II STEADY CURRENTS AND MAGNETISM

A. MOSTLY REVISION

1. currents, conservation of charge
2. force on a current-carrying wire
3. Biot-Savart law
4. div $\underline{B} = 0$
5. Ampère's law
6. magnetostatic boundary conditions
7. summary of Maxwell with no time-dependence

II STEADY CURRENTS + MAGNETISM

A. MOSTLY REVISION

1. currents, charge conservation

$\underline{J}$ - current density

charge per unit area per unit time flowing across a plane ⊥ʳ to the line of flow

charge per carrier

$$\underline{J} = \rho \underline{v} = n e \underline{v}$$

charge density

no. of charge carriers per unit volume

I - current

charge per unit time flowing across a surface S

$$I = \int \underline{J} \cdot d\underline{S}$$

continuity equation : expresses conservation of charge

rate of decrease of charge in V = charge leaving V per unit time

$$-\frac{\partial}{\partial t}\int_V \rho \, dV = \int_S \underline{J} \cdot d\underline{S} = \int \text{div}\, \underline{J} \, dV$$

divergence thm.

true ∀ V ∴ $\boxed{\text{div}\, \underline{J} = -\frac{\partial \rho}{\partial t}}$

Ohm's law

experimentally for many materials, e.g. metals at constant temperature

$$\underline{J} = \sigma \underline{E}$$

$\uparrow$ conductivity (material property)

for a wire, length d, cross sectional area A, with a voltage V across it

$$E = \frac{V}{d} \qquad J = \frac{I}{A}$$

$$\therefore \frac{I}{A} = \frac{\sigma V}{d}$$

$$\therefore V = I \frac{d}{A\sigma}$$

resistance R

$$R = \frac{d}{A\sigma} = \frac{d\rho}{A}$$

ρ ← resistivity

σ ← conductivity

2. force on a current carrying wire

Lorentz force $\underline{F}_{mag} = q(\underline{v} \wedge \underline{B})$

force on a volume element $\delta\tau$ containing charge density ρ is

$$\delta \underline{F}_{mag} = \rho (\underline{v} \wedge \underline{B})\, \delta\tau$$

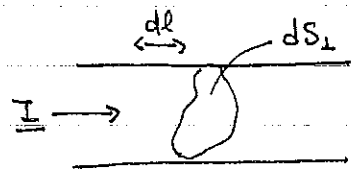
$$= (\underline{J} \wedge \underline{B})\, \delta\tau$$

$$\therefore \underline{F}_{mag} = \int_\tau \underline{J} \wedge \underline{B}\, d\tau$$

for a wire $\quad d\tau = dS_\perp \, dl$

element of area $\perp^r$ to wire

element of length along wire



$$\therefore \underline{F}_{mag} = \int_\tau (\underline{J} \wedge \underline{B}) \, dS_\perp \, dl$$

$$= \int_{wire} (\underline{I} \wedge \underline{B}) \, dl$$

$$\equiv \int_{wire} I \, (\underline{dl} \wedge \underline{B})$$

(whether we write $\underline{I}\, dl$ or $I\, \underline{dl}$ just a matter of convenience)

3. Biot - Savart law

starting point for magnetostatics (ie steady currents and fields constant in time) cf Coulomb's law for electrostatics

$$\underline{B}(\underline{r}) = \frac{\mu_0 I}{4\pi} \int_{wire} \frac{\underline{dl}' \wedge (\widehat{\underline{r}-\underline{r}'})}{|\underline{r}-\underline{r}'|^2}$$

magnetic field units Tesla $1\,T = 1\,N A^{-1} m^{-1}$

permeability of free space $\mu_0 = 4\pi \times 10^{-7}\, N A^{-2}$

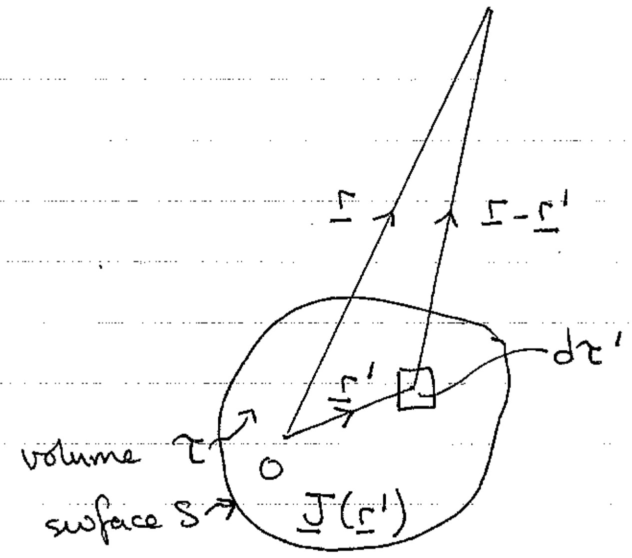
vector from element of current at $\underline{r}'$ to point where we are calculating the field $\underline{r}$.

NB1 for a volume distribution of charge

$$\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \int_\tau \frac{\underline{J}(\underline{r}') \wedge \widehat{|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|^2} \, d\tau'$$

NB2 superposition applies for magnetic fields

4. <u>to prove div $\underline{B} = 0$</u>



$$\operatorname{div} \underline{B} = \frac{\mu_0}{4\pi} \int_\tau \operatorname{div} \left\{ \frac{\underline{J}(\underline{r}') \wedge \widehat{(\underline{r}-\underline{r}')}}{|\underline{r}-\underline{r}'|^2} \right\} d\tau'$$

div is w.r.t. unprimed co-ordinates

integral is over the primed co-ordinates

(V2) $$\operatorname{div} \left\{ \underline{J}(\underline{r}') \wedge \frac{\widehat{(\underline{r}-\underline{r}')}}{|\underline{r}-\underline{r}'|^2} \right\} = \frac{\widehat{(\underline{r}-\underline{r}')}}{|\underline{r}-\underline{r}'|^2} \cdot \operatorname{curl} \underline{J}(\underline{r}') - \underline{J}(\underline{r}') \cdot \operatorname{curl} \left\{ \frac{\widehat{(\underline{r}-\underline{r}')}}{|\underline{r}-\underline{r}'|^2} \right\}$$

zero because $\underline{J}$ depends on the primed co-ordinates and curl is taken w.r.t. the unprimed ones.

$= 0$ (V6)

$$\boxed{\therefore \operatorname{div} \underline{B} = 0}$$

no magnetic monopoles (charges)

5. Ampère's law (curl $\underline{B} = \mu_0 \underline{J}$)

$$\text{curl}\, \underline{B}(\underline{r}) = \frac{\mu_0}{4\pi}\int_\tau \text{curl}\left\{\frac{\underline{J}(\underline{r}') \wedge (\widehat{\underline{r}-\underline{r}'})}{|\underline{r}-\underline{r}'|^2}\right\} d\tau'$$

V4: $\text{curl}\left\{\frac{\underline{J}(\underline{r}') \wedge (\widehat{\underline{r}-\underline{r}'})}{|\underline{r}-\underline{r}'|^2}\right\}$

$$= \underline{J}(\underline{r}')\,\text{div}\underbrace{\left\{\frac{(\widehat{\underline{r}-\underline{r}'})}{|\underline{r}-\underline{r}'|^2}\right\}}_{\alpha} - \underbrace{(\underline{J}(\underline{r}').\,\text{grad})\left\{\frac{(\widehat{\underline{r}-\underline{r}'})}{|\underline{r}-\underline{r}'|^2}\right\}}_{\beta}$$

+ zero terms which are derivatives w.r.t. x, y, z of $\underline{J}(\underline{r}')$

α (using V7) gives a contribution to curl $\underline{B}(\underline{r})$ of

$$\frac{\mu_0}{4\pi}\int_\tau \underline{J}(\underline{r}')\, 4\pi\, \delta^3(\underline{r}-\underline{r}')\, d\tau' = \mu_0 \underline{J}(\underline{r})$$

x component of β

$$+ \underline{J}(\underline{r}').\,\text{grad}'\left\{\frac{x-x'}{|\underline{r}-\underline{r}'|^3}\right\}$$

reason for doing this is so I can use divergence thm.
allowed because I am differentiating a function of $\underline{r}-\underline{r}'$

V3: $$= \text{div}'\left\{\underline{J}(\underline{r}')\frac{x-x'}{|\underline{r}-\underline{r}'|^3}\right\} - \frac{(x-x')}{|\underline{r}-\underline{r}'|^3}\,\text{div}'\,\underline{J}(\underline{r}')$$

put back into curl $\underline{B}$ formula and use divergence thm

zero from continuity equation for steady currents

$$\sim \int_S \frac{(x - x')}{|\underline{r} - \underline{r}'|^3} \; \underline{J}(\underline{r}') \cdot d\underline{S}'$$

zero because, by construction τ is the volume that includes all the currents $\therefore$ none are flowing through S

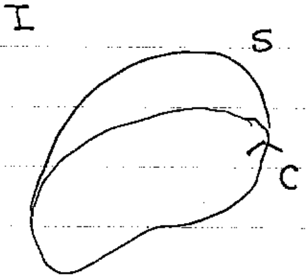
$$\boxed{\therefore \; \text{curl } \underline{B} = \mu_0 \underline{J}(\underline{r})}$$ Ampères law

$$\int_S \text{curl } \underline{B} \cdot d\underline{S} = \mu_0 \int_S \underline{J} \cdot d\underline{S}$$

Stokes' thm

definition of I

$$\boxed{\oint_C \underline{B} \cdot d\underline{l} = \mu_0 I}$$

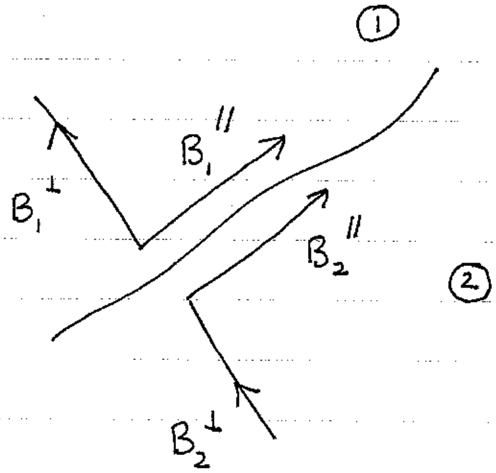


total current passing through any surface S spanning C

N.B. to calculate $\underline{B}$ in general need Biot-Savart, but in situations of high symmetry, Ampère much easier

cf -- $\underline{E}$ . . . - Coulomb . . .
.. -- - Gauss much easier

6. magnetostatic boundary conditions

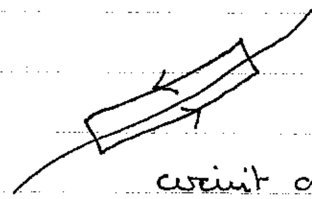


1. $\operatorname{div} \underline{B} = 0$

$$\therefore \int_S \underline{B} \cdot d\underline{S} = 0$$

Gaussian cylinder across surface $\Rightarrow \boxed{B_1^{\perp} = B_2^{\perp}}$

2. $\oint \underline{B} \cdot d\underline{l} = \mu_0 I$ (Ampère's law)



circuit across surface

$$\boxed{B_2^{\parallel} - B_1^{\parallel} = \mu_0 I^{s}}$$

surface current threading loop
(out of paper +ve ; r.h. rule)

7. summary so far (Maxwell without time dependence)

$$\operatorname{div} \underline{E} = \frac{\rho}{\varepsilon_0} \qquad \text{(Gauss)}$$

$$\operatorname{curl} \underline{E} = 0$$

$$\operatorname{div} \underline{B} = 0$$

$$\operatorname{curl} \underline{B} = \mu_0 \underline{J} \qquad \text{(Ampère)}$$