YEAR 2: ELECTRICITY AND MAGNETISM

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PROBLEM SET 1: ELECTROSTATICS

† Trickier problems, for people who have finished all the others.

A. Revision (repeated from set 0)

1. (a) Calculate the electric field a distance r_0 above the midpoint of a straight line segment of length $2L$ which carries a charge density λ.
(b) Check the limits of your expressions for $r_0 \ll L$ and $r_0 \gg L$ and explain why they are sensible.
(c) For $r_0 \ll L$ rederive the result for the electric field using Gauss' theorem.

2. (a) The electric potential of a spherically symmetric charge distribution $\rho(r)$ is

$$V(r) = A \exp(-\lambda r).$$

Calculate (i) the electric field, (ii) $\rho(r)$.
(b) For a charge distribution $\rho(r) = \epsilon_0 \lambda A \left(\frac{2}{r} - \lambda\right) \exp(-\lambda r)$ calculate (without any reference to part (a)) (i) the electric field, (ii) the electric potential.

3. A metal sphere of radius r_1 is surrounded by a thick concentric metal shell of inner and outer radii r_2 and r_3. The sphere carries charge q and the shell is uncharged.
(a) Find the surface charge density at r_1, r_2 and r_3.
(b) Find the potential at the centre, relative to infinity.
(c) The outer surface is grounded. How do the answers to (a) and (b) change?

B. Poisson and Laplace equations

4. The vertical potential gradient above the earth is 110 $\mathrm{Vm^{-1}}$ at 100m and 25 $\mathrm{Vm^{-1}}$ at 1000m. Estimate the mean electrostatic charge density of the atmosphere between these heights?

5. An infinitely long rectangular pipe running parallel to the z-axis has three grounded metal sides, at $x = 0$, $y = 0$ and $y = a$. The fourth side, at $x = b$, is maintained at a potential $V_0(y)$.
(a) Find the potential inside the pipe for general $V_0(y)$.
(b) Calculate the potential explicitly for $V_0(y) = V_0$ (constant).
(c) Calculate the net charge per unit length on the side of the pipe at $x = 0$.

6. (a) Solve Laplace's equation in cylindrical coordinates using separation of variables and assuming cylindrical symmetry (no dependence on z). (Don't forget the solution for which the separation constant in zero – give an example where this solution will be needed.)
(b) Find the potential outside an infinitely long metal pipe of radius R placed at right angles to a uniform electric field $\vec{E}_0$. Where have you chosen the zero of potential?

7. (a) A charge density $\sigma = k \cos\theta$ is glued over the surface of a spherical shell of radius R. Find the potential inside and outside the sphere.
(b) Calculate the dipole moment of this charge distribution. Hence find the approximate potential at points far from the sphere and comment on the higher multipoles of the charge distribution.

8†. (Finals 1995) (i) A solid conducting sphere of radius a is surrounded by a concentric conducting spherical shell of internal radius b. The shell is earthed and the space between the sphere and the shell is empty. When the sphere is raised to potential V_0, find the potential function in the space between the conductors and thus calculate the total charge Q_0 on the sphere.
(ii) The centre of the outside shell is then displaced through a small distance δ while keeping V_0 constant. The magnitude of δ is such that the inner surface of the shell is adequately described by the equation $r = b + \delta\cos\theta$, where r is the magnitude of the radius vector $\vec{r}$ from the centre of the sphere to a point on the shell, and θ is the angle between $\vec{r}$ and the direction of the displacement vector $\vec{\delta}$. Show that, to leading order in δ, the surface charge density on the sphere is

$$\sigma = \frac{Q_0}{4\pi}\left(\frac{1}{a^2} - \frac{3\delta\cos\theta}{b^3 - a^3}\right).$$

Sketch the electric field. Using your sketch or otherwise discuss whether the system described in (i) is stable with respect to small displacements of its components.

C. Multipole expansion

9. Show that the electric field of a dipole can be written in the coordinate-free form

$$\vec{E}(r) = \frac{\{3(\vec{p}.\hat{\vec{r}})\hat{\vec{r}} - \vec{p}\}}{4\pi\epsilon_0 r^3}$$

where $\hat{\vec{r}}$ is the unit vector along $\vec{r}$.

10. An electric quadrupole is formed by a charge $-2q$ at the origin and charges $+q$ at the points $(0, 0, \pm a)$.

(a) Show that the potential for $r \gg a$ is $V = qa^2(3\cos^2\theta - 1)/4\pi\epsilon_0 r^3$.

(b) Show that there is no translational force or couple on the quadrupole in a uniform field.

(c) Show that the couple on the quadrupole due to a point charge Q placed a large distance r away from it is $\Gamma = 3qa^2 \sin 2\theta \; Q/4\pi\epsilon_0 r^3$.

11†. An electric dipole is situated at the origin and points along $\vec{z}$. An electric charge is released from rest at a point in the (x, y) plane. Show that it swings back and forth in a semicircular arc about the origin.

D. Polarisable materials

12. A parallel plate capacitor with plates of area A and separation d and carrying a charge Q, has a block of dielectric of relative permittivity ϵ, cross sectional area A and thickness t inserted next to the positive plate.

(a) Find the values of, and sketch, $\vec{E}$, $\vec{D}$ and $\vec{P}$ in the space between the plates, in both air and dielectric.

(b) Write down an expression for the bound charge on the surface of the dielectric and check that your answers obey Gauss' theorem.

(c) Derive an expression for the capacitance of the capacitor.

13. A sphere of radius R carries a polarization

$$\vec{P}(\vec{r}) = k\vec{r}$$

where k is a constant.

(a) Calculate the bound charges.

(b) Find the electric field inside and outside the sphere.

(c) Write down $\vec{D}$ inside and outside.

(d) Check that your answers are consistent with Gauss's law for $\vec{D}$.

14. The field inside a large piece of dielectric is $\vec{E}_0$ and the polarisation is $\vec{P}$.

(i) A long, needle-shaped cavity running parallel to $\vec{P}$ is hollowed out of the material. Find the field and displacement at the centre of the cavity in terms of $\vec{E}_0$ and $\vec{P}$.

(ii) Repeat for a thin, wafer-shaped cavity perpendicular to $\vec{P}$.

(iii) Repeat for a small spherical cavity.

(Assume that the cavities are small enough that $\vec{E}_0$ and $\vec{P}$ are essentially uniform.)

15. (from Finals 1993) (a) Derive the boundary conditions that must be satisfied by the electric field $\vec{E}$ and the electric displacement $\vec{D}$ at an interface between two uniform dielectric media.

(b) A sphere of radius R and relative permittivity ϵ_1 is placed in an infinite medium of relative permittivity ϵ_2. A point dipole $\vec{p}$ is placed at the centre of the sphere. Find expressions for the electric potential inside and outside the sphere.

(c) Verify that if $\epsilon_1 = \epsilon_2$ your expressions reduce to the usual dipole potential.

16. Two long, coaxial, cylindrical conducting surfaces of radii a and b are lowered vertically into a liquid dielectric. If the liquid rises an average height h between the electrodes when a potential difference V is established between them show that the electric susceptibility of the liquid is

$$\chi_e = \frac{(b^2 - a^2)\rho g h \ln(b/a)}{\epsilon_0 V^2}$$

where ρ is the density of the liquid, g is the acceleration due to gravity and the susceptibility of air is neglected.