# COMPLEX NUMBERS AND DIFFERENTIAL EQUATIONS

## PROBLEM SET 4

more challenging problems for eg the vacation or revision

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## Complex Numbers

1. (i) Obtain and sketch the locus in the complex plane defined by Re $z^{-1} = 1$. On the same picture sketch the locus defined by Im $z^{-1} = 1$. At what angle do these loci intersect one another? Show that the unit circle touches both loci but crosses neither of them.

(ii) Make a sketch of the complex plane showing a typical pair of complex numbers $z_1$ and $z_2$ which satisfy the equations

$$z_2 - z_1 = (z_1 - a)e^{2\pi i/3},$$
$$a - z_2 = (z_2 - z_1)e^{2\pi i/3}$$

where $a$ is a real positive constant. Describe the geometrical figure whose vertices are $z_1$, $z_2$ and $a$.

2. The polynomial $f(z)$ is defined by

$$f(z) = z^5 - 6z^4 + 15z^3 - 34z^2 + 36z - 48.$$

Show that the equation $f(z) = 0$ has two purely imaginary roots. Hence, or otherwise, factorize $f(z)$, and find all of its roots. Check that the sum and product of the roots take the expected values.

3. Show that the equation $(z+1)^n - e^{2in\theta}(z-1)^n = 0$ has roots $z = -i\cot(\theta + r\pi/n)$, $\quad r = 0, n-1$. Hence show that

$$\Pi_{r=1}^{n} \cot\left(\theta + \frac{r\pi}{n}\right) = \begin{cases} (-1)^{n/2}, & \text{for n even,} \\ (-1)^{(n-1)/2}\cot n\theta, & \text{for n odd.} \end{cases}$$

4. Find all the roots, real and complex, of the equation $z^3 - 1 = 0$. If $\omega$ is one of the complex roots prove that $1 + \omega + \omega^2 = 0$. Find the sums of the following series:

$$S_1 = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \ldots; \qquad S_2 = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \ldots; \qquad S_3 = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \ldots.$$

## Differential Equations

5. When a varying couple $I\cos nt$ is applied to a torsional pendulum with natural period $2\pi/m$ and moment of inertia $I$, the angle of the pendulum satisfies the equation of motion $\frac{d^2\theta}{dt^2} + m^2\theta = \cos nt$. The couple is first applied at time $t = 0$ when the pendulum is at rest in equilibrium. Show that in the subsequent motion the root mean square angular displacement is $1/\left|m^2 - n^2\right|$ when the average is taken over a time large compared to $1/\left|m - n\right|$. Discuss the motion as $|m - n| \mapsto 0$.

6. Prove that

$$\frac{d^2\theta}{dt^2} = \frac{1}{2}\frac{du}{d\theta}$$

where $u = \left(\frac{d\theta}{dt}\right)^2$.

A simple pendulum with damping proportional to the square of its velocity is described by the equation

$$2\frac{d^2\theta}{dt^2} + k\left(\frac{d\theta}{dt}\right)^2 = -\lambda \sin\theta$$

where $\theta$ is the angular displacement from the downwards vertical and $k$ and $\lambda$ are constants. By writing this equation in terms of the variable $u$, or otherwise, obtain an expression for the square of the angular velocity of the pendulum as a function of $\theta$.

The pendulum is given an initial angular velocity $\omega_0$ at its equilibrium position $\theta = 0$. Show that it will just reach the horizontal if

$$\omega_0^2 = \frac{\lambda}{1+k^2}\left(ke^{\frac{k\pi}{2}} + 1\right).$$

7. The position vector of a particle of charge $q$ and mass $m$ that moves in a region of space where there is a magnetic field $\mathbf{B}$ and an electric field $\mathbf{E}$ obeys the equation of motion

$$m\frac{d^2\mathbf{r}}{dt^2} = q\left(\mathbf{E} + \frac{\mathbf{dr}}{\mathbf{dt}} \times \mathbf{B}\right).$$

If $\mathbf{B}$ is along the x-axis and $\mathbf{E}$ is along the z-axis, write down the equations of motion for each component of $\mathbf{r}$, and solve the resulting coupled differential equations.

At time t = 0 the charge is at the origin with velocity
(a) $v = (0; E = B; 0)$,
(b) $v = (0; E = 2B; 0)$,
(c) $v = (0; E = B; E = B)$.
Find, and sketch, its trajectory for each initial condition.

8. A mass $m$ is constrained to move in a straight line and is attached to a spring of strength $\lambda^2 m$ and a dashpot which produces a retarding force $\alpha m v$ where $v$ is the velocity of the mass. Find the displacement of the mass when an amplitude-modulated periodic force $Am\cos pt \sin\omega t$ with $p \ll \omega$ and $\alpha \ll \omega$ is applied to it.

Show that for $\omega = \lambda$ the displacement is the amplitude-modulated wave

$$x = -2\frac{\cos\omega t \sin(pt+\phi)}{\sqrt{4\omega^2p^2 + \alpha^2\omega^2}} \quad \text{where} \quad \tan\phi = \alpha/2p.$$

9. $y(x) = 1/x$ is one of the solutions of the differential equation

$$F(x,y) = x(x+1)\frac{d^2y}{dx^2} + (2-x^2)\frac{dy}{dx} - (2+x)y = 0.$$

Find a second linearly independent solution by setting $y_2(x) = y_1(x)u(x)$ (or by noting the sum of the coefficients in the equation).

Hence, using the variation of parameters method, find the general solution of

$$F(x,y) = (x+1)^2.$$