Lecture notes: 3rd year fluids

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C. Lubrication approximation

(also known as the thin film approximation)

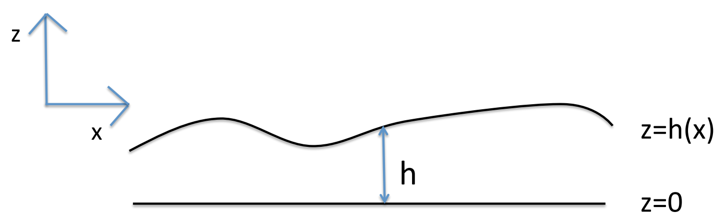

Figure 1: Geometry for the lubrication approximation. $h \ll$ other dimensions.

(i) set-up

Consider the geometry in Figure 1. Let

• U be a typical horizontal (ie along x) flow speed,
• L be a typical horizontal length scale,
and assume that $\boxed{h \ll L}$.

(h can be a function of x and y in general, but we will just consider x to shorten the writing, and we assume steady flow.)

This geometry is relevant for the flow of thin films eg lubriction of a bearing, coating flows, paint spreading, printing.

(ii) How do velocities and their derivatives scale?

$$\begin{aligned} u_x &\sim U \\ \frac{\partial u_x}{\partial z} &\sim \frac{U}{h} && \text{because the flow has magnitude } U \text{ in the bulk, but is no-slip at the surfaces} \\ \frac{\partial^2 u_x}{\partial z^2} &\sim \frac{U}{h^2} \\ \frac{\partial u_x}{\partial x} &\sim \frac{U}{L} \\ \frac{\partial^2 u_x}{\partial x^2} &\sim \frac{U}{L^2} && \text{note that } x\text{-gradients of velocity} \ll z\text{-gradients of velocity} \end{aligned}$$

To see how u_z scales we use incompressibility, $\nabla \cdot \mathbf{u} = 0$.

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0 \quad \Rightarrow \quad \frac{U}{L} \sim \frac{u_z}{h} \quad \Rightarrow \quad u_z \sim \frac{Uh}{L} \ll u_x$$

(iii) Identify the leading order terms in the Navier-Stokes equation

assume steady flow

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \left(u_x \frac{\partial}{\partial x} + u_z \frac{\partial}{\partial z}\right)(u_x, u_z) \sim \left(\frac{U}{L} + \frac{Uh/L}{h}\right)(U, \frac{Uh}{L}) \sim \frac{U^2}{L} + \text{smaller terms}$$

$$\nu \nabla^2 \mathbf{u} \approx \nu \frac{\partial^2 u_x}{\partial z^2} + \text{smaller terms} \sim \frac{\nu U}{h^2}$$

so

$$\frac{(\mathbf{u}\cdot\nabla)\mathbf{u}}{\nu\nabla^2\mathbf{u}} \sim \frac{Uh^2}{\nu L} \sim Re\left(\frac{h^2}{L^2}\right)$$

So for low Re, (but the h^2/L^2 terns helps to satisfy the inequality) the non-linear term in N-S can be ignored. Writing out the components of N-S retaining only the dominant viscous term gives

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu \frac{\partial^2 u_x}{\partial z^2}, \qquad \frac{1}{\rho}\frac{\partial p}{\partial z} = 0.$$

So p is just a function of x and the two equations we need to solve in the lubrication approximation are

$$\frac{1}{\rho}\frac{dp}{dx} = \nu \frac{\partial^2 u_x}{\partial z^2}, \tag{1}$$

and the continuity equation

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0. \tag{2}$$

(iv) An example: the sticky ruler

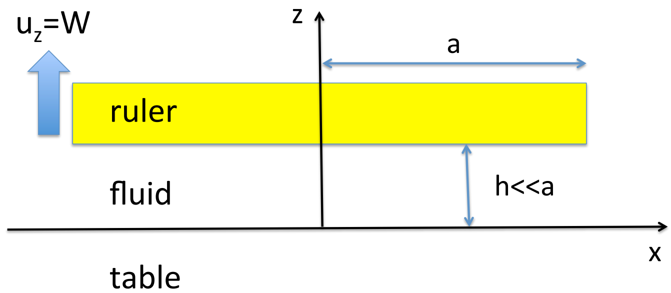

Figure 2: A ruler being pulled off a table at speed W. (We are looking at the ruler edge on and assuming that it is infinite along y; the z-scale is magnified.)

1. Start from Equation (1). Integrate twice wrt z to find how u_x depends on z remembering that $\frac{dp}{dx}$ is independent of z. Put in the boundary conditions, $u_x = 0$ at $z = 0$ and $z = h$ to give

$$u_x = -\frac{1}{2\eta}\frac{dp}{dx}z(h-z). \tag{3}$$

2. Differentiate Equation (3) wrt x and substitute into the continuity equation (2)

$$\frac{1}{2\eta}\frac{d^2p}{dx^2}z(h-z) = \frac{\partial u_z}{\partial z} \tag{4}$$

and then integrate wrt z to obtain

$$u_z = \frac{1}{2\eta}\frac{d^2p}{dx^2}\left(\frac{hz^2}{2} - \frac{z^3}{3} + C\right). \tag{5}$$

The constant $C = 0$ because $u_z = 0$ at $z = 0$.

3. At $z = h$, $u_z = W$ so Equation (5) gives

$$W = \frac{1}{2\eta}\frac{d^2p}{dx^2}\frac{h^3}{6}. \tag{6}$$

4. Integrate Equation (6) twice to get the pressure distribution in the fluid. The boundary conditions are $p = p_0$ (atmospheric pressure) at $x = \pm a$.

$$p - p_0 = \frac{6\eta W}{h^3}(x^2 - a^2). \tag{7}$$

5. Substituting back into Equation (3) gives u_x,

$$u_x = -\frac{6Wx}{h^3} z(h - z). \tag{8}$$

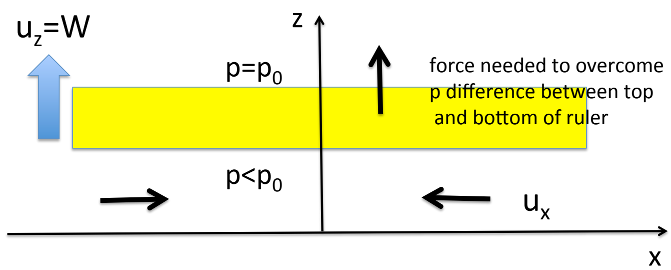

Figure 3: To check that the signs make sense.

D. Waves

<u>Linear sound waves</u>

Our aim is to derive the equation for sound propagation in the *linear* approximation. Sound is a wave motion that couples p, ρ and $\mathbf{u}$. Therefore we will use the *compressible* Navier-Stokes equations, and we will assume inviscid flow.

We will need

• the vector identity

$$\nabla(\nabla \cdot \mathbf{u}') = \nabla^2 \mathbf{u}' + \nabla \wedge (\nabla \wedge \mathbf{u}') \equiv \nabla^2 \mathbf{u}' \quad \text{for irrotational flow.} \tag{9}$$

• the equation of state of the gas $p(\rho)$. Taylor expanding about a reference density ρ_0

$$p(\rho_0 + \rho') = p(\rho_0) + \left.\frac{dp}{d\rho}\right|_{\rho_0} \rho' + \dots$$

$$\nabla p = \left.\frac{dp}{d\rho}\right|_{\rho_0} \nabla \rho'. \tag{10}$$

The equations of motion are

$$\text{Euler:} \quad \rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p. \tag{11}$$

$$\text{continuity:} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho = 0. \tag{12}$$

We will expand Equations (11) and (12) for small deviations from the equilibrium state $p = p_0$, $\rho = \rho_0$, $\mathbf{u} = 0$,

$$\begin{aligned} \rho &= \rho_0 + \rho'(\mathbf{r}, t), \\ p = p_0 + p'(\mathbf{r}, t) &\equiv p(\rho_0 + \rho'), \\ \mathbf{u} &= \mathbf{u}'(\mathbf{r}, t), \end{aligned}$$

and keep terms linear in small ($'$) quantities. We will choose to write the resulting equations in terms of the variables ρ' and $\mathbf{u}'$.

Expanding the Euler equation and using Equation (10)

$$(\rho_0 + \rho') \left\{ \frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{u}' \cdot \nabla)\mathbf{u}' \right\} = -\nabla p \Big|_{\rho_0 + \rho'} = -\frac{dp}{d\rho}\Big|_{\rho_0} \nabla \rho'.$$

So to leading order

$$\rho_0 \frac{\partial \mathbf{u}'}{\partial t} = -\frac{dp}{d\rho}\Big|_{\rho_0} \nabla \rho'. \tag{13}$$

Expanding the continuity equation

$$\frac{\partial(\rho_0 + \rho')}{\partial t} = -(\rho_0 + \rho')\nabla \cdot \mathbf{u}' + \mathbf{u}' \cdot \nabla(\rho_0 + \rho')$$

and to leading order

$$\frac{\partial \rho'}{\partial t} = -\rho_0 \nabla \cdot \mathbf{u}'. \tag{14}$$

Differentiating Equation (13) with respect to time

$$\rho_0 \frac{\partial^2 \mathbf{u}'}{\partial t^2} = -\frac{dp}{d\rho}\Big|_{\rho_0} \frac{\partial}{\partial t}(\nabla \rho'). \tag{15}$$

Taking the grad of Equation (14) and using the identity (9)

$$\frac{\partial}{\partial t}(\nabla \rho') = -\rho_0 \nabla^2 \mathbf{u}'. \tag{16}$$

Comparing Equations (15) and (16) gives

$$\boxed{\frac{\partial^2 \mathbf{u}'}{\partial t^2} = \left.\frac{dp}{d\rho}\right|_{\rho_0} \nabla^2 \mathbf{u}'.} \tag{17}$$

This is the wave equation with velocity $c^2 = \frac{dp}{d\rho}|_{\rho_0}$. Small oscillations in p, ρ and $\mathbf{u}$ propagate as a wave. The wave is undamped (as we have set $\nu = 0$). It is non-dispersive, as the speed is independent of frequency. Thermal conduction is normally negligible for sound waves in air so $p\rho^{-\gamma}$ =constant and $c^2 = \gamma p/\rho_0$ where γ is the ratio of specific heats.

Incompressibility

This is just a short aside to check the condition for incompressibility.

For it to be a good approximation to treat a fluid as incompressible we require

pressure variation due to flow $\ll$ pressure variation due to sound waves

From Bernoulli the pressure variation due to the flow satisfies

$$\frac{\mathbf{u}^2}{2} + \frac{p}{\rho} = \text{constant} \quad \Rightarrow \quad \Delta p \sim u^2 \Delta\rho.$$

For sound waves

$$\Delta p \sim \frac{dP}{d\rho}\Delta\rho \sim c^2 \Delta\rho.$$

Therefore, for incompressibility,

$$u^2 \ll c^2 \quad \text{or} \quad Ma \ll 1$$

where the Mach number, $Ma = u/c$.

Surface waves

Consider a deep body of fluid with a surface at $y = 0$ and let the small displacement of the surface be $\eta(x, t)$. We want to find the dispersion relation for waves on the surface.

(i) Equation of motion:

Assume irrotational, incompressible flow so

$$\nabla^2 \phi = 0 \tag{18}$$

where ϕ is the velocity potential.

(ii) Boundary conditions:

Assuming small velocities and amplitudes –

1. The time-dependent Bernoulli's function is constant at the free surface.

$$\frac{\partial \phi}{\partial t} + \frac{u^2}{2} + \frac{p_0}{\rho} + g\eta = \text{constant} \qquad \text{at} \quad y = 0.$$

Ignoring the non-linear term in u^2, because we are assuming u is small, and choosing the constant to be p_0/ρ this becomes

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \qquad \text{at} \quad y = 0.$$

(Taking $p = p_0$ (atmospheric pressure) means that we are ignoring surface tension which is a good approximation for waves with wavelength more than a few cm.)

2. Assume that the y component of the velocity is equal to the vertical velocity of the interface:

$$u_y = \frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \qquad \text{at} \quad y = 0.$$

3. Velocities tend to zero far below the surface:

$$u_y = \frac{\partial \phi}{\partial y} = 0 \qquad \text{as} \quad y \rightarrow -\infty.$$

(iii) Solving the equation

Look for a solution

$$\phi = A \ \cos(kx - \omega t) \ g(y).$$

Substituting in to Laplace's equation (18), solving, and using boundary condition 3 gives

$$\phi = A \ cos(kx - \omega t) \ e^{ky}.$$

Boundary condition 2 then gives

$$\eta = -\frac{A\omega}{g} \ sin(kx - \omega t)$$

which is consistent with boundary condition 1 if the dispersion relation is

$$\omega^2 = gk.$$

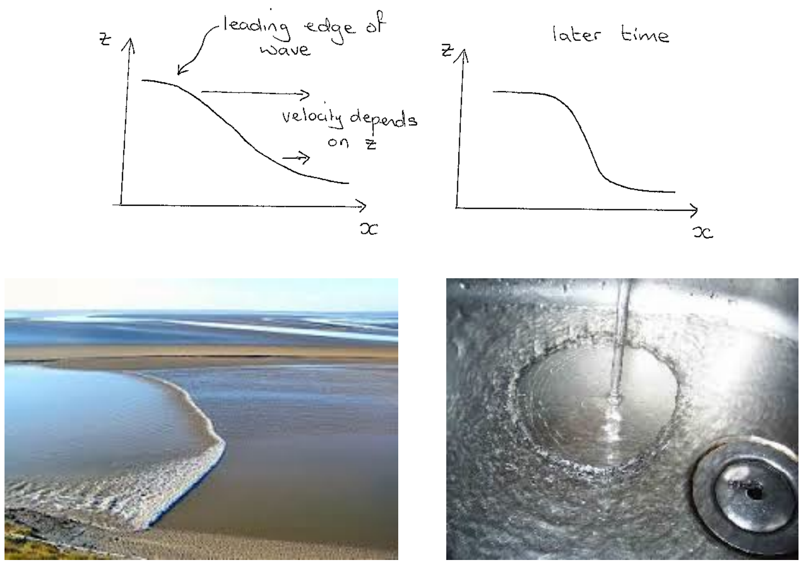

Figure 4: Non-linear waves

Non-linear waves

The equations for non-linear waves are more complicated. A ‘toy model’ that gives the right physics is

$$\frac{\partial z}{\partial t} + z\frac{\partial z}{\partial x} = 0. \tag{19}$$

The general solution of this equation is

$$z = F(x - zt). \tag{20}$$

Let’s check that this is a solution:

$$
\begin{aligned}
\frac{\partial z}{\partial t} &= F'\left\{-z - \frac{\partial z}{\partial t}t\right\} \\
\frac{\partial z}{\partial x} &= F'\left\{1 - \frac{\partial z}{\partial x}t\right\} \\
\Rightarrow \quad \frac{\partial z}{\partial t} + z\frac{\partial z}{\partial x} &= F'\left\{-\frac{\partial z}{\partial t}t - z + z - z\frac{\partial z}{\partial x}t\right\} \\
&= F'\left\{-t\left(\frac{\partial z}{\partial t} + z\frac{\partial z}{\partial x}\right)\right\} = 0.
\end{aligned}
$$

Compare the wave (20) to the solution of the linear wave equation $z = F(x-ct)$. In the linear case the wave velocity is a constant, c. In the non-linear case the velocity is $z(x)$ ie it depends on the wave displacement z at any point x. This means that points of larger displacement move faster, and the wave steepens (Figure 4). Physically the slope of the displacement cannot become infinite because viscosity must eventually become important (the viscous dissipation increases with gradients). However, wave profiles can become very steep.

E. Zero Reynolds number

At low Re viscosity dominates. Examples of viscous flows are:
slow movements of the earth's mantle
creep of a glacier
treacle dripping from a spoon
microfluidics
bacterial swimming

At Re=0 we can ignore the inertial terms in the Navier Stokes equations giving the *Stokes equation*

$$\boxed{\nabla p = \eta \nabla^2 \mathbf{u}.}$$

There is no explicit time dependence in the Stokes equations which implies kinematic *reversibility*. A striking example of this is provided by a blob of dye suspended in a high viscosity fluid contained between two cylinders. When the fluid is sheared by rotating the outer cylinder the dyed blob can be stretched to wrap several times around the inner cylinder. If the motion of the cylinders is reversed the dyed fluid returns to its original shape. See
https://www.youtube.com/watch?v=p08_KlTKP50

Stokes drag

Stokes flow is linear, forces are proportional to velocities.

In particular the force needed to pull a colloid of radius a through a fluid at velocity $\mathbf{u}$, called the Stokes drag, is

$$\mathbf{F} = 6\pi\eta a\mathbf{u}.$$

(see problem set)

(b) bacterial swimming

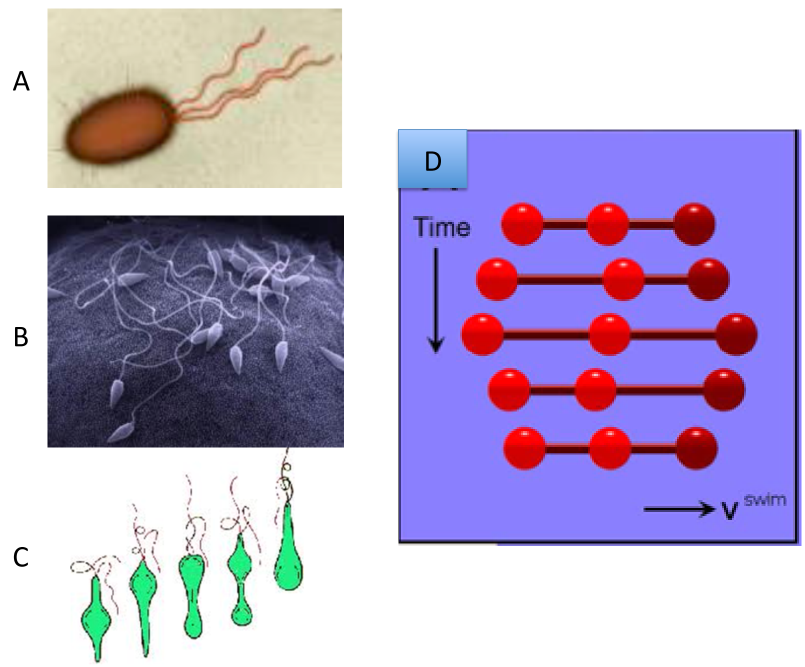

Figure 5: Microscopic swimmers that have evolved to overcome the Scallop Theorem. A: *E. coli*, B: sperm cells, C: *Euglena metaboly*, D: a model 3-sphere swimmer.

A far-reaching consequence of the Re = 0 limit is Purcell's Scallop Theorem:

To achieve propulsion at zero Reynolds number in Newtonian fluids a swimmer must deform in a way that is not invariant under time reversal.

As a swimmer moves through a low Re fluid there is nothing in the equations of motion of the surrounding fluid that picks out a preferred direction. If the boundary conditions, i.e. the swimmer's stroke cycle, is invariant under time reversal as well, its net displacement after each cycle must be zero – otherwise reversing time would give the same physical system, but a different displacement.

Therefore at zero Re the stroke must be non-invariant under time reversal to allow swimming.

Microscopic swimmers have evolved several strategies to overcome the Scallop Theorem (Figure 5). Many move by using long, thin appendages called flagella. Bacterial flagella (e.g. *E. coli*) are driven by rotary motors which induce helical waves to overcome the Scallop Theorem. Eukaryotic flagella tend to oscillate through bending waves. *Paramecium* is covered by cilia, shorter but similar in structure to eukaryotic flagella that have distinct power and recovery strokes. Cilia also act as pumps in the body, clearing mucus from the lungs and initiating asymmetry in morphogenesis. *Euglena metaboly* produces a suitable swimming stroke by altering its body shape so that the relative motion of its ends is coupled to changes in their mass.

At least three degrees of freedom are needed to define a suitable stoke for zero Re swimming. A simple model swimmer is the three-sphere swimmer, which comprises three spheres coupled by rods. The rods have no hydrodynamic coupling to the fluid, but specifying their lengths as a function of time serves to define the swimming stroke. The stroke and the swimmer displacement as a function of time are shown in Figure 5D. The advantage of building the swimmer out of spheres is that the flow field around a sphere is known analytically in the Stokes limit. Hence exact results can be obtained for the swimming speed if the sphere radii are small compared to their separation. (See An introduction to the hydrodynamics of swimming microorganisms, J.M. Yeomans, D.O. Pushkin and H. Shum, European Physics Journal: Special Topics **223** 1771-1785 (2014) for more - not on the syllabus - details.)

F. Instabilities

Rayleigh-Bernard convection

Consider a fluid between two horizontal plates. The top plate is at temperature T_1 and the bottom plate is at temperature T_2. If $T_2 < T_1$ the colder fluid, at the bottom, has higher density and nothing happens. If $T_2 > T_1$ it is reasonable to expect that the lighter fluid will rise if it gains enough gravitational energy by doing so. This would mean that the zero velocity state is unstable, and that the instability leads to a different flow configuration. This is indeed what happens for a sufficiently large temperature difference: for $T_2 - T_1 < \Delta T_c$ the zero velocity state is stable; for $T_2 - T_1 > \Delta T_c$ it is replaced by counter rotating convection cells (Figure 6). The diameter of the rolls $\sim$ the distance between the plates and the velocity of the rotating fluid increases with distance from the threshold. At higher ΔT more complicated flow patterns occur, eg non-stationary rolls.

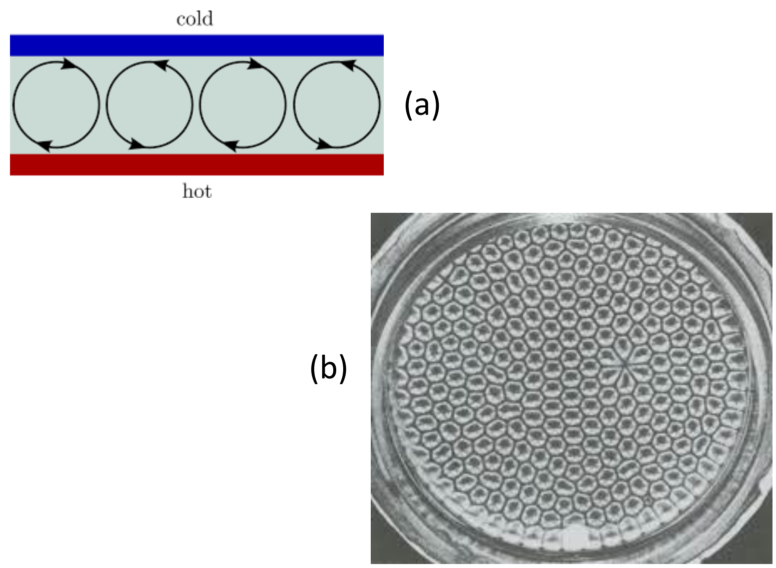

Figure 6: Rayleigh-Bernard convetion rolls: (a) from the side, (b) from above.

To understand the mechanism which drives the instability consider a fluid particle which, due to a local random fluctuation, starts to move upwards. If it doesn't cool down too quickly it will be surrounded by a fluid of increasingly high density as it moves so the buoyancy force acting on it will increase and its velocity will be amplified, setting up a positive feedback loop and driving the instability. This will work if the thermal diffusivity, κ, which equalises the temperature, and the viscosity, ν, which damps out any velocity differences, are not too large. The control parameter is the Rayleigh number

$$Ra = \frac{\alpha \Delta T g d^3}{\nu \kappa}, \qquad \text{where} \quad \frac{1}{\rho_0}\frac{d\rho}{dT} = -\alpha$$

and d is the distance between the plates.

Scaling argument for the Rayleigh number

A fluctuation gives a fluid particle of volume V and linear dimensions $\sim r_0$ an upwards velocity. Is the flow amplified or does it die out? To answer this question we will compare the upthrust, due to buoyancy forces, to the Stokes drag:

characteristic time for thermal relaxation of the particle is $\tau_Q \sim \dfrac{r_0^2}{\kappa}$,
distance moved by particle in this time is $v\tau_Q$,
resulting difference in temperature between particle and fluid is $\delta T = \dfrac{\partial T}{\partial y}\delta y \sim \dfrac{\Delta T}{d} v\tau_Q$,
so difference in density between particle and surrounding fluid is $\delta \rho = \dfrac{\partial \rho}{\partial T}\delta T \sim \rho_0 \alpha \dfrac{\Delta T}{d} v \dfrac{r_0^2}{\kappa}$,
therefore the upthrust is $\sim \delta \rho V g \sim \rho_0 \alpha \dfrac{\Delta T}{d} v \dfrac{r_0^2}{\kappa} r_0^3 g$.

The Stokes drag is $\sim \eta r_0 v \sim \rho_0 \nu r_0 v$.

So buoyancy is greater than drag if $\rho_0 \alpha \dfrac{\Delta T}{d} v \dfrac{r_0^5 g}{\kappa} > \rho_0 \nu r_0 v$.

Rearranging $Ra = \dfrac{\alpha \Delta T g d^3}{\nu \kappa} > \left(\dfrac{d}{r_0}\right)^4$.

Remember that we have ignored all factors of order unity so this tells us that the Rayleigh number is a suitable control parameter for this instability, that there is a critical Rayleigh number and that $Ra_c > 1$. It does not tell us the value of Ra_c which experiments show is 1708.

<u>Other instabilities</u>

Rayleigh-Plateau
The Rayleigh-Plateau instability (Figure 7) causes the break-up of fluid filaments eg the stream of water falling from a tap. Small fluctuations on the surface of the fluid can be decomposed into Fourier components. Some wavelengths die out, but some grow, and the most unstable (fastest growing) wavelength determines the final spacing of the drops.

The physics driving the instability is the Laplace pressure. There is a pressure drop across a curved interface: $\Delta P = \sigma \left(\frac{1}{r_1} + \frac{1}{r_2}\right)$ where σ is the surface tension and r_1 and r_2 are the principal radii of curvature. It turns out that the pressure inside the filament is higher at the necks than in the bulges, so fluid is pushed out of the necks, which break, forming a line of droplets.

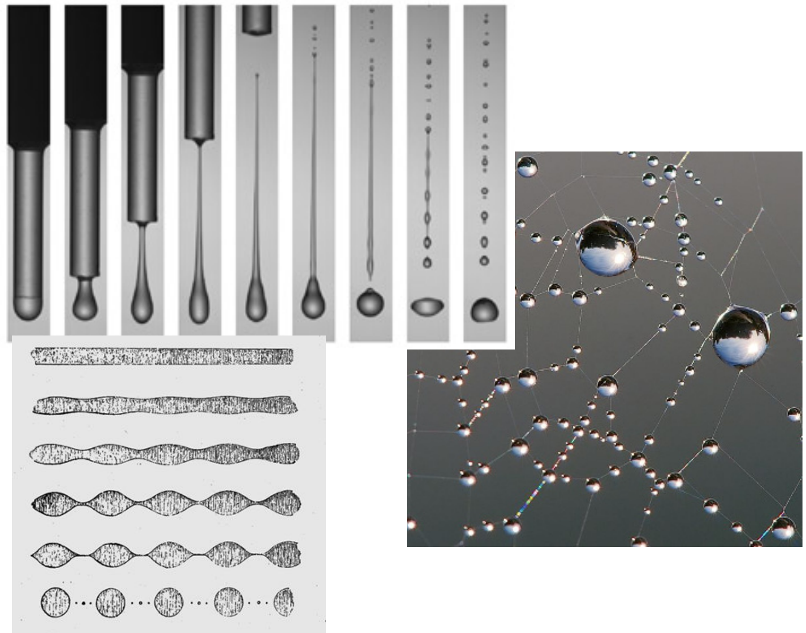

Figure 7: Examples of the Rayleigh-Plateau instability. A falling stream of water breaks up into drops. A series of tiny satellite drops can form between the main droplets. Dew on a spider's web forms drops rather than coating the fibres because of the Rayleigh-Plateau instability.

Rayleigh-Taylor
Instability of an interface between two fluids of different densities. Consider a dense fluid on top of a less dense one under gravity. A small perturbation of the interface will decrease the gravitational energy but increase the surface tension energy. For long enough wavelengths gravity wins; for water over air 'long enough' is $\sim$ 1cm.

Kelvin-Helmholtz
Caused by shear across a fluid or at the interface between two fluids. Wind blowing over the ocean leads to waves as a result of the Kelvin-Helmholtz instability. Certain cloud patterns and the patterning around the great red spot on Jupiter are other examples.

G. Turbulence

Understanding turbulence is a hard problem We do not have a complete, predictive, theory of turbulence.

• deterministic chaos ... can be described by the N-S equation (we assume) but there is little understanding how.

• in any real fluid thermal fluctuations may be important as there is sensitive dependence on initial conditions.

• the velocity and vorticity fluctuate 'randomly' but this is not just white noise; there is structure and there are correlations in the velocity field.

<u>Statistical description of turbulence</u>

Write (using index notation)

$$v_i = \overline{v}_i + v_i' \tag{21}$$

where $\overline{v}_i$ is the mean flow and v_i' are velocity fluctuations about the mean. Note that, by definition, $\overline{v}_i' = 0$. The average, denoted by an overbar, is taken at a point over a long time, or over a large system at a given time. If the system is *ergodic* these will give the same answer.

N-S written in index notation is

$$\frac{\partial v_i}{\partial t} + (v_j \partial_j) v_i = -\frac{1}{\rho}\partial_i p + \nu \partial_j^2 v_i. \tag{22}$$

Substituting for v_i from Equation (21) gives

$$\frac{\partial \overline{v}_i}{\partial t} + \frac{\partial v_i'}{\partial t} + (\overline{v}_j + v_j')\partial_j(\overline{v}_i + v_i') = -\frac{1}{\rho}\partial_i p + \nu \partial_j^2 (\overline{v}_i + v_i'). \tag{23}$$

Averaging, remembering that averages and derivatives commute and that the average of a product is not equal to the product of averages, gives

$$\frac{\partial \overline{v}_i}{\partial t} + (\overline{v}_j \partial_j)\overline{v}_i + \overline{(v_j' \partial_j) v_i'} = -\frac{1}{\rho}\partial_i p + \nu \partial_j^2 \overline{v}_i \tag{24}$$

where we have used $\overline{v}_i' = 0$.

Equation (24) is N-S for the mean flow plus a non-linear term $\overline{(v_j' \partial_j) v_i'}$. This can be considered to produce an additional stress acting on the mean flow. Attempts to deal with it include modelling it as an additional velocity, the *eddy viscosity*. But this just shifts to problem to understanding how the eddy viscosity depends on the local flow properties.

Kolmogorov scaling

Kolmogorov scaling is a useful way of describing turbulence that approximately matches experiment and probably contains some of the correct physics.

Kolmogorov described turbulence in terms of an *energy cascade*. Energy injected into the flow field is transmitted to the largest vortices. These transmit energy sequentially to smaller and smaller vortices with no dissipation. Energy is finally dissipated by the viscosity at the scale of the smallest vortices.

Let

ϵ be the rate at which the energy per unit mass is flowing down the cascade,
ℓ be a characteristic vortex size,
v_ℓ be the velocity of vortices of size ℓ,
so the energy per unit mass of a vortex $\sim v_\ell^2$.

By dimensional analysis

$$\epsilon \sim \frac{v_\ell^3}{\ell} \equiv v_\ell^2/(\ell/v_\ell).$$

The range of ℓ over which this scaling holds is called the *inertial subrange*. It is bounded above by the length scale at which energy is injected. It is bounded below by the breakdown of the assumption of no dissipation.

Let ξ be the characteristic vortex size at which dissipation starts to become important. For vortices of size ξ

$$\begin{array}{ccl} \text{rate of energy flow} & \sim & \text{rate of energy dissipation} \\ \epsilon \sim \dfrac{v_\xi^3}{\xi} \quad \text{(as before)} & \sim & \nu\dfrac{v_\xi^2}{\xi^2} \quad \text{(expected to depend linearly on the viscosity, and} \\ & & \text{then a (time scale)}^2 \text{ is needed to get the dimensions right)} \end{array}$$

which can be rearranged to give the

$$\boxed{\xi \sim \nu^{3/4}\epsilon^{-1/4}} \qquad \boxed{v_\xi \sim (\nu\epsilon)^{1/4}}$$

Kolmogorov length Kolmogorov velocity

As a consistency check, we would expect the energy dissipation to become appreciable at $Re \sim 1$:

$$Re = \frac{\xi v_\xi}{\nu} = \frac{\nu^{3/4}\epsilon^{-1/4}(\nu\epsilon)^{1/4}}{\nu} = 1.$$

Comparison to experiment

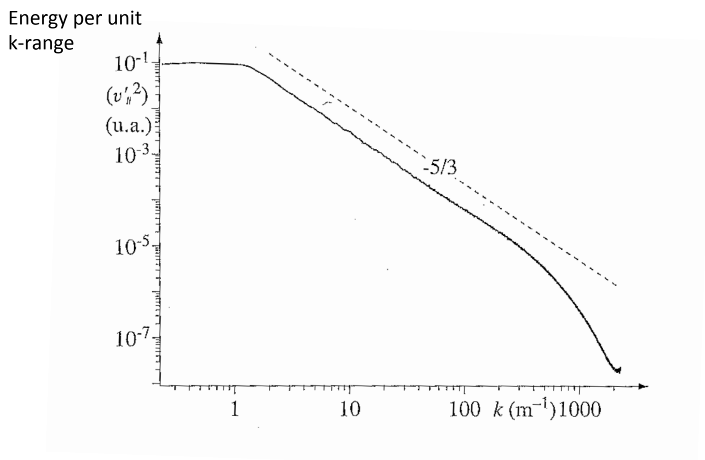

Figure 8: Scaling in the inertial subrange of bulk turbulence.

The experimental results in Figure 8 are a reason to believe that Kolmogorov scaling is at least a partial description of the physics of turbulence. To obtain these results the velocity spectrum is measured by putting an array of fine wire sensors in the fluid and heating them – the rate of heat transfer to the fluid is a measure of the local velocity. The spectrum of the velocity squared is then Fourier transformed to give the energy per unit mass per unit k-range, where k is the wavevector.

Note that Figure 8 is a log-log plot, and that the dependence is linear (except at high and low k). Thus the experiments show that the energy has a power law dependence on k, that is $E(k) \sim k^{\alpha}$ where here $\alpha = -5/3$. The scaling is built in to the Kolmogorov picture. The theory can also explain the value of the exponent, $-5/3$, as follows:

• $E(k)$ cannot depend on ℓ as the Kolmogorov assumption is that the energy flow between different length scales or, equivalently, different k is independent of ℓ in the inertial range.
• $E(k)$ cannot depend on ν as there is no viscous dissipation in the inertial range.
• $E(k)$ can depend on ϵ and k and, by dimensional analysis,

$$\begin{array}{ccccc} E(k) & \sim & \epsilon^{2/3} & \times & k^{-5/3} \\ [length]^3[time]^{-2} & & ([length]^2[time]^{-3})^{2/3} & & (length^{-1})^{-5/3} \end{array}$$