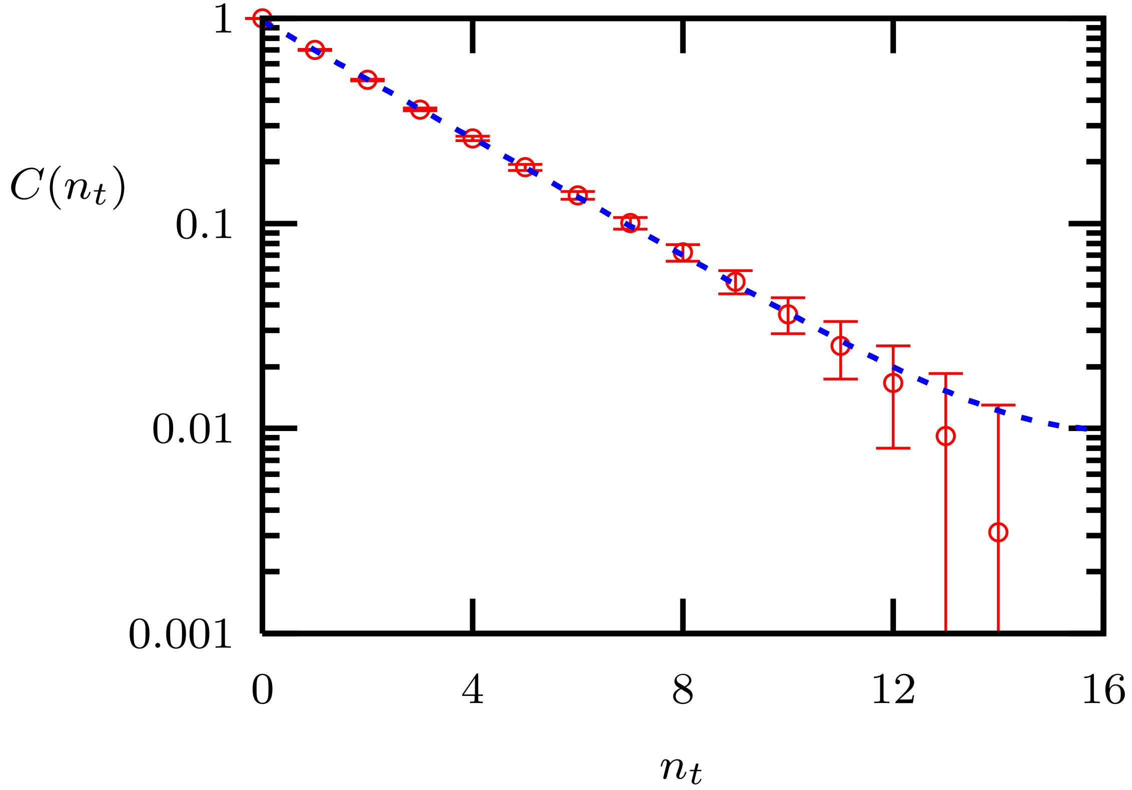
• write the Euclidean correlator of an operator $\phi(t)$:

$$\langle \phi^\dagger(t = an_t)\phi(0)\rangle = \langle \phi^\dagger e^{-Han_t}\phi\rangle = \sum_i |c_i|^2 e^{-aE_i n_t} \stackrel{t\to\infty}{=} |c|^2 e^{-man_t}$$

where am is lightest mass (in lattice units) with quantum numbers of ϕ. In particular, take $\vec{p} = 0$, colour singlet, and some particular J^{PC}.

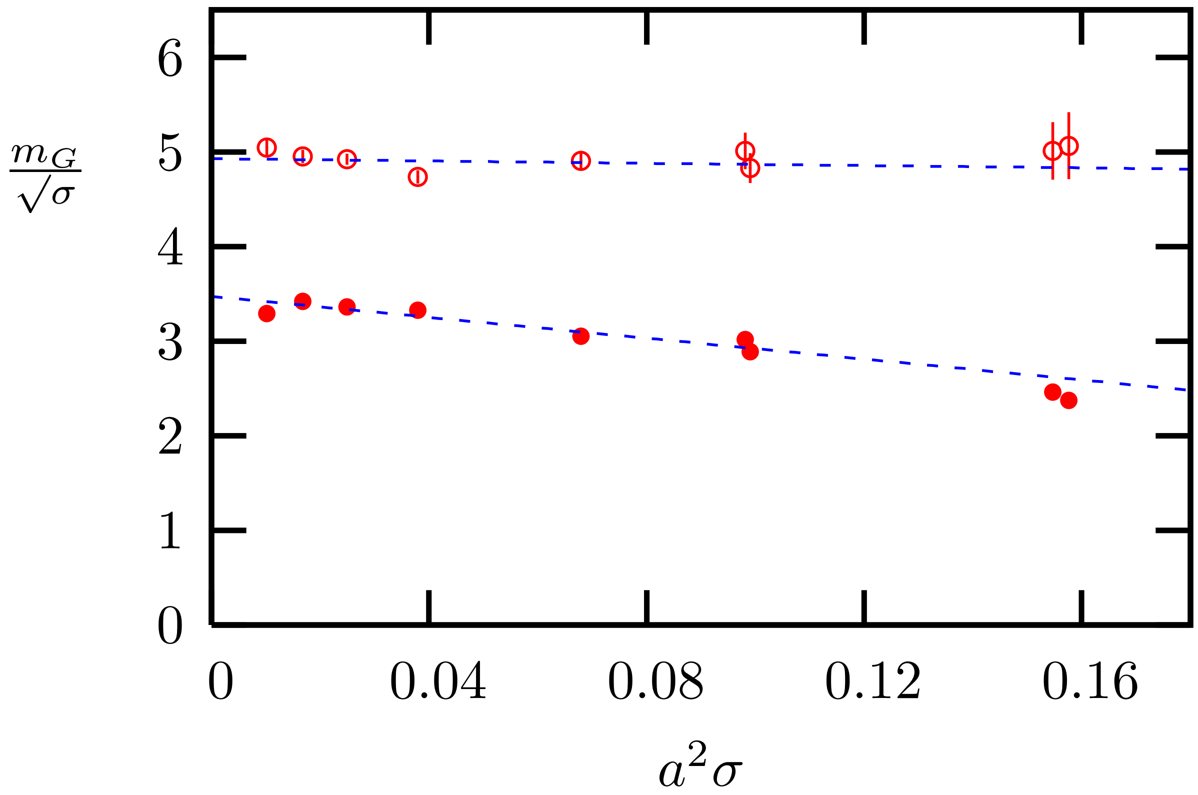
• in a numerical calculation, with finite errors, we need to be able to calculate am at small t before the 'signal' has become too small, so that we have *significant* evidence for the exponential $\propto e^{-man_t}$ over some range of n_t – i.e. we need (normalised) $|c|^2 \simeq 1$ $\leftrightarrow$ ϕ is a good wavefunctional for the desired ground state

SU(3), 32^4, $\beta = 6.515$ best blocked/smeared glueball operator



$C(t = an_t) \overset{t\uparrow}{\simeq} |c|^2 e^{-man_t} \quad \Rightarrow \quad$ fit : $am_{0^{++}} = 0.330(7)$ with $|c|^2 \simeq 0.97$

SU(3) continuum limit: B.Lucini et al: hep-lat/0404008



(●) $\frac{m_{0^{++}}}{\sqrt{\sigma}}$; (○) $\frac{m_{2^{++}}}{\sqrt{\sigma}}$

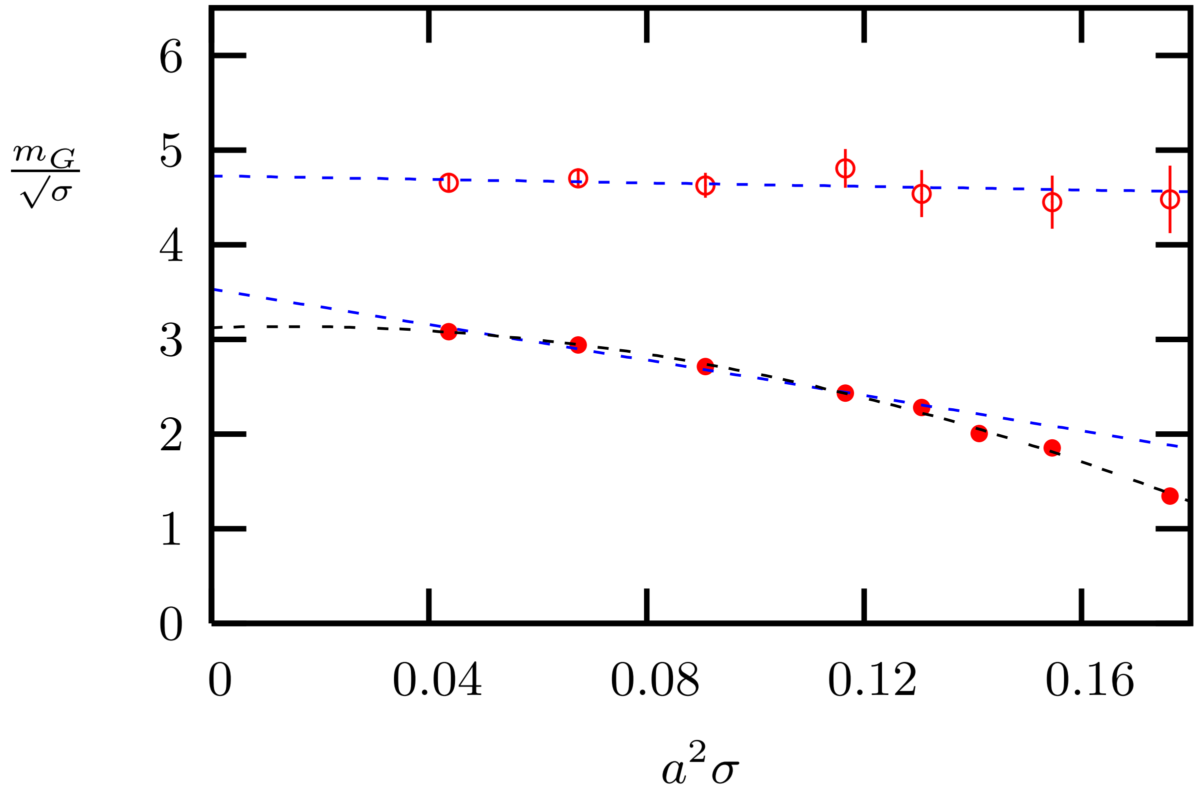
$O(a^2)$ extrapolations to $a = 0$: $\frac{m_{0^{++}}}{\sqrt{\sigma}} = 3.47(4) - 5.52(75)a^2\sigma$

$\longrightarrow$

$$m_{0^{++}} \simeq 3.5\sqrt{\sigma} \simeq 1.6 \text{ GeV}$$

which fits in with the three observed $J^{PC} = 0^{++}$ flavour 'singlet' states $f_0(1350), f_0(1500), f_0(1700)$ coming from mixing of nearby $u\overline{u} + d\overline{d}$, $s\overline{s}$ and glueball states

SU(8) continuum limit



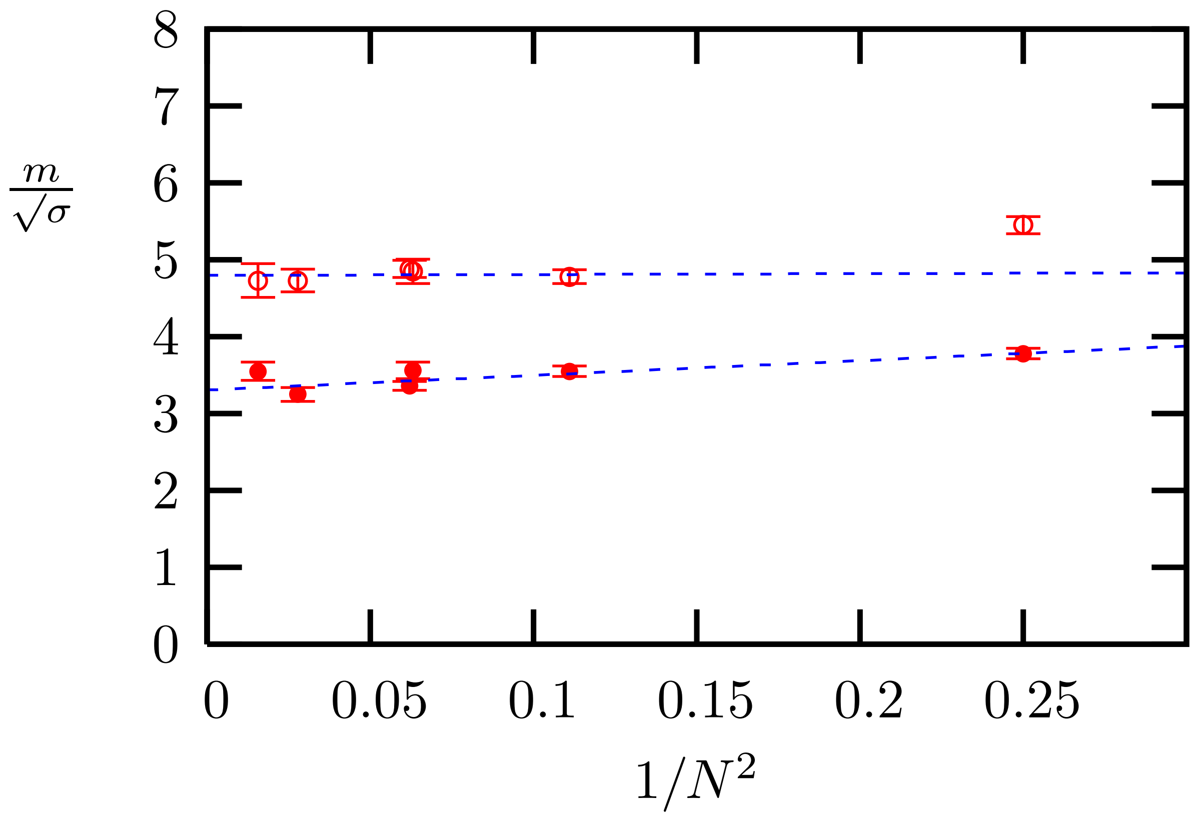
$O(a^2)$ extrapolation: $\frac{m_{0^{++}}}{\sqrt{\sigma}} = 3.53(8) - 9.3(1.0)a^2\sigma$

$O(a^4)$ extrapolation: $\frac{m_{0^{++}}}{\sqrt{\sigma}} = 3.13(25) + 1.66a^2\sigma - 66.0(a^2\sigma)^2$

but beware bias ‘nearby’ bulk critical point!

Glueball mass spectrum: large-N limit

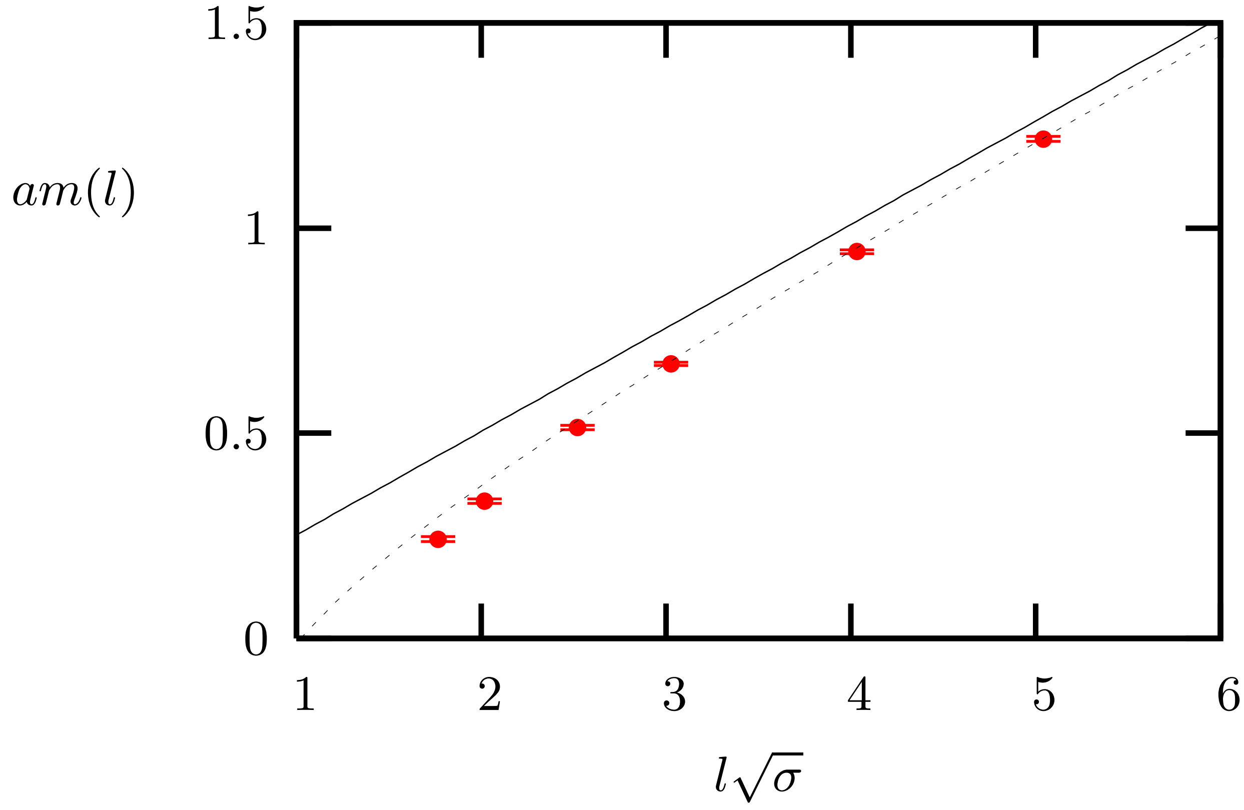
B.Lucini, M.Teper, U.Wenger: hep-lat/0404008



(•) 0^{++}; (○) 2^{++} $\longrightarrow$ SU(3) is 'close to' SU(∞) for many quantities

SU(6) : energy of flux loop closed around a spatial torus

H. Meyer, M. Teper: hep-lat/0411039



$\longrightarrow$ linear confinement: $am(l) \simeq \sigma l - \frac{\pi}{3l}$ at large N

QCD at $N = \infty$

- L. Del Debbio, B. Lucini, A. Patella and C. Pica: arXiv:0712:3036.
- G. Bali and F. Bursa, arXiv:0806:2278; arXiv:0708:3427.
- A. Hietanen, R. Narayanan, R. Patel and C. Prays: arXiv:0901:3752.

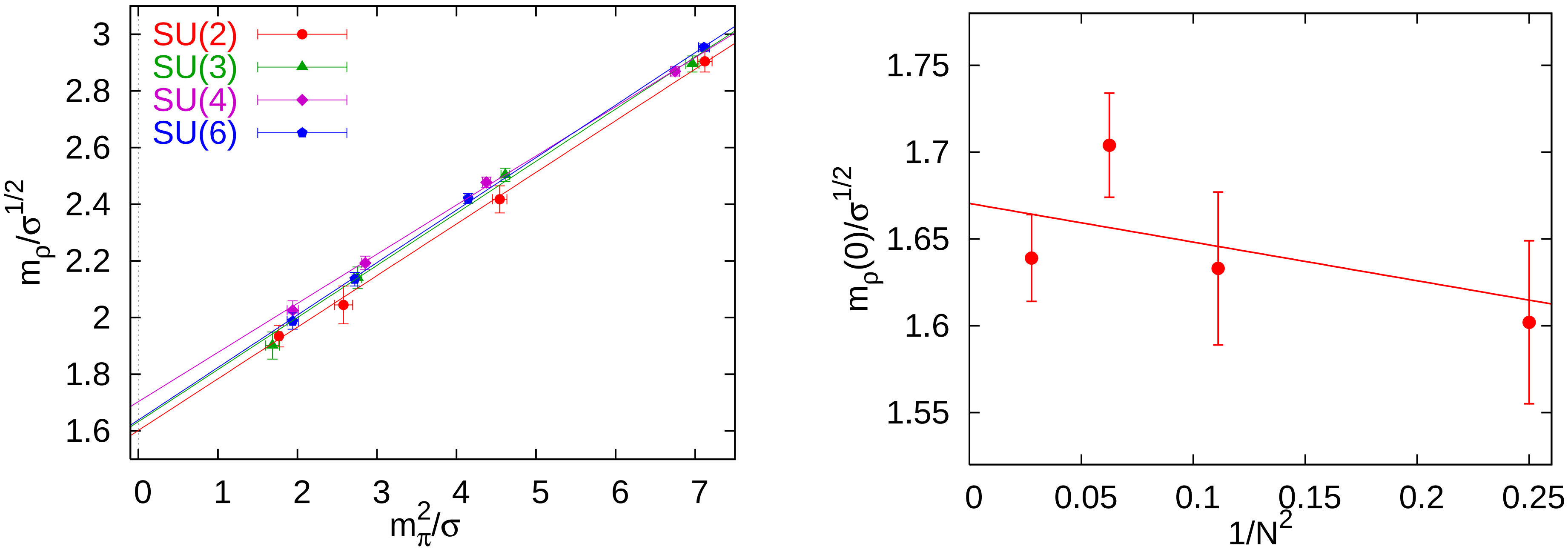
Strategy (ideal):

$$\text{quenched } QCD_N \stackrel{N\to\infty}{\longrightarrow} \text{full } QCD_{N=\infty}$$

- perform quenched QCD calculations at various N, a and m_q
- extrapolate at fixed a and m to $N = \infty$, with $O(1/N^2)$ corrections
- at each a do conventional (full QCD) chiral extrapolation in m_q
- extrapolate to continuum limit

- now compare to full QCD (or expt!) with SU(3)

G. Bali and F. Bursa, arXiv:0806:2278



m_ρ versus m_π (left); m_ρ for $m_q = 0$ versus $1/N^2$ (right).

Del Debbio et al: $\lim_{N\to\infty} \frac{m_\rho}{\sqrt{\sigma}} = 1.627(10) \quad ; \qquad a\sqrt{\sigma} = 0.335$

$+$

Bali and Bursa: $\lim_{N\to\infty} \frac{m_\rho}{\sqrt{\sigma}} = 1.688(25) \quad ; \qquad a\sqrt{\sigma} = 0.209$

$\longrightarrow$

$$\lim_{N\to\infty, a\to 0} \frac{m_\rho}{\sqrt{\sigma}} = 1.79(5)$$

versus, in the real world :

$$\frac{m_\rho}{\sqrt{\sigma}} \simeq \frac{770\mathrm{MeV}}{440\mathrm{MeV}} \simeq 1.75$$

$\longrightarrow$

$N = 3$ is 'close to' $N = \infty$ for full QCD ...

BUT:

using small V, but $N = 17, 19$ large enough for finite-V corrections to be (presumably) small, Narayanan et al find

$$m_\rho \simeq 5.86 T_c \simeq 3.5\sqrt{\sigma} \simeq 2 m_\rho^{QCD} \quad !!$$

Note:

- the value of a is roughly the same as Bali and Bursa if expressed in units of T_c or $\sqrt{\sigma}$, so it is not a corrections
- the $O(1/N^2)$ corrections observed for $N \in [2, 6]$ are weak, so it is not N corrections either

$\Longrightarrow$ What is going on?

My bet is that Hietanen are getting a large admixture of excited states ... but this needs urgent resolution.

$g^2 N$ fixed as $N \to \infty$?

What does this mean?

- D=2+1

Recall:

$$g^2 = [m] \quad \Longrightarrow \quad \beta = \frac{2N}{ag^2}$$

so:

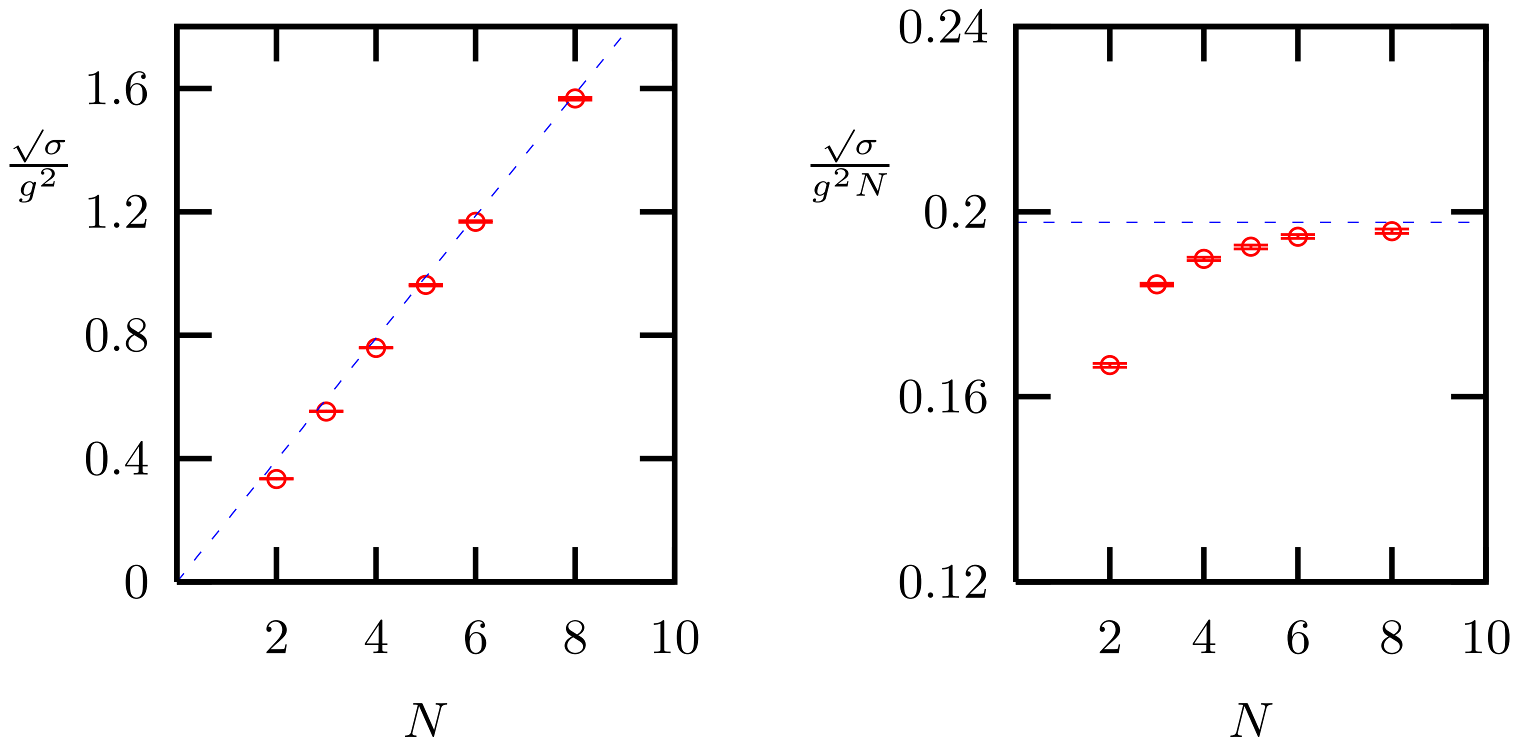
$$\beta a \sqrt{\sigma} = \frac{2N}{ag^2} a \sqrt{\sigma} \stackrel{a \to 0}{\longrightarrow} 2N \frac{\sqrt{\sigma}}{g^2}$$

and the question then is:

$$\frac{\sqrt{\sigma}}{g^2} \propto N \quad ?$$

$D = 2 + 1$

MT, hep-lat9804008; B. Bringoltz, MT, hep-th/0611286



$\Longrightarrow$

- smooth physics at large N $\longrightarrow$ $g^2 N$ fixed
- $\frac{\sqrt{\sigma}}{g^2 N} = c_0 + \frac{c_1}{N^\gamma}$ $\longrightarrow$ $\gamma = 2.01 \pm 0.20$

$g^2 N$ fixed as $N \to \infty$: D=3+1 ?

- Coupling runs, so what does this mean?

- at each N define length scales l in physical units, e.g. mass gap m_G:

$$l \longrightarrow l m_G$$

- define the 't Hooft running coupling at each N:

$$\lambda_N = g^2 N$$

- then what we mean is:

$$\lambda_N(l m_G) \stackrel{N\to\infty}{\longrightarrow} \lambda_\infty(l m_G)$$

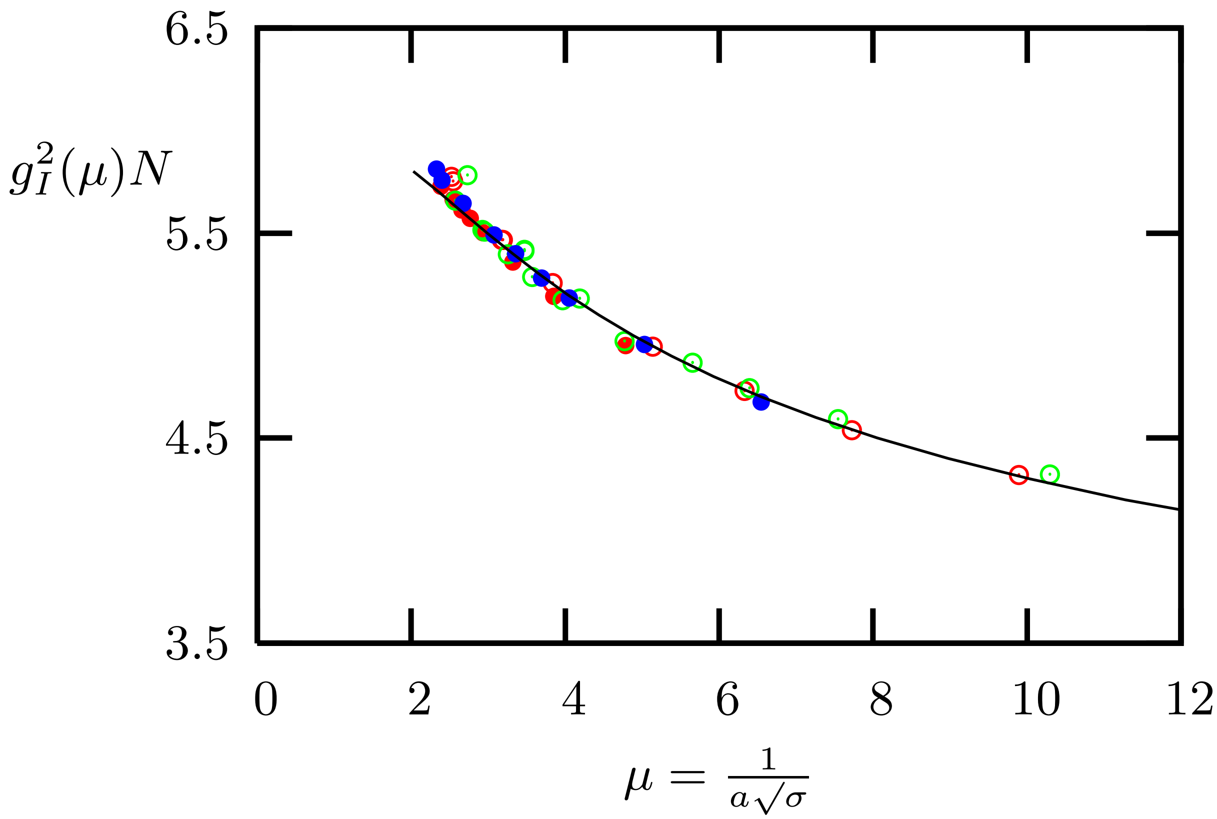
Note: on small scales, where we know the β-function, this follows if Λ_N has a smooth large-N limit.

- start with the bare lattice coupling, $g_L^2(a)$

D=3+1

MT, Lat 08 , arXiv:0812.0085; B. Lucini, MT hep-lat/0103027

bare coupling (Parisi mean field improvement): $g_I^2(a) = \frac{g_L^2(a)}{u_p} = \frac{2N}{\beta}\frac{1}{u_p}$



SU(2) ○ ; SU(3) ○; SU(4) ● ; SU(6) ○ ; SU(8) ●

Fit relationship $a\surd\sigma(a)$ to $g^2(a)$ by:

- continuum running
- lattice spacing corrections

$$
\begin{aligned}
a\sqrt{\sigma}(a) &= \text{lattice running} \times \text{3loop continuum running} \\
&= \frac{\sqrt{\sigma}(0)}{\Lambda_I}\left(1+ca^2\sigma\right) e^{-\frac{1}{2\beta_0 g_I^2}} \left(\frac{\beta_1}{\beta_0^2}+\frac{1}{\beta_0 g_I^2}\right)^{\frac{\beta_1}{2\beta_0^2}} e^{-\frac{\beta_2^I}{2\beta_0^2} g_I^2}
\end{aligned}
$$

Now from the 2-loop β-function we see that:

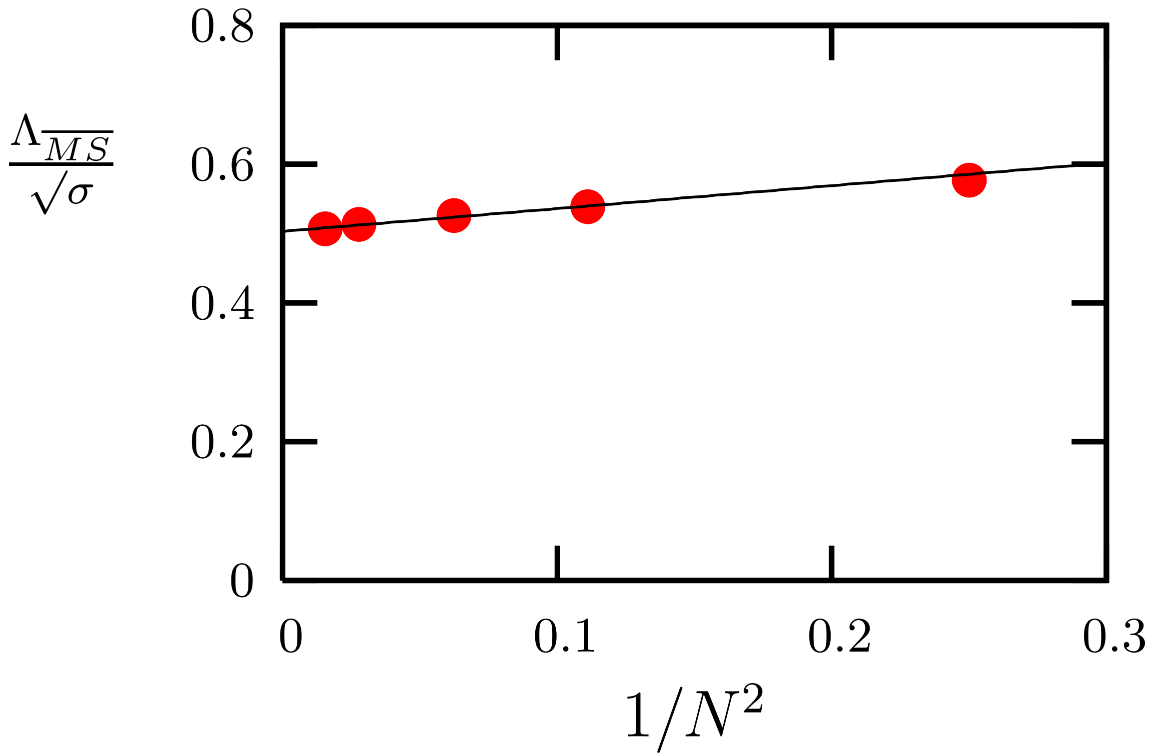
$$g^2 N = \text{constant} \quad \Leftrightarrow \quad \text{physics} = \text{ind of } N$$

$\equiv$

$$\frac{\Lambda_{\overline{MS}}}{\surd\sigma} = \text{ind of } N$$

Fitting the bare coupling at various N one finds:

C. Allton, M. Teper, A. Trivini, arXiv:0803.1092



$$\frac{\Lambda_{\overline{MS}}}{\sqrt{\sigma}} = 0.503(2)(40) + \frac{0.33(3)(3)}{N^2} \qquad \text{QED}$$

∘ So much for the cheap and dirty calculation ...

cheap $\leftrightarrow$ no extra work

dirty $\leftrightarrow$ lattice corrections

∘ clearly it would be better to calculate some coupling on a scale l at some a and then to send $a \to 0$ while keeping l fixed in 'physical units' e.g. keeping $l\sqrt{\sigma} = l/a \times a\sqrt{\sigma(a)}$ fixed:
i.e. extract a continuum running coupling from the lattice calculation.

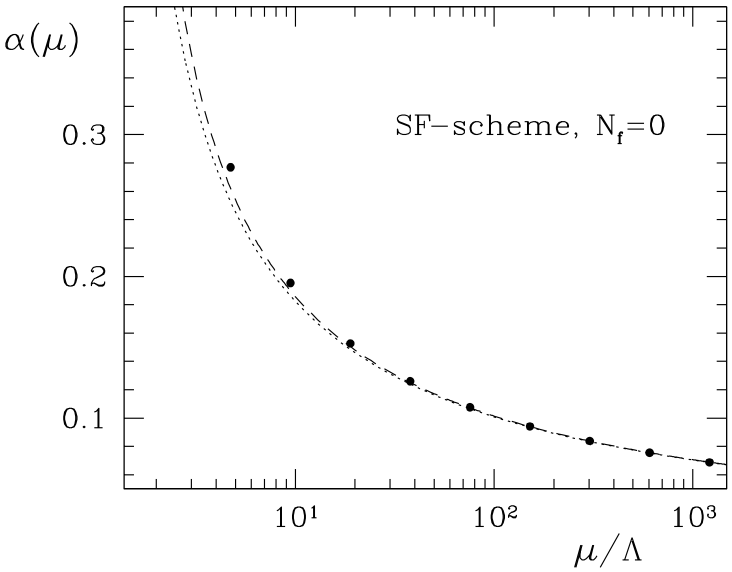
∘ a nice calculation of this kind is the

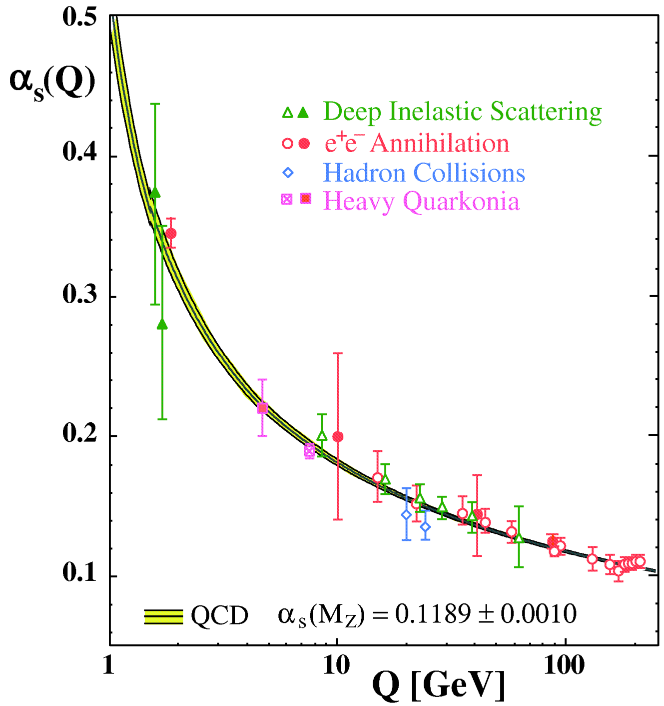
step-scaling technique + 'Schrodinger Functional' scheme

of the *Alpha* Collaboration:

continuum SF coupling in SU(3)

Alpha collaboration , hep-lat/9810063 S. Bethke, hep-ex/0606035

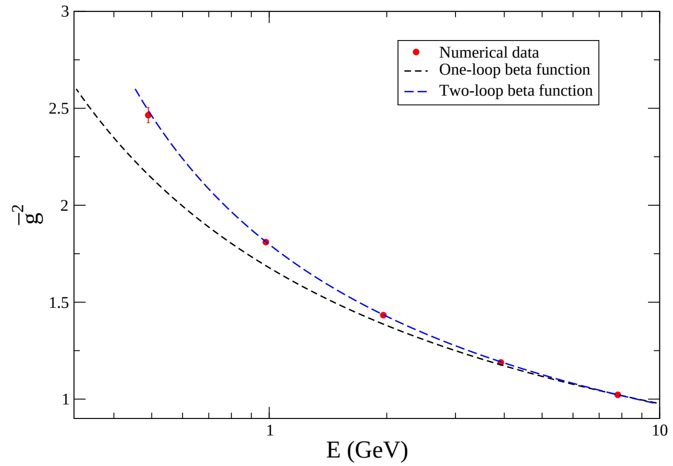


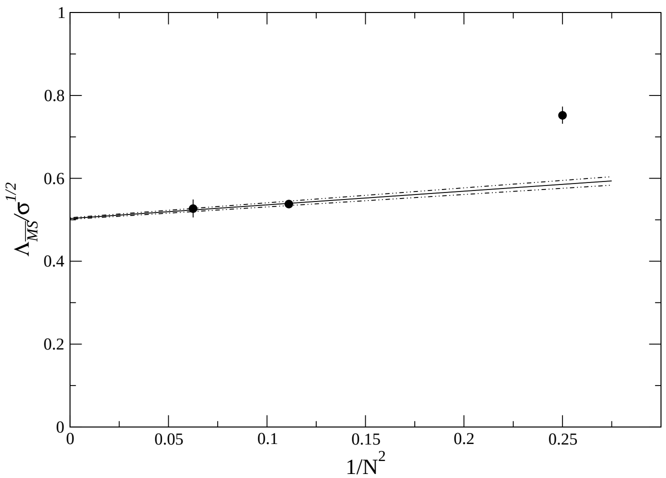


comparable range to (real) experiment ($\Lambda \sim 125\,\mathrm{MeV}$) and much more accurate!

recently for SU(4):

B. Lucini, G. Moraitis, arXiv:0805.2913, 0710.1533





new SU(4) plus older *Alpha* SU(2) and SU(3) →

$$\frac{\Lambda_{\overline{MS}}}{\sqrt{\sigma}} = 0.528(40) + \frac{0.18(36)}{N^2}$$

which is consistent with bare coupling

QUESTION: does the SF coupling acquire non-perturbative jumps at the Narayanan-Neuberger $N = \infty$ phase transitions?

Calculating as $N \to \infty$: how much harder?

• $\propto N^3$ factor coming from matrix multiplication;
partly offset by smaller finite V corrections at larger N.

• We calculate masses from connected correlators i.e. correlations between fluctuations

but

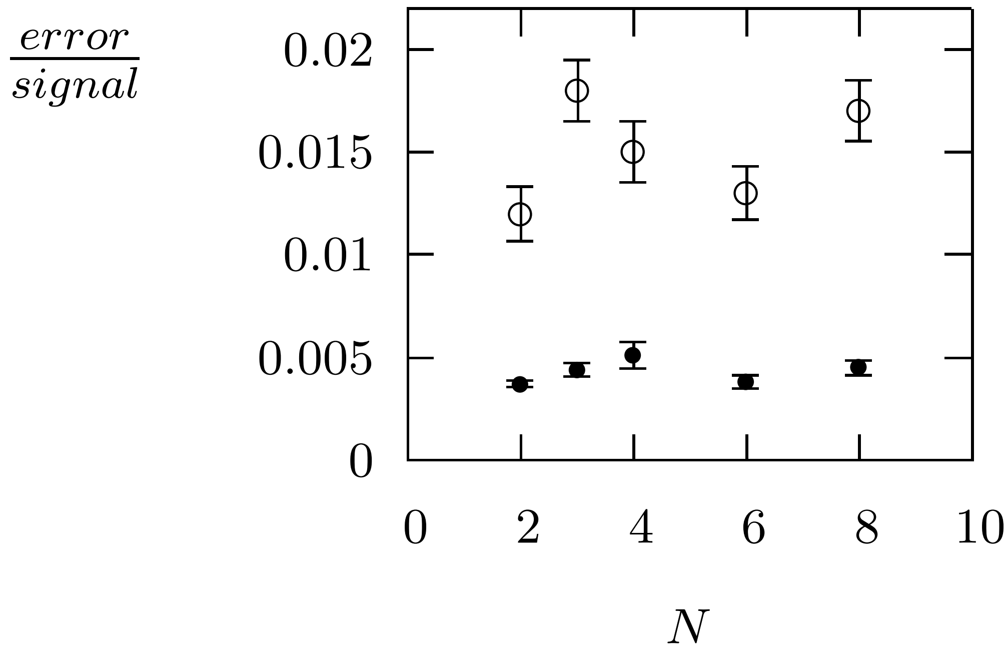
as $N \to \infty$ all fluctuations vanish

$\Rightarrow$

mass calculations become impossible as $N \to \infty$?

NO

the errors on the fluctuations are themselves determined by higher order correlators, which generically vanish at the same rate the observed ratio of error to signal for the same number of Monte Carlo field configurations is roughly independent of N



Error to signal ratio for $C_2(t)$ after 10^5 sweeps on 10^4 lattices at fixed lattice spacing, $a \simeq 1/5T_c$, and for $t = 0$ ($\bullet$) and $t = a$ ($\circ$).

• For QCD with quarks:
the most expensive part of current calculations is matrix *times* vector multiplication (e.g. in propagators) and this is $\propto N^2$;
in principle also *partly offset* by smaller finite V corrections at larger N.

• As for glueballs, we calculate masses from connected correlators

but

in this case the errors on the correlators are determined by higher order correlators, which generically vanish *not* at the same rate, but as $O(1/N)$ – which translates into an effective improvement of $\propto N^2$ in statistics

$\Rightarrow$

ideally, increasing N has no extra cost!

but

in practice things are not ideal and the cost grows as $\propto N$
Bali,Bursa

Some Conclusions

∘ large N gauge theories are linearly confining at low T

∘ moreover for many basic physical quantities, e.g. the lightest glueball masses, the deconfining temperature, the string tension, one finds

$$SU(3) \simeq SU(\infty)$$

and if the Bursa ... DelDebbio calculation is right

$$QCD_{N=3} \simeq QCD_{\infty}$$

∘ increasing N is surprisingly inexpensive ... particularly for fermions

$\Longrightarrow$

the hadron spectrum of QCD_{∞} will be beautiful and full of phenomenologically useful insights - worth doing!

What is the effective string action that describes confining flux tubes?

$\longrightarrow$

What string theory describes SU(∞) gauge theories ?

- Veneziano amplitude
- 't Hooft large-N – genus diagram expansion
- Polyakov action
- Maldacena ... AdS/CFT/QCD ...

Consider a confining flux tube that closes upon itself by winding once around a spatial torus of length l

D=2+1 and D=3+1

- Let $E_n(l)$ be the energy spectrum of the flux tube
- Confining flux is localised in 'tubes' $\forall l \geq l_c = 1/T_c$

– at $l = l_c$ there is a phase transition to a small-l 'deconfined' phase

- Q : what is the effective string theory describing low-energy modes of a very long flux tube, i.e. $E_n(l) - E_0(l) \ll \sqrt{\sigma} \quad ; \quad l \gg 1/\sqrt{\sigma}$?
- Q : is there an effective string theory describing the spectrum all the way down to $l \sim O(l_c)$ where the energy gaps typically become large?

all this is most plausible at $N \to \infty$ where complications such as decays and mixing, e.g string $\to$ string + glueball go away

Note:

All the numerical results you shall see are for closed flux tubes:

- spectrum of fundamental flux tubes in D=2+1

B. Bringoltz, M. Teper : hep-th/0611286.
A. Athenodorou, B. Bringoltz, M. Teper : arXiv:0709.0693

- spectrum of $k = 2$ flux tubes in D=2+1

B. Bringoltz, M. Teper : arXiv:0802.1490 A. Athenodorou, B. Bringoltz, M. Teper : arXiv:0812.0334

- spectrum of fundamental flux tubes in D=3+1

A. Athenodorou, B. Bringoltz, M. Teper : Lat09 contribution and in preparation

There has also been a great deal of work on open flux tubes (potentials) and on (ratios) of Wilson loops and on ... which I will not talk about here.

Ground state energy

from smeared Polyakov loops: $\langle l_p^\dagger(t) l_p(0) \rangle \overset{t\to\infty}{\propto} \exp\{-m_p(l)t\}$

$\Longrightarrow$ we expect

$$m_p(l) \overset{l\to\infty}{=} \sigma l - \frac{\pi(D-2)}{6l} + O\left(\frac{1}{l^3}\right)$$

∘ linear confinement $\Rightarrow$ σl

∘ spontaneous breaking translation invariance $\Rightarrow$ $-\frac{\pi(D-2)}{6l}$
from the sum of zero-point energies of the massless (Goldstone) transverse modes – the 'universal' Luscher correction

∘ any other massless modes on flux tube $\Rightarrow$ further $O(1/l)$ contributions

$\Longrightarrow$ determine the coefficient of the $O(1/l)$ Luscher correction in order to determine whether a long flux tube is described by an effective bosonic string theory

Nambu-Goto in flat space-time :

– simplest example of a bosonic string theory

– a free string theory and 'sick' outside $D = 26$

but

diseases invisible in sector of states built on a single long string

e.g. P. Olesen, PLB160 (1985) 144; J. Polchinski, A. Strominger, PRL67 (1991) 1681.

$\Longrightarrow$

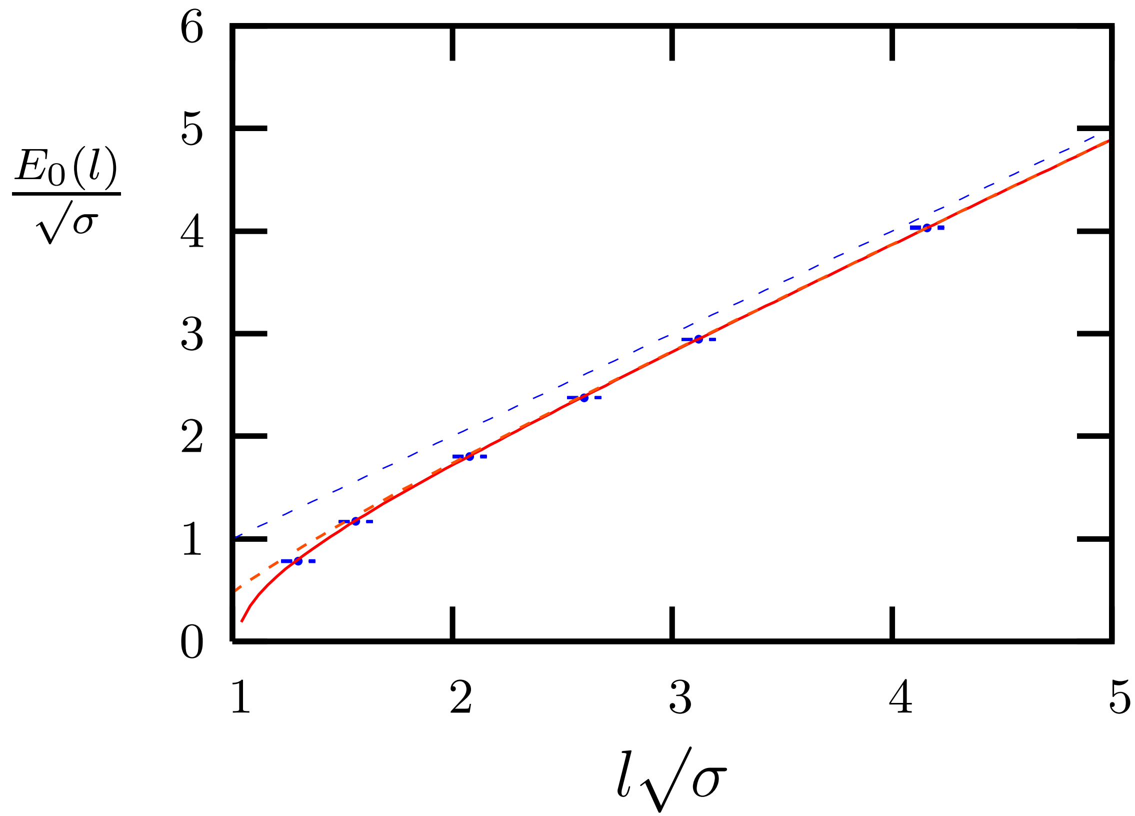
the ground state energy: J. Arvis, PLB 127 (1983) 127

$$E_0(l) = \sigma l \left(1 - \frac{\pi(D-2)}{3\sigma l^2}\right)^{\frac{1}{2}} \stackrel{l\to\infty}{=} \sigma l - \frac{\pi(D-2)}{6l} + O\left(\frac{1}{l^3}\right)$$

Recall: flux tubes are free at $N = \infty$

Ground state energy

D=2+1 SU(5) : $a\sqrt{\sigma} \simeq 0.130, \quad l_c\sqrt{\sigma} \simeq 1.07$



...Luscher: $E_0(l) = \sigma l - \frac{\pi}{6l}$; —Nambu-Goto: $E_0(l) = \sigma l \left(1 - \frac{\pi}{3\sigma l^2}\right)^{\frac{1}{2}}$

bosonic universality class?

fit to each pair of neighbouring values of l

- effective Luscher correction

$$E_0(l) = \sigma_{\rm eff} l - c_{\rm eff} \frac{\pi(D-2)}{6l}$$

- effective Nambu-Goto

$$E_0(l) = \sigma_{\rm eff} l \left(1 - c_{\rm eff} \frac{\pi}{3\sigma_{\rm eff} l^2}\right)^{\frac{1}{2}}$$

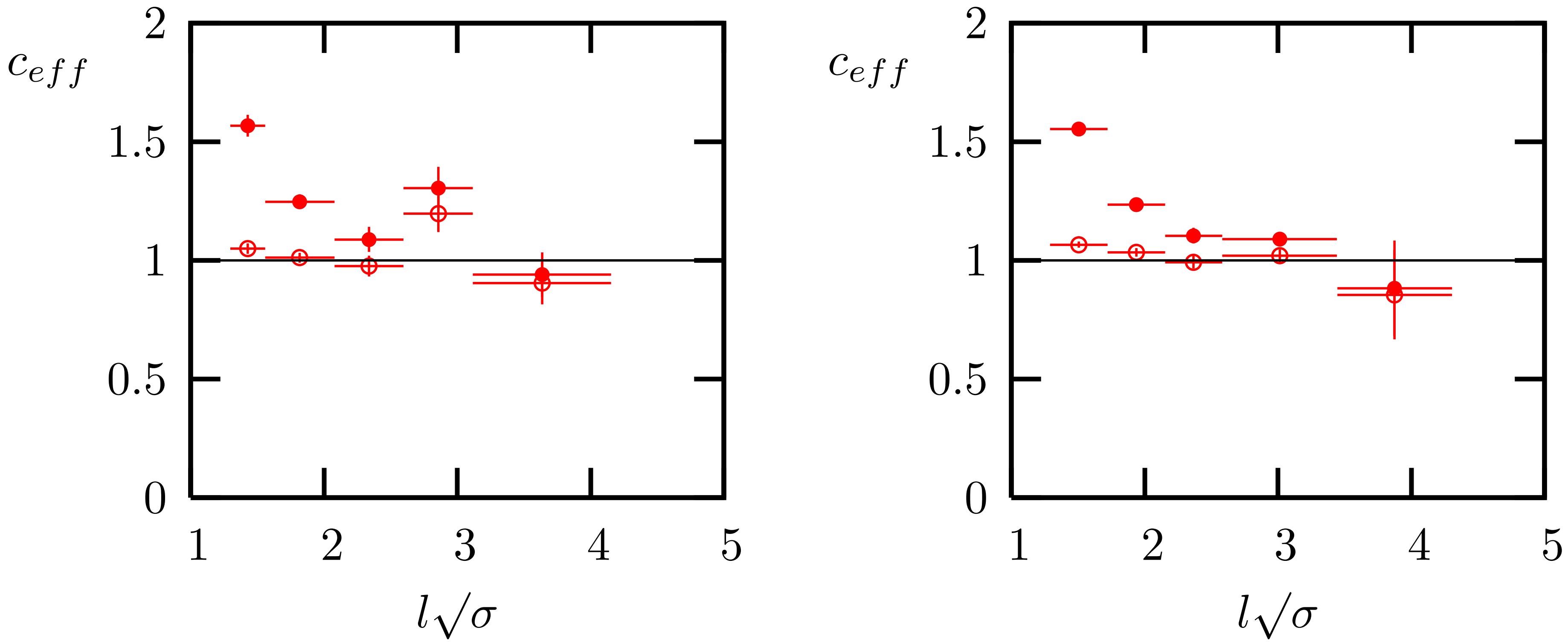
then if

$$c_{eff}(l) \stackrel{l\to\infty}{\longrightarrow} c = 1$$

the effective string action belongs to the universality class of a simple bosonic where the only massless modes of the string are the transverse translations

if $c \neq 1$ then there must be other massless modes on the string, e.g. $c = 1,\ 1.5,\ 0$ for bosonic, Neveu-Schwartz, Ramond strings respectively

D=2+1 : SU(5) : $l_c\sqrt{\sigma} \simeq 1.07$ SU(4) : $l_c\sqrt{\sigma} \simeq 1.08$



c_{eff} : from Luscher ● , and from Nambu-Goto ○

$\Rightarrow$

- Luscher fit:

$$c_{eff}(l \to \infty) \to 1$$

i.e. universality class of simple bosonic string

but $c_{eff}(l)$ far from unity at smaller l $\longleftrightarrow$large higher order corrections to Luscher term

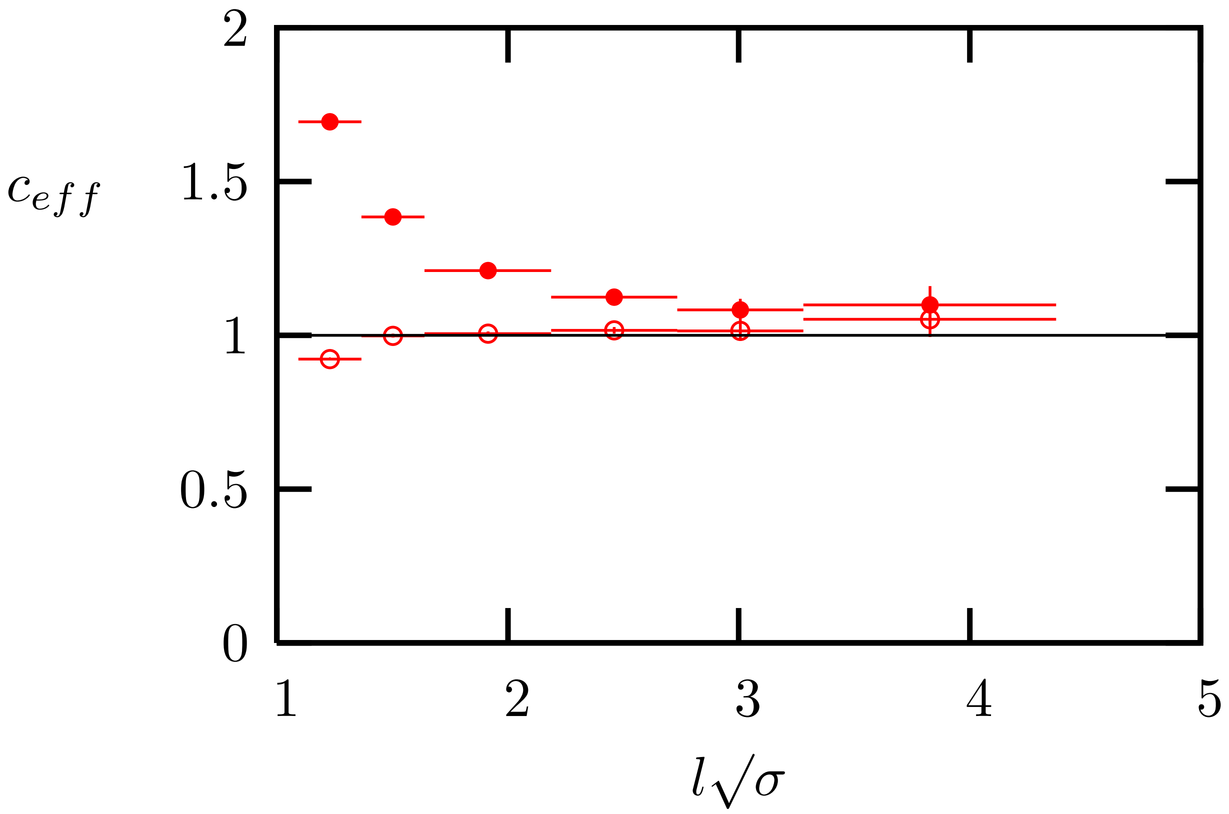
- Nambu-Goto fit:

$$c_{eff}^{NG}(l) \simeq 1 \quad \forall l \geq l_c$$

so i.e. flux tube behaves like an ideal free bosonic string, even when it is hardly longer than it is wide

$\Rightarrow$ is this a large N effect?

small $N \longrightarrow$ SU(2) : $l_c\sqrt{\sigma} \simeq 0.95$



c_{eff} : from Luscher ● , and from Nambu-Goto ○

Flux tubes as strings

∘ 4-torus , Euclidean time extent τ;

∘ sources at $\vec{x} = (0, y, z)$, $\vec{x} = (r, y, z)$;

∘ partition function with sources

$$Z_{s\bar{s}}(r,\tau) = \int dA\, l^\dagger(x=0,y,z) l(x=0,y,z) \exp\{-S[A]\}$$

where $l(\vec{x})$ is a Polyakov loop encircling the t-torus.

∘ $E_n(r)$ = energy eigenstate of two sources separated by r

$\tilde{E}_n(p,\tau)$ = energy eigenstate of flux tube with transverse momentum p and winding around spatial torus of length τ;

then

$$\frac{1}{Z} Z_{s\bar{s}}(r,\tau) = \sum_n e^{-E_n(r)\tau} = \sum_{n,p} c_n(p,\tau) e^{-\tilde{E}_n(p,\tau) r}$$

$\longrightarrow$ open-closed string duality

∘ the flux tube between the sources sweeps out a surface S bounded by the cylinder $r \times \tau$;

∘ if $S_{eff}(S)$ is the effective string action, then:

$$\begin{aligned} Z_{cyl}(r,\tau) &= \int_{cyl=r\times\tau} dS e^{-S_{eff}[S]} = \frac{1}{Z} Z_{s\bar{s}}(r,\tau) \\ &= \sum_n e^{-E_n(r)\tau} = \sum_{n,p} c_n(p,\tau) e^{-\tilde{E}_n(p,\tau)r} \end{aligned}$$

this duality has long been known, but in 2004 Luscher and Weisz : hep-th/0406205 realised that it provided a powerful contraint on the form of S_{eff};

∘ this year Aharony and Karzbrun : arXiv:0903.1927 added constraints from closed-closed string duality on a torus, $T^2 = r \times \tau$,

$$Z_{torus}(r,\tau) = \int_{T^2=r\times\tau} dS e^{-S_{eff}[S]} = \sum_{n,p} e^{-\tilde{E}_n(p,\tau)r} = \sum_{n,p} e^{-\tilde{E}_n(p,r)\tau}$$

Aside

there have been two main approaches to $S_{eff}(S)$:

• static gauge approach

M. Luscher, K. Symanzik, P. Weisz : Nucl. Phys. B173 (1980) 365;

M. Luscher : Nucl. Phys. B180 (1981) 317;

M. Luscher, P. Weisz : hep-th/0406205;

O. Aharony, E. Karzbrun : arXiv:0903.1927.

• covariant approach

J. Drummond : hep-th/0411017;

N. Hari Dass, P. Matlock : hep-th/0612291;

N. Hari Dass et al : arXiv:0910.5615; arXiv:0910.5620; arXiv:0911.3236.

I will describe the former approach, but will summarise both sets of results.

‘static gauge’ strategy

∘minimal surface S_{min}: $x \in [0,r]$ and $t \in [0,\tau]$

∘general surface S : given by displacement vector $h(x,t)$ from S_{min}

∘write S_{eff} in terms of this field h; schematically

$$S_{eff}[S] \longrightarrow S_{eff}[h]$$

∘$S_{eff}[h]$ cannot depend on h, only ∂h - Goldstone mode - so can do derivative expansion for long modes; schematically

$$S_{eff} = \sigma r \tau + \int_0^\tau dt \int_0^r dx \frac{1}{2}\partial h \partial h + \sum_{n=2} c_n \int_0^\tau dt \int_0^r dx (\partial h)^{2n}$$

∘for long wavelength modes it is clear that

$$\int_0^\tau dt \int_0^r dx (\partial h)^{2n} \longrightarrow \left(\frac{1}{\sigma l^2}\right)^{(n-1)}$$

in the energies

∘ (massless) oscillation of periodic string of length l

$$\lambda = \frac{l}{k} \longrightarrow p = \frac{\pi k}{l} = \text{energy}$$

∘ quantisation ⇒ phonons along a string, with creation operators a_k and a_{-k} that arise from the fourier decompostion of $h(x,t)$.

∘ since $p = \pi k/l$ we expect for a long string that the low-lying stringy excitations be given by

$$E_n(l) \stackrel{l\to\infty}{=} E_0(l) + O\left(\frac{\pi}{l}\right)$$

∘ but because the flux tube has a finite width, $\sim 1/\sqrt{\sigma}$, there should also be massive excitations leading to non-stringy excitations

$$E_n(l) = E_0(l) + O(\sqrt{\sigma})$$

a pause for reflection

• integrate over single sheet spanning the cylinder
$\longleftrightarrow$ single flux tube propagating around cylinder
$\longleftrightarrow$ no mixing (= handles) or decays
$\Rightarrow$ can only be valid in $N \to \infty$ limit

• parameterisation of a string in terms of $h(x, t)$ is not general – a string with an overhang requires a multivalued field h which we do not allow
$\Rightarrow$ can only be valid for smooth, low energy fluctuations of the string

• derivative expansion is probably only asymptotic with corrections $O(e^{-c\sigma l^2})$, and will only be valid if $1/\sigma l^2 \ll 1$, i.e. for modes

$$E_n(r) - E_0(r) \ll \sqrt{\sigma}$$

the Gaussian approximation

first non-trivial term in our effective string action:

$$S_{eff} = \sigma r\tau + \int_0^\tau dt \int_0^r dx \frac{1}{2}\partial_\alpha h \partial_\alpha h$$

neglecting an $O(\tau)$ boundary/self-energy piece

$\Longrightarrow$

$$Z_{cyl}(r,\tau) = e^{-\sigma r\tau}|\eta(q)|^{-(D-2)} \quad : \quad q = e^{-\pi\tau/r}$$

where η is the Dedekind function

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n).$$

so, matching this to

$$Z_{cyl}(r,\tau) = \int_{cyl=r\times\tau} dS e^{-S_{eff}[S]} = \frac{1}{Z} Z_{s\bar{s}}(r,\tau) = \sum_n \omega_n e^{-E_n(r)\tau}$$

$\Longrightarrow$

$$E_n(r) = \sigma r + \frac{\pi}{r}\left\{n - \frac{1}{24}(D-2)\right\}$$

is an *exact* result for the level spectrum of open strings.

Now, the Dedekind eta function possesses a well-known modular invariance:

$$\eta(q) = \left(\frac{2r}{\tau}\right)^{\frac{1}{2}} \eta(\tilde{q}) \qquad ; \quad \tilde{q} = e^{-4\pi r/\tau}.$$

but if we try to match to

$$Z_{cyl}(r, \tau) = \sum_{n,p} c_n(p, \tau) e^{-\tilde{E}_n(p,\tau) r}$$

we find that only an approximate match is possible:

$$\tilde{E}_n(\tau) = \sigma\tau + \frac{4\pi}{\tau}\left\{n - \frac{1}{24}(D-2)\right\} + O\left(\frac{1}{\tau^2}\right)$$

i.e. the Gaussian approximation does not possess exact open-closed string duality.

So: the Gaussian approximation produces the correct $O(1/l)$ Luscher correction to the ground and excited states.

S_{eff} at order $(\partial h)^4$

M. Luscher, P. Weisz : hep-th/0406205

open-closed string duality constraints

$\Rightarrow$

$$E_n = \sigma l + \frac{4\pi}{l}\left(\tilde{n} - \frac{D-2}{24}\right) - \frac{8\pi^2}{\sigma l^3}\left(\tilde{n} - \frac{D-2}{24}\right)^2 + O(l^{-4})$$

and the $O(1/l^3)$ correction is also universal

simultaneously using PS covariant approach

J. Drummond : hep-th/0411017, later N. Hari Dass, P. Matlock : hep-th/0612291

for D=2+1 (LW,JD); for D=3+1 (JD)

S_{eff} at order $(\partial h)^6$

O. Aharony, E. Karzbrun : arXiv:0903.1927

add torus closed-closed string duality constraints

$\Rightarrow$

the $O(1/l^5)$ correction is also universal in D=2+1 and partially so in D=3+1

AND

in a class of confining theories with a gauge-gravity duals

$\Rightarrow$

in $D = 2 + 1$ and $D = 3 + 1$ the $O(1/l^5)$ term is universal

$\Rightarrow$

there may indeed be extra constraints not captured in the effective field theory approach.

... and very recently using PS covariant approach

N. Hari Dass et al, arXiv:0910.5615, 0910.5620, 0911.3236

$\Rightarrow$ universal to all orders?

Nambu-Goto free string theory

$$\int \mathcal{D}X e^{-\frac{i}{\sigma}\times \mathrm{Area}}$$

spectrum:

$$E_n^2(l;q) = (\sigma\, l)^2 + 8\pi\sigma\left(\frac{N_L+N_R}{2} - \frac{D-2}{24}\right) + \left(\frac{2\pi q}{l}\right)^2 .$$

$2\pi q/l =$ total momentum along string;
$N_L, N_R =$ sum left and right oscillators ('phonons');
state $= \prod_k a_k^{n_k} \prod_{k'} \tilde{a}_{,}^{n_{k'}} k|0\rangle$

$$N_L = \sum_{k>0} n_L(k)\, k, \quad N_R = \sum_{k'>0} n_R(k')\, k', \quad N_L - N_R = q$$

J. Arvis, Phys. Lett. 127B(1983)106

- Also: open-closed string duality is *exact* (LW)

$\Longrightarrow$ any universal terms of S_{eff} are precisely those of Nambu-Goto expanded to the same order.

Strings in D=2+1 : quantum numbers

○ length of string, l

○ non-zero momentum $p = 2\pi q/l$ along string

$\longrightarrow$ requires a deformation along the string

$\longrightarrow$ so need non-trivial phonon excitation: $q = N_L - N_R$

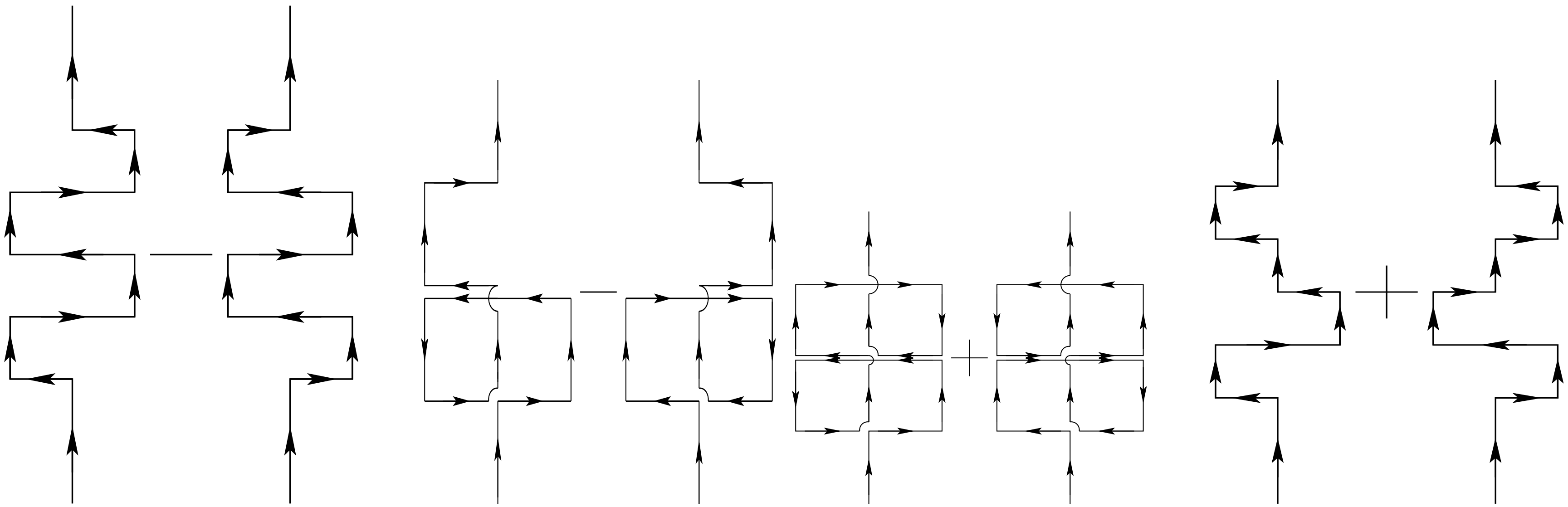
○ parity: $h(x) \rightarrow -h(x) \quad \leftrightarrow \quad a_k \rightarrow -a_k, \tilde{a}_k \rightarrow -\tilde{a}_k$

$\longrightarrow P = (-1)^{number\, of\, phonons}$

○ no rotations (2 space dimensions); transverse momentum uninteresting; $C = \pm$ sectors degenerate, so charge conjugation uninteresting.

Excited States

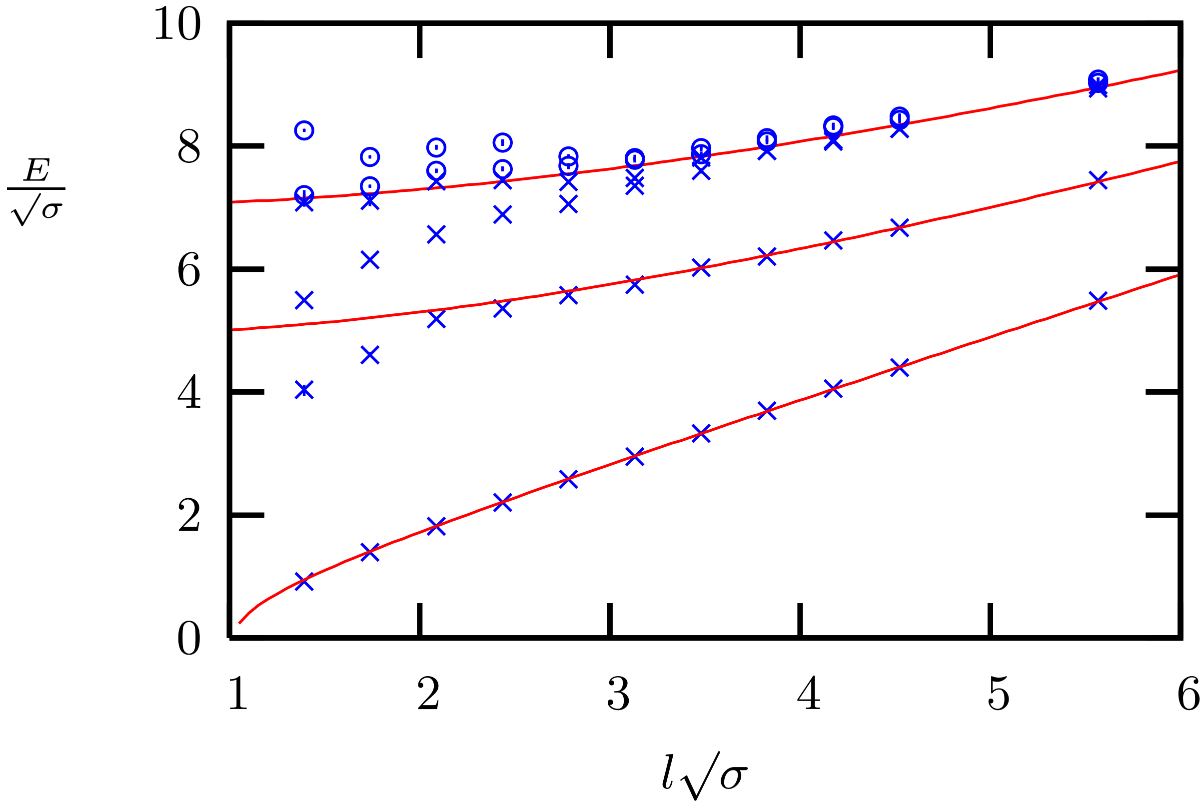
to have good overlaps onto excited string states, we need to include many more operators in our variational basis – in particular operators that 'look' excited and ones that have an intrinsic handedness so that we can construct $P = -$ as well as $P = +$, e.g.



typically we have 100-200 operators in our basis ...

SU(3) : $q = 0$ closed string spectrum

$a\sqrt{\sigma} = 0.17395(7) \quad ; \quad l_c\sqrt{\sigma} \simeq 1.0$



— : Nambu-Goto : σ from ground state

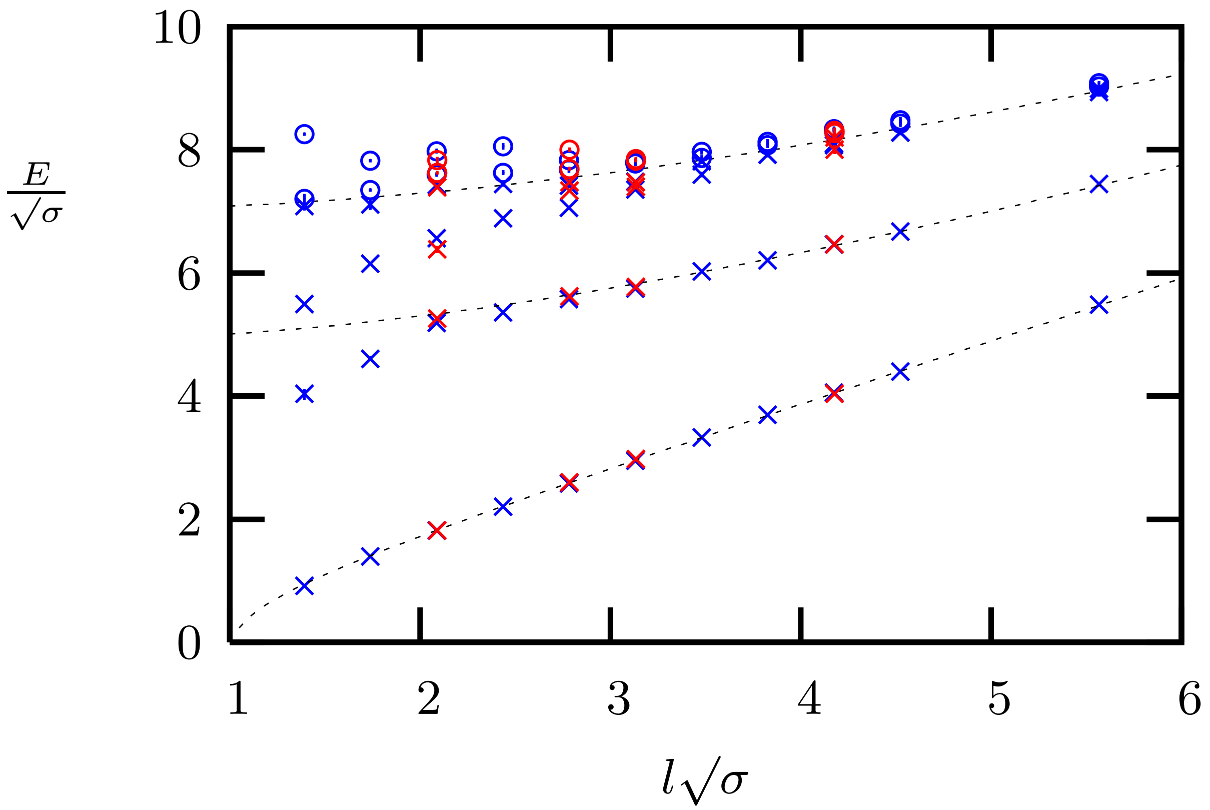
$\times$: +ve parity $\circ$: -ve parity content of lightest $q=0$ NG states:

$\vert 0\rangle$	P=+, q=0
$a^R(k=1)a^L(k=1)\vert 0\rangle$	P=+, q=0
$a^R(k=2)a^L(k=2)\vert 0\rangle$	P=+, q=0
$a^R(k=1)a^R(k=1)a^L(k=1)a^L(k=1)\vert 0\rangle$	P=+, q=0
$a^R(k=2)a^L(k=1)a^L(k=1)\vert 0\rangle$	P=−, q=0
$a^R(k=1)a^R(k=1)a^L(k=2)\vert 0\rangle$	P=−, q=0

Since our lightest states have energies and degeneracies as in Nambu-Goto down to $l\sqrt{\sigma} \sim 2$, they are well-described by the above states.

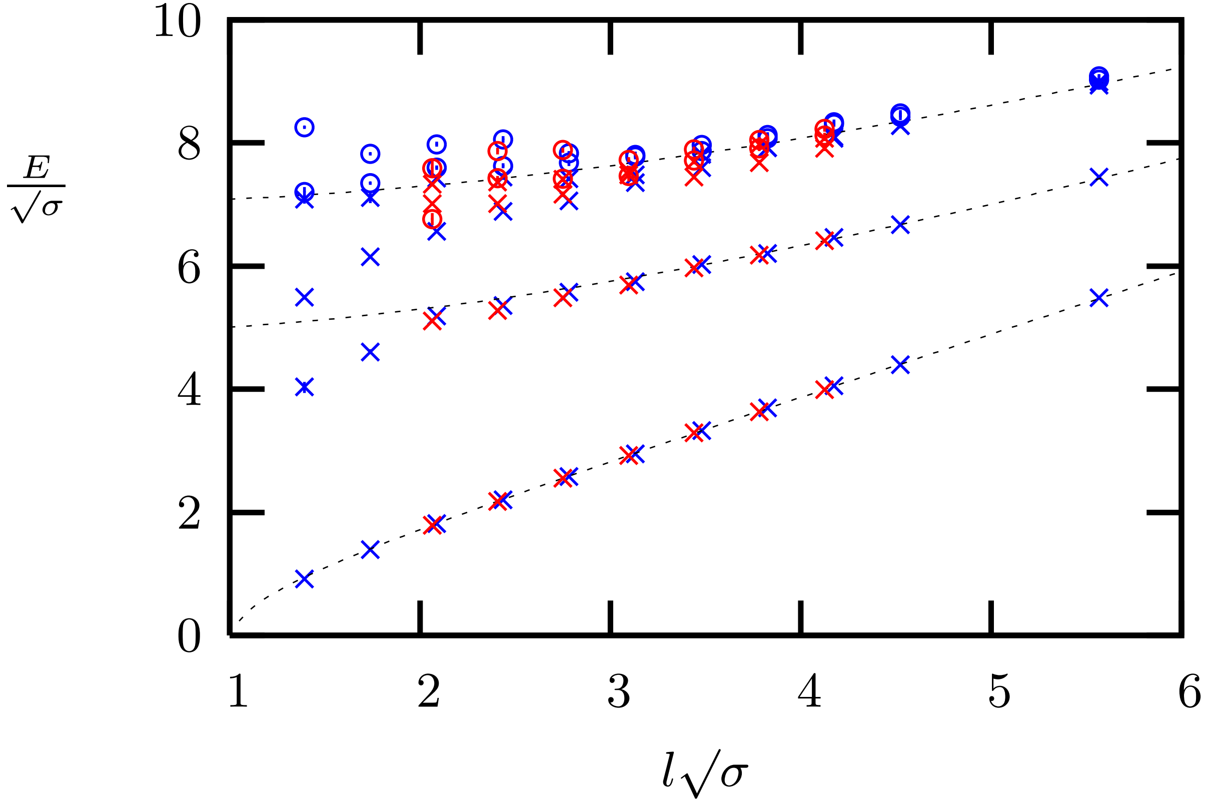
SU(3) : how close to continuum limit?

compare $a\sqrt{\sigma} \simeq 0.174 \quad vs \quad a\sqrt{\sigma} \simeq 0.087$



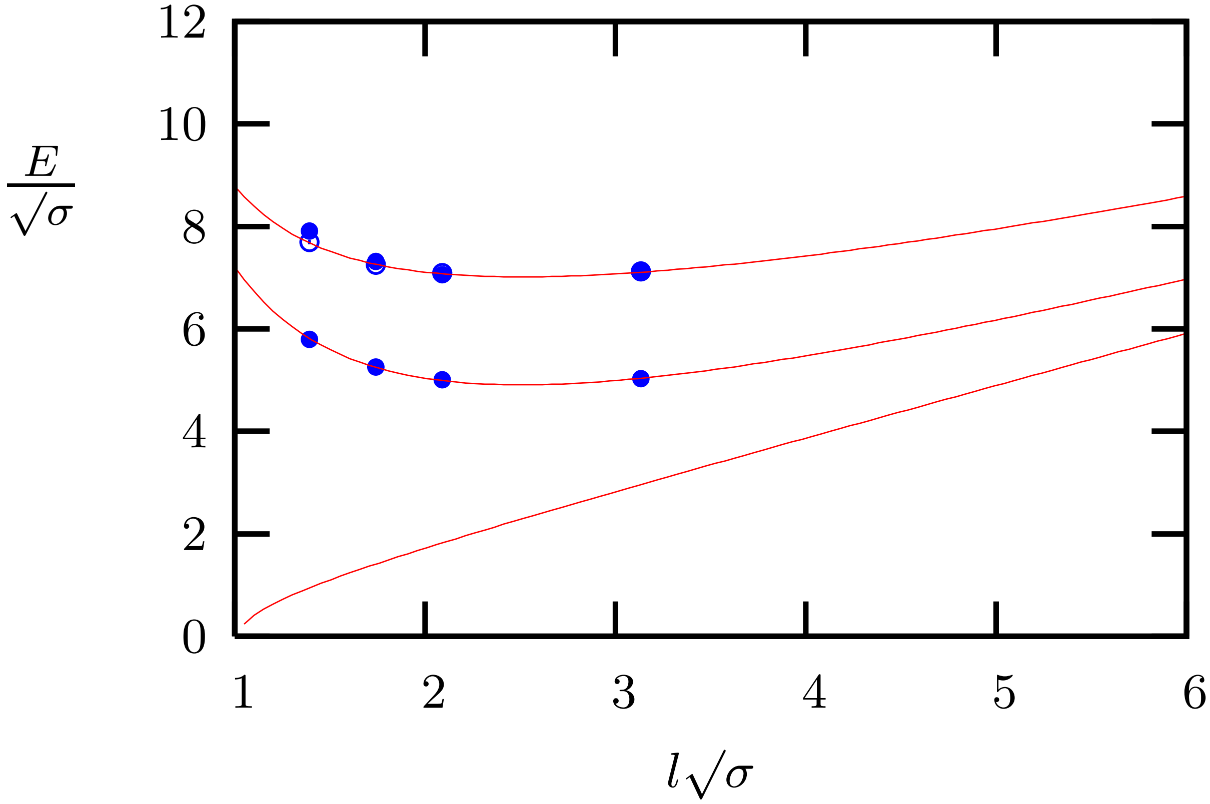
no significant difference as $a \to a/2 \Rightarrow$ we have ‘continuum’ physics

SU(3) vs SU(6) : same a



$\Rightarrow \quad SU(3) \simeq SU(\infty)$

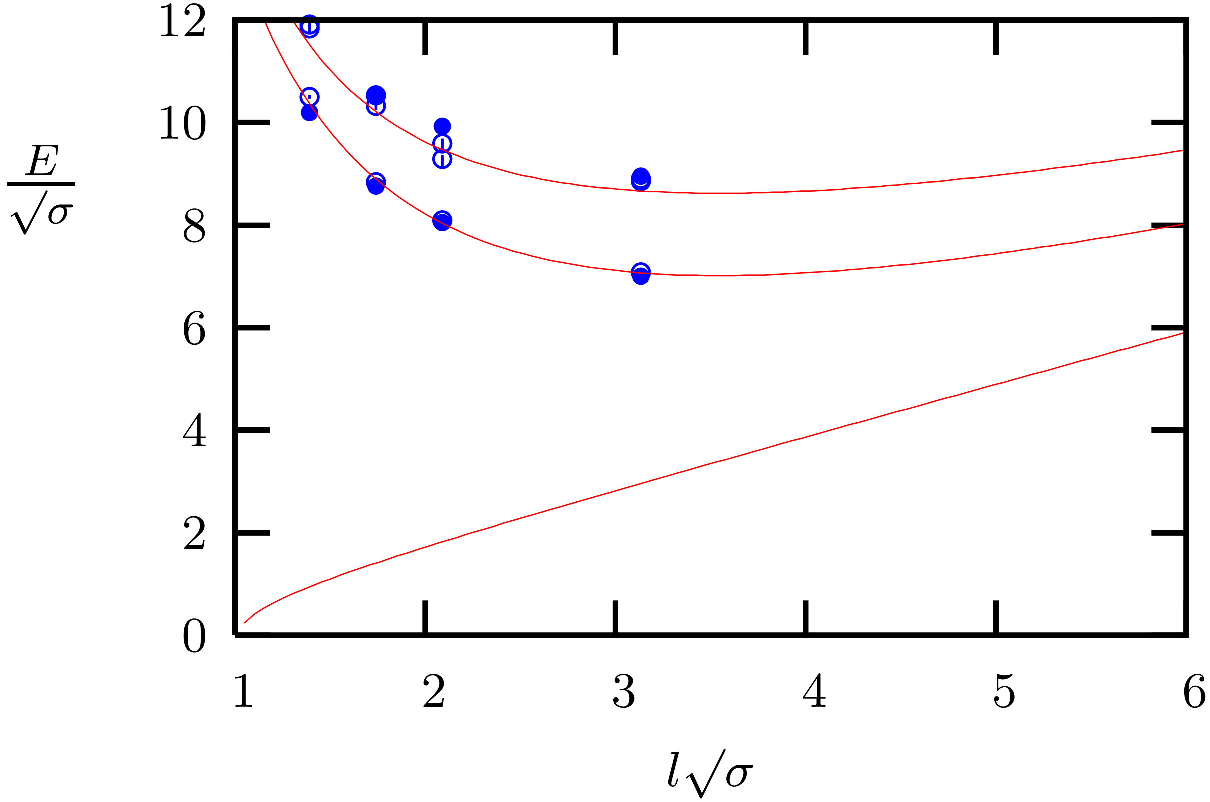
$q = 1$ spectrum : SU(3) at smaller a



$\bullet$ $P = -$; $\circ$ $P = +$ $gs:\ a_1|0>$; $es:\ a_2 a_{-1}|0>,\ a_1 a_1 a_{-1}|0>$

curves: NG predictions (σ fixed from $q = 0$ fit)

$q = 2$ spectrum : SU(3) at smaller a



$\bullet\ P = -\quad ;\quad \circ\ P = +$ $\qquad gs:\ a_1 a_1|0>,\ a_2|0>$

curves: NG predictions (σ fixed from $q = 0$ fit)

◇ Striking agreement with free string model, down to $l\sqrt{\sigma} \simeq 2$.

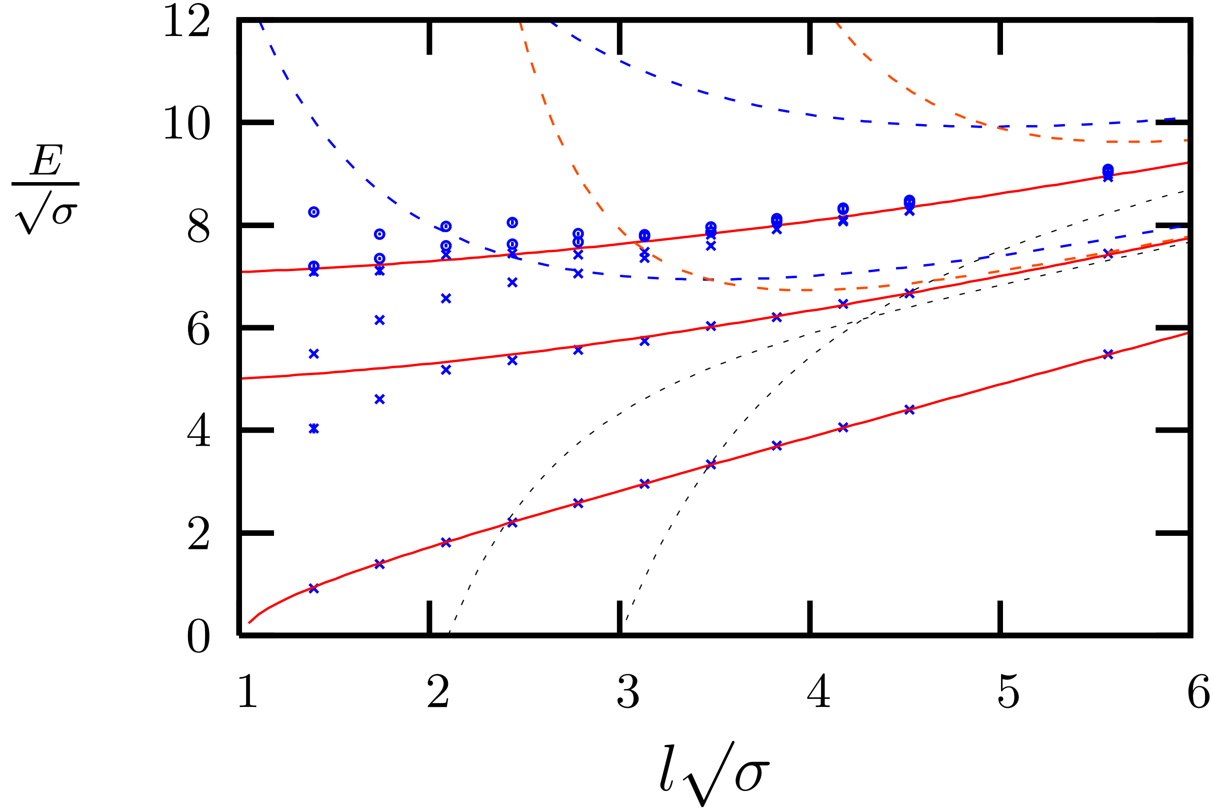
◇ Remarkable since $l\sqrt{\sigma} \simeq 2$

$\Longrightarrow$ length flux tube $\sim$ 1-2 $\times$ width

$\Longrightarrow$ a 'fat blob' rather than an ideal 'string'.

◇ Is this just a manifestation of the fact that we know (Luscher, Weisz, Drummond, Aharony, ...) that the first 3 or 4 terms in an expansion of $E_n(l)$ in powers of $1/\sigma l^2$ must be the same as Nambu-Goto?

$S_{Nambu-Goto}$ vs derivative expansion of $S_{eff}[h]$



— Nambu-Goto

- - LSW, '80; ... LW, '04; JD, '04 ... AK, '09 wild oscillations as we increase order of ∂h

$\Longrightarrow$ an expansion that is divergent

e.g. consider Nambu-Goto expression (for $q = 0$)

$$E(l) = \sigma l \left(1 + \frac{8\pi}{\sigma l^2}\left(n - \frac{D-2}{24}\right)\right)^{\frac{1}{2}}$$

it can only be expanded as a power series in $1/l\sqrt{\sigma}$ when

$$\frac{8\pi}{\sigma l^2}\left(n - \frac{1}{24}\right) \leq 1 \quad \leftrightarrow \quad l\sqrt{\sigma} \geq \sqrt{8\pi n} \sim 5\sqrt{n}$$

whereas in practice we have a very good fit by Nambu-Goto even down to

$$l\sqrt{\sigma} \sim 2\ ,\ n = 1 \qquad\qquad l\sqrt{\sigma} \sim 3\ ,\ n = 2$$

which is well outside its radius of convergence

$\Longrightarrow$

what we are observing is agreement with Nambu-Goto at distances where the derivative expansion no longer makes any prediction

flux tubes and strings in D=2+1

• flux tubes are in the universality class of a simple bosonic string theory

• Theory $\Longrightarrow$ the spectrum of long flux tubes is the same as Nambu-Goto up to $O(1/l^5)$ at least

• Lattice $\Longrightarrow$ the spectrum of long flux tubes is very close to Nambu-Goto for values of l well outside the radius of convergence of the derivative expansion

$\Longrightarrow$

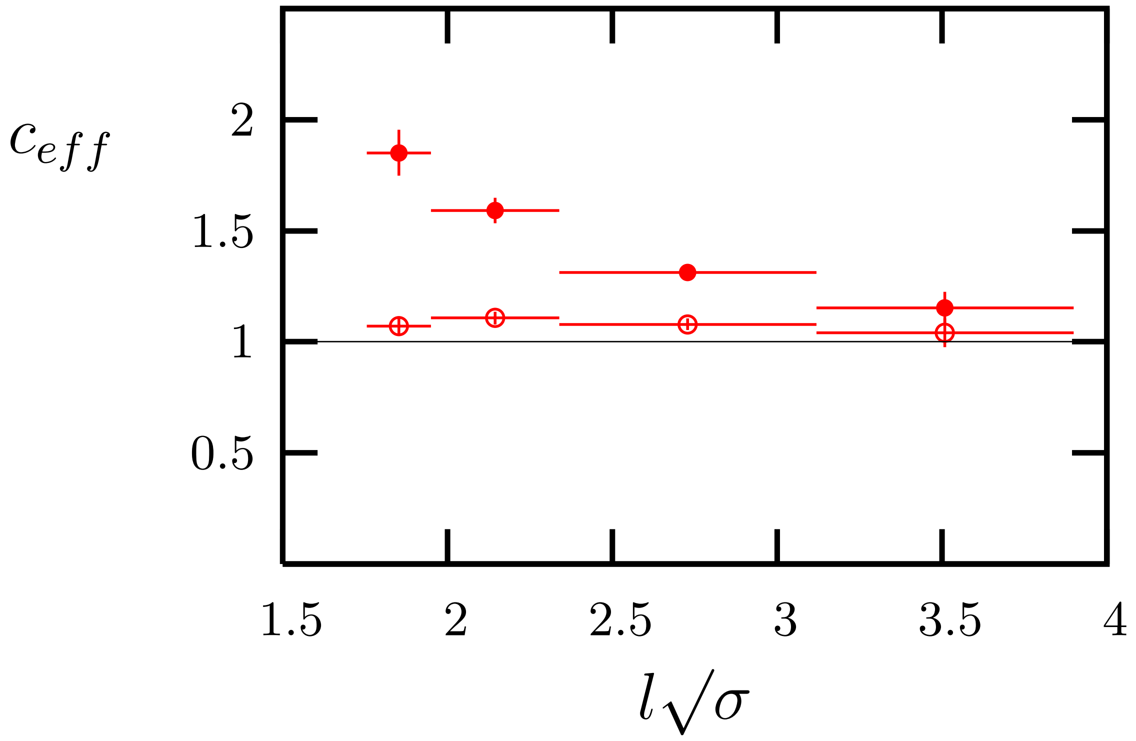
there is a useful effective string action for confining flux tubes not only at very large l, but at almost all values of l, and that an accurate first approximation to such an action is the Nambu-Goto free string theory

where are the massive modes, $E(l) \sim E_0(l) + O(\sqrt{\sigma})$ *?*

What about D=3+1?

Athenodorou, Bringoltz, MT: Lat09 and in preparation

[SU(3) ; $\beta = 6.0625$; $l_c\sqrt{\sigma} \sim 1.6$] – ground state energy



○ Nambu-Goto; ● Luscher

◦ apparently

$$c_{eff} \stackrel{l\to\infty}{\longrightarrow} 1$$

i.e. the confining flux tube is in the simple bosonic string universality class

◦ Nambu-Goto has much smaller corrections than Luscher, just as in D=2+1, but the corrections to NG are now significantly larger, albeit still very small.

Checks:

- for SU(3) compare a and $a' \simeq 2a/3$ to control continuum limit
- for fixed a compare SU(3) and SU(5) to control large-N limit

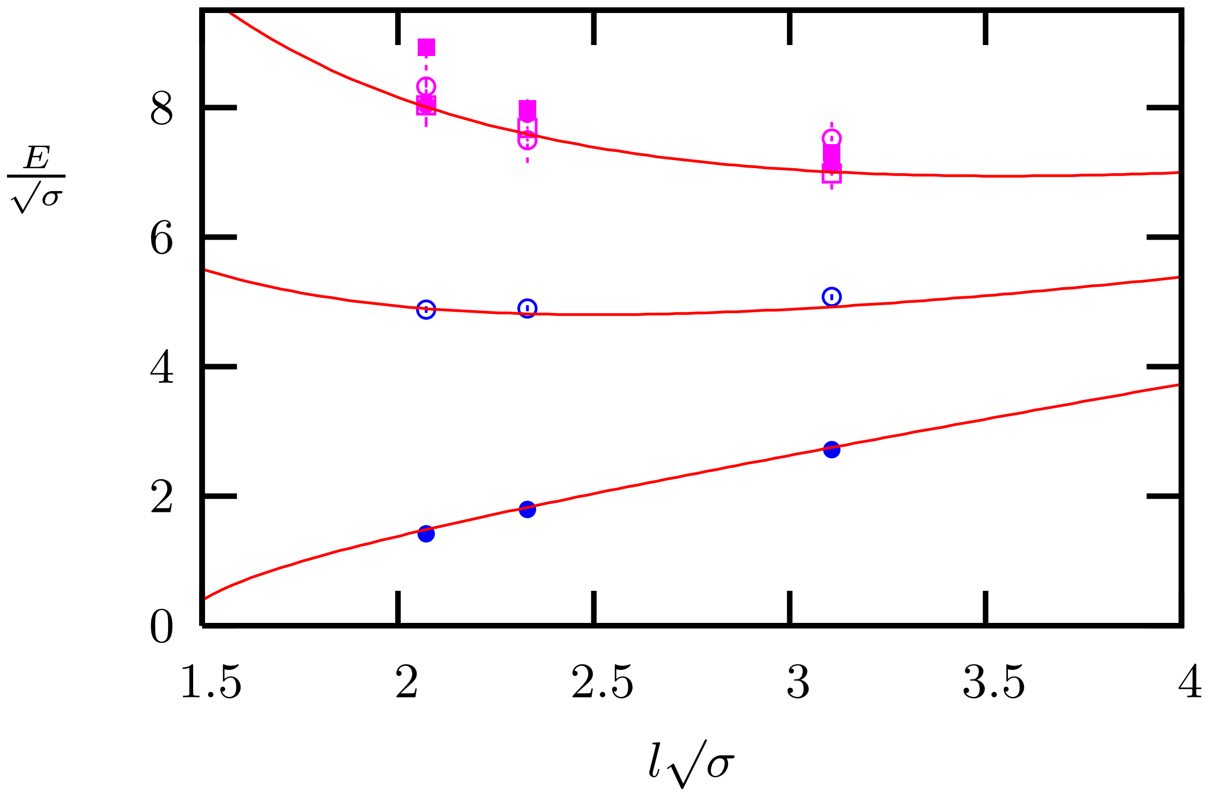
relevant string quantum numbers in 3+1 dimensions:

- ∘ length, l.
- ∘ momentum along string, $p = 2\pi q/l$.
- ∘ angular momentum around string axis, $J = 0, 1, 2...$
- ∘ $D = 2 + 1$ parity in plane orthogonal to string axis, P_ρ
- ∘ reflection 'parity' across this same plane, P_r

⇒ Nambu-Goto prediction

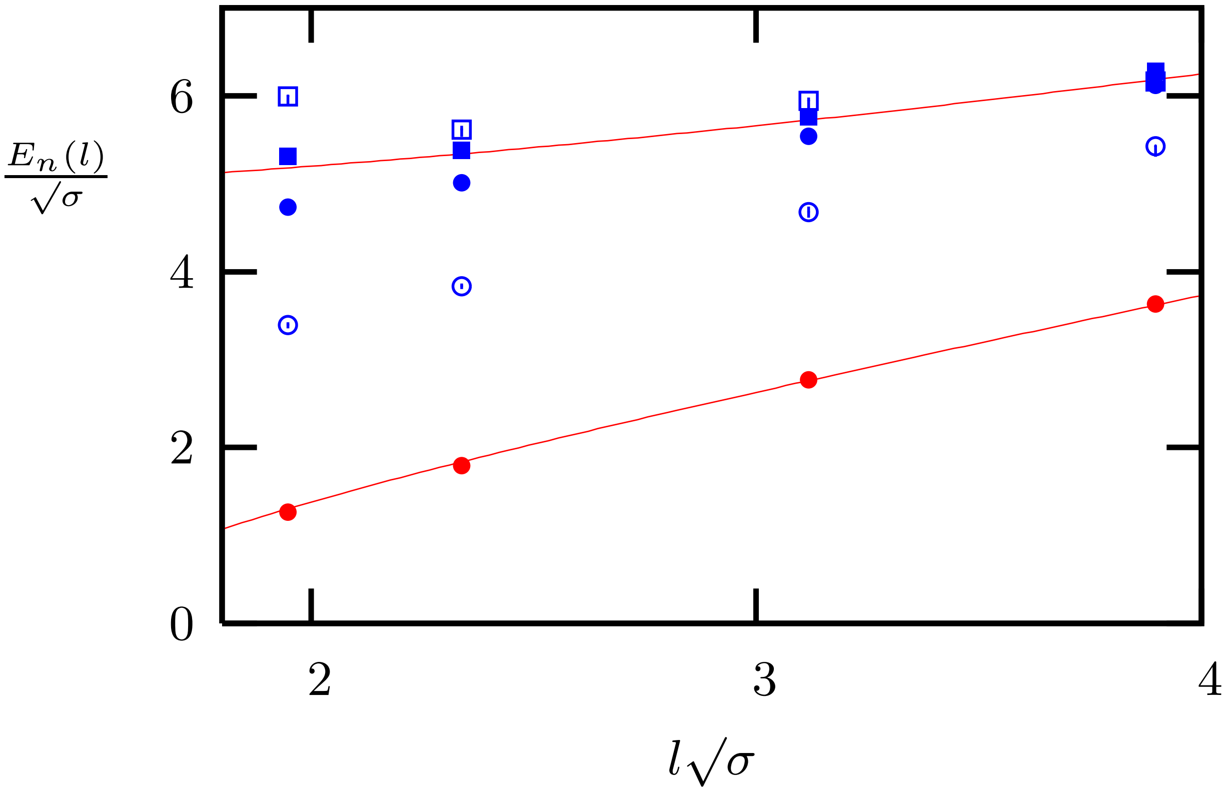
$$E_{NG}(l;q) = \sqrt{(\sigma\, l)^2 + 4\pi\sigma\left(N_L^+ + N_L^- + N_R^+ + N_R^- - \frac{1}{6}\right) + \left(\frac{2\pi q}{l}\right)^2}.$$

$q = 0, 1, 2$ ground states SU(3)∼SU(5)



→ excellent agreement with Nambu-Goto

$q = 0$ spectrum SU(3)~SU(5)

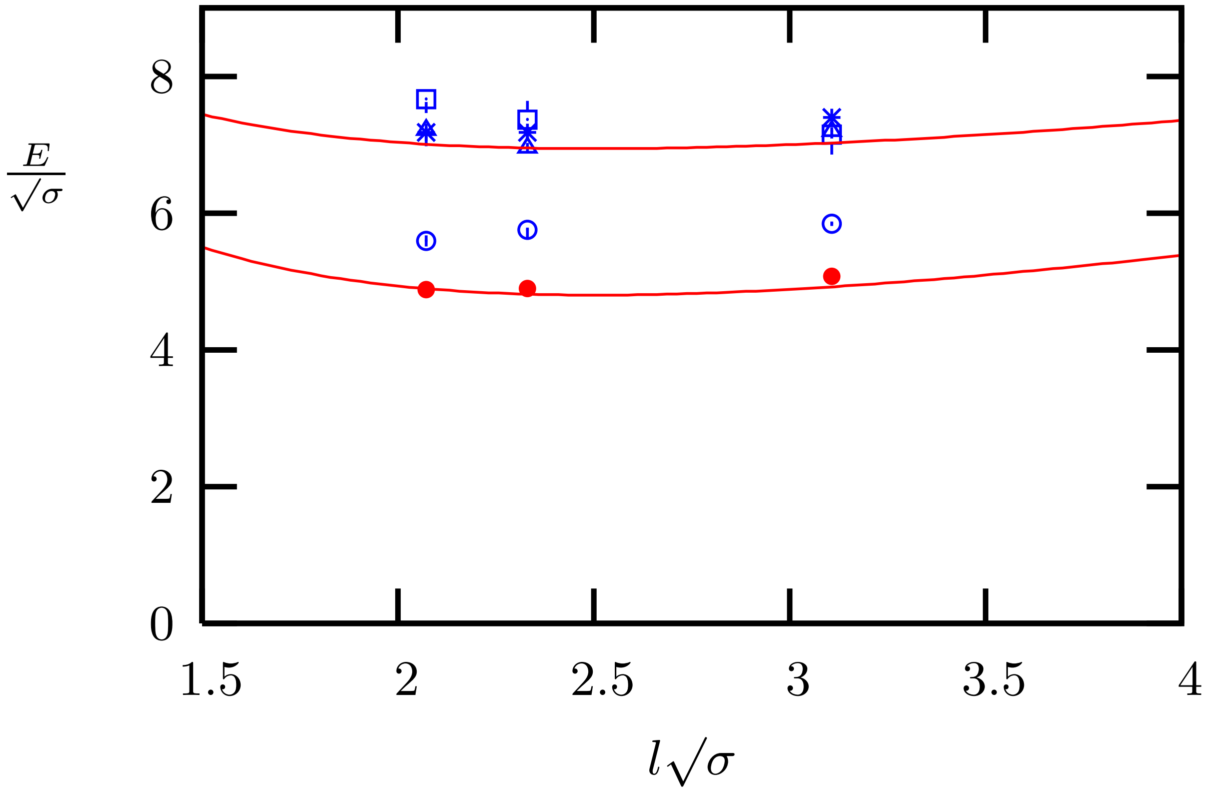


$n = 0$: ● : $J = 0, P_\rho = P_r = +$

$n = 1$: ○ : $J = 0, P_\rho = -, P_r = -$ ⟸

● : $J = 0, P_\rho = P_r = +$ □ : $J = 2, P_\rho = P_r = +$ □ : $J = 2, P_\rho = -, P_r = +$

$q = 1$ spectrum SU(3)~SU(5)



$\bullet$: $J = 1, P_\rho = +$

$\circ$: $J = 0, P_\rho = -$ $\Longleftarrow$

$*$: $J = 0, P_\rho = +$ $\square$: $J = 2, P_\rho = +$

⇒

• in $D = 3 + 1$ most states are remarkably close to Nambu-Goto, very much like $D = 2 + 1$

• but a few, with particular quantum numbers, e.g. $J = 0$ and $P_\rho = -$, are *very* far from Nambu-Goto, sometimes ($q = 1$) showing no sign of approaching the latter;
and the anomalous behaviour strengthens as a decreases and N increases.

⇒

Are these anomalous states related to the massive excitations of the flux tube that have eluded us, so far, in two spatial dimensions? This remains to be understood. The fact that the overall picture has this clear binary character, gives us some confidence that it can be simply understood.

Some Conclusions

•

– For many basic quantities $SU(3) \simeq SU(\infty)$

– Large N calculations are not much harder than $N = 3$

•

– Recent advances wrt the effective string action for long flux tubes $\longrightarrow$ to at least $O(1/l^5)$ the spectrum of closed flux tubes is identical to that of the the free-string Nambu-Goto model

– Our numerical work $\longrightarrow$ Nambu-Goto (mostly) works well at shorter l, where the tube is a 'fat blob' rather than a 'string', and where the derivative expansion diverges

– $\Longrightarrow$ the starting point for S_{eff} should be the Nambu-Goto free string theory with small corrections for most l – *plus massive breathing modes?*