YEAR 2: ELECTRICITY AND MAGNETISM

JULIA YEOMANS

PROBLEM SET 4: ELECTROMAGNETIC WAVES II

$^*$ Standard bookwork, included because you need to understand and learn this material, but your tutor may not want it handed in.
$^\dagger$ Trickier problems, for people who have finished all the others.

A. Reflection and refraction of electromagnetic waves

1.$^*$ A plane electromagnetic wave is incident along the normal to a plane interface separating dielectic media 1 and 2 with relative permittivities $\epsilon_1$ and $\epsilon_2$ respectively. Use the approprate boundary conditions on $\vec{E}$ and $\vec{H}$ to determine the relative amplitudes of the electric vectors of the incident, reflected and transmitted waves, $E_i$, $E_r$ and $E_t$. Use Poynting's vector to determine the relation between the incident, reflected and transmitted intensity and check that energy is conserved.

2. Light is normally incident from a medium 1 with impedance $Z_1$, through a layer of medium 2 of uniform thickness $l$ and impedance $Z_2$, into medium 3 of impedance $Z_3$. Obtain an expression for the total reflected intensity of light when the thickness $l$ corresponds to
(a) $\lambda_2/4$,
(b) $\lambda_2/2$,
where $\lambda_2$ is the wavelength of the light in medium 2.
Show that in case (a) the condition for zero reflectance corresponds to $Z_1Z_3 = Z_2^2$.
Comment on the uses of single and multiple layer coatings in the fabrication of optical components.

3.$^*$ A plane wave with arbitrary polarisation and direction, travelling in a medium of relative permittivity $\epsilon_1$ is incident on a plane boundary with a second medium of relative permittivity $\epsilon_2$. Both media are non-conducting with a relative permeability of unity.
(a) Show that any wave reflected from the boundary has the same frequency and lies in the same

plane as the incident wave and the normal to the boundary, that the angle of incidence is equal to the angle of reflection, and that Snell's law holds.

(b) Find the ratios of the amplitudes of the electric fields of the reflected and transmitted waves to that of the incident wave if the electric field lies (i) in the plane of incidence, (ii) normal to the plane of incidence.

(c) Check that the components of energy flow perpendicular to the surface balance.

(d) Define, and find an expression for, the Brewster angle.

(e) Write down the condition for total internal reflection. Discuss the behaviour of the wave in the less dense medium and show that there is no net energy flow into this medium.

4. Polaroid sunglasses reduce glare by transmitting only one polarisation of light. By what factor may the glare off water be reduced when the sun is (i) $45°$, (ii) $60°$ above the horizon?

5. Optical fibres guide light by using total internal reflection. The evanescent fields outside the guiding core can cause 'crosstalk' (coupling) between adjacent fibres. To estimate the size of this coupling consider a plane model with a silica core with refractive index $n = 1.535$ and external cladding of a borosilicate glass with $n = 1.525$. The optical signal has a free space wavelength of $0.85\mu$m. How far away must neighbouring cores be placed if the field is to be $10^{-3}$ of the surface value, if the incident angle of waves within the core is (a) $85°$, (b) $89°$?

6. An electromagnetic wave falls normally on the surface of a good conductor. Show the fraction of power transmitted is approximately $4\sqrt{\epsilon_0\omega/2\sigma}$. Estimate the magnitude of this quantity at radio and at optical frequencies.

B. Plasmas

7. (a) What is a plasma?

(b) Find the velocity and displacement of a free electron responding to an oscillating electric field of the form $E = E_0 e^{j\omega t}$.

(c) Hence show that the conductivity of a neutral plasma with $N$ free electrons per unit volume is $\sigma = -jNq^2/m\omega$ where $m$ is the mass and $q$ is the charge of an electron.

(d) Show that the dispersion relation for a plane wave

$$\vec{E}(z,t) = E_0 e^{j(\omega t - \tilde{k}z)} \ \hat{\vec{x}}$$

moving in a conducting medium is

$$\tilde{k}^2 = \mu\mu_0\epsilon\epsilon_0\omega^2 \left(1 - \frac{j\sigma}{\epsilon\epsilon_0\omega}\right).$$

(e) Putting together (c) and (d), and taking $\mu = \epsilon = 1$, show that the refractive index of a plasma can be written

$$n^2 = 1 - \frac{\omega_p^2}{\omega^2}$$

and write down an expression for the plasma frequency $\omega_p$.

(f) Show that the product of the phase and group velocities in the plasma is equal to $c^2$.

(g) Why may the positive ions be ignored in this calculation?

8. An alternative approach to the same dispersion relation is to treat the plasma as a dielectric ($\sigma = 0$) with an induced polarisation. From the displacement of a free electron, derived in question 7b, find the polarisation $\vec{P}$, relative permittivity $\epsilon$ and hence the refractive index $n$ of a neutral plasma of free electrons of density $N$, together with an equal number of ions.

9. The region between the plates of a parallel plate capacitor is evacuated and filled with a uniform space charge of free electrons of density, $N$, together with the same density of positive ions. An AC potential $V = V_0 e^{j\omega t}$ is applied. Show that the system behaves like a capacitor (in free space) in parallel with an inductor and find the value of the effective inductance.

10. During a re-entry of the space shuttle it was observed that radio communications at frequencies below 10GHz were interrupted. Calculate the electron density in the neighbourhood of the shuttle.

11. A system is designed to measure the density of free electrons in the ionosphere. The density of free electrons is negligible below a height $h_1$ and above this height it increases with height $h$ as

$$N = a(h - h_1).$$

A trasmitter on the ground sends short pulses of radiation of frequency $\nu$ vertically upwards. These are reflected at the point where the refractive index $n = 0$ and are then received on the ground at a time $\Delta\tau$ after emission. Use the measurements in the table below to calculate $a$ and $h_1$.

| $\nu(MHz)$ | $\Delta\tau(ms)$ |
|---|---|
| 5 | 1.01 |
| 6 | 1.16 |

F. <u>Radiation</u>[†]

12. Explain what is meant by electric dipole radiation. Calculate V, $\vec{A}$, $\vec{E}$, $\vec{B}$ and the Poynting vector $\vec{P}$ at large distances from an electric dipole aerial. Sketch the angular dependence of $\vec{P}$ and show that the total power radiated by the aerial is $\mu_0 p_0^2 \omega^4 / 12\pi c$ where $p_0$ is the amplitude and $\omega$ the frequency of the oscillating dipole.

13. Repeat question 12 for magnetic dipole radiation. Hence show that the ratio of the power radiated by an electric dipole aerial to that radiated by a magnetic dipole aerial is $(m_0/p_0 c)^2$, where $m_0$ is the amplitude of the magnetic dipole moment. Argue that this ratio $\ll 1$