4. GUIDED WAVES

A. Transmission lines

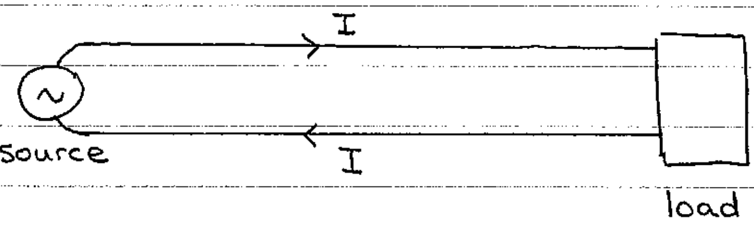
1. Introduction
2. Telegraph equations v, Z
3. boundary between transmission lines
4. termination by a load
5. coaxial line
6. e.m. waves on a coaxial line

B. Waveguides

GUIDED WAVES

A. Transmission Lines

1. **Introduction** Consider the circuit

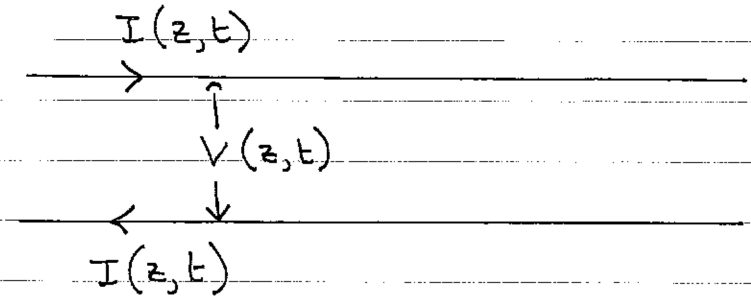


unless length of line ≪ λ of radiation flowing along it
or, equivalently, time signal takes to travel along wire ≪ $\frac{1}{\omega}$ ← freq. of source

need to consider wires as an explicit element in the circuit called a transmission line
(we shall assume that the distance between the wires remains constant)
the necessity to connect source and load by a transmission line causes

- delays (e.g. ~ 5 ns m^{-1} in typical dielectric cable connecting 2 computers, say)
- reflections if line not matched to load ← i.e. same impedance
- dispersion

2. **Telegraph equations**



L, C inductance, capacitance per unit length of line

$$V(z+dz,t) = V(z,t) - L\,dz\,\frac{\partial I(z,t)}{\partial t}$$

$\downarrow$ Taylor expansion

$$= V(z,t) + \frac{\partial V}{\partial z}(z,t)\,dz$$

$$\therefore \frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t} \quad (1)$$

Let Q be the charge between z and $z+\delta z$

$$C\,\delta z\,V = Q$$

$$C\,\delta z\,\frac{\partial V}{\partial t} = \frac{\partial Q}{\partial t}$$

but $\frac{\partial Q}{\partial t} = I(z,t) - I(z+\delta z,t) = -\frac{\partial I}{\partial z}\delta z$

$$\therefore \frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t} \quad (2)$$

mpairing $\frac{\partial}{\partial z}$ (1) and $\frac{\partial}{\partial t}$ (2) $\Longrightarrow$

$$\boxed{\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2}}$$

wave equation with velocity $v = \frac{1}{\sqrt{LC}}$

$$V(z,t) = f(z-vt) + g(z+vt)$$

NB $f'(u) \equiv \frac{df}{du}$
ie differential of f with respect to its argument

using (2)

$$\frac{\partial I}{\partial z} = Cvf' - Cvg'$$

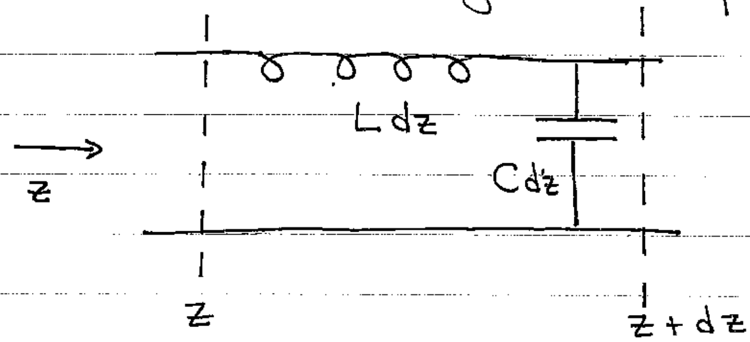
$$\therefore I = Cvf - Cvg = \sqrt{\frac{C}{L}}\,f - \sqrt{\frac{C}{L}}\,g$$

$$\boxed{\therefore \; Z = \frac{V}{I} = \pm\sqrt{\frac{L}{C}}}$$

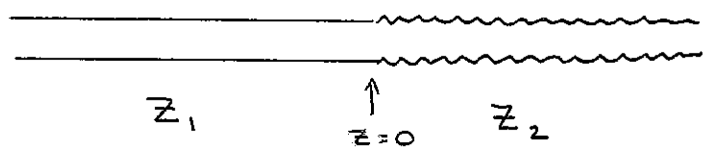
+ve for waves travelling to $+\hat{z}$
−ve " " " " $-\hat{z}$

characteristic impedance of the transmission line

N.B. equivalent circuit for a length dz of loss free line:



3. <u>boundary between transmission lines</u>



$V_i e^{j(\omega t - k_1 z)} \rightarrow$ $\qquad$ $V_t e^{j(\omega t - k_2 z)} \rightarrow$

$\leftarrow V_r e^{j(\omega t + k_1 z)}$

At $z = 0$ $\quad V_1 = V_2$

$$\therefore V_i + V_r = V_t$$

$$I_1 = I_2$$

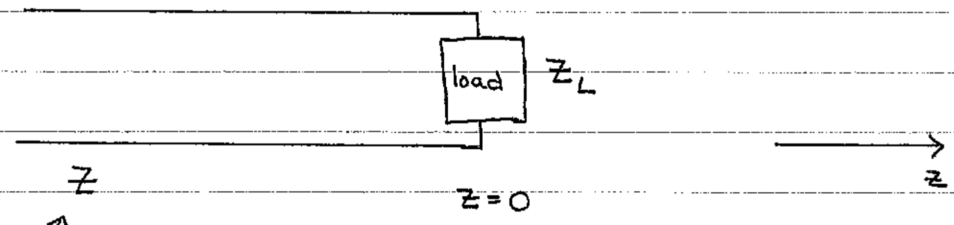
$$\therefore \frac{V_i}{Z_1} - \frac{V_r}{Z_1} = \frac{V_t}{Z_2}$$

$$\therefore \frac{V_r}{V_i} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \qquad \frac{V_t}{V_i} = \frac{2Z_2}{Z_2 + Z_1}$$

$$\text{Power} = IV = \frac{V^2}{Z}$$

$$\therefore \text{expect} \quad \frac{V_i^2}{Z_1} = \frac{V_r^2}{Z_1} + \frac{V_t^2}{Z_2} \qquad \checkmark$$

4. Termination by a load



characteristic impedance of line

$\longrightarrow$ $V_i e^{j(\omega t - kz)}$

$\longleftarrow$ $V_r e^{j(\omega t + kz)}$

$$\therefore V = V_i e^{j(\omega t - kz)} + V_r e^{j(\omega t + kz)}$$

$$I = \frac{V_i}{Z} e^{j(\omega t - kz)} - \frac{V_r}{Z} e^{j(\omega t + kz)}$$

at $z = 0$ "$V = I Z_L$"

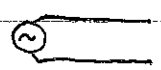
$$\therefore V_i + V_r = \left(\frac{V_i}{Z} - \frac{V_r}{Z} \right) Z_L$$

$$\boxed{\therefore \frac{V_r}{V_i} = \frac{Z_L - Z}{Z_L + Z}} \qquad (1)$$

impedance matched: $Z_L = Z$ $\therefore V_r = 0$

max. power transfer to load

open circuit $Z_L = \infty$ $I = 0$ at $z = 0$ $\therefore V_r = V_i$



short closed circuit $Z_L = 0$ $V = 0$ at $z = 0$ $\therefore V_r = -V_i$

(phase change of π)

input impedance

if line has length ℓ

$$Z_{in} \equiv \frac{V(-\ell)}{I(-\ell)} = \left\{ \frac{V_i e^{jk\ell} + V_r e^{-jk\ell}}{V_i e^{jk\ell} - V_r e^{-jk\ell}} \right\} Z$$

$$\underset{(\text{using } ①)}{=} \left\{ \frac{(Z_L + Z)\, e^{jkl} + (Z_L - Z)\, e^{-jkl}}{(Z_L + Z)\, e^{jkl} - (Z_L - Z)\, e^{-jkl}} \right\} Z$$

for a $\lambda/2$ line $\quad e^{\pm jkl} = e^{\pm j\pi} = -1$

$$\therefore Z_{in} = Z_L$$

for a $\lambda/4$ line $\quad e^{\pm jkl} = e^{\pm j\pi/2} = \pm i$

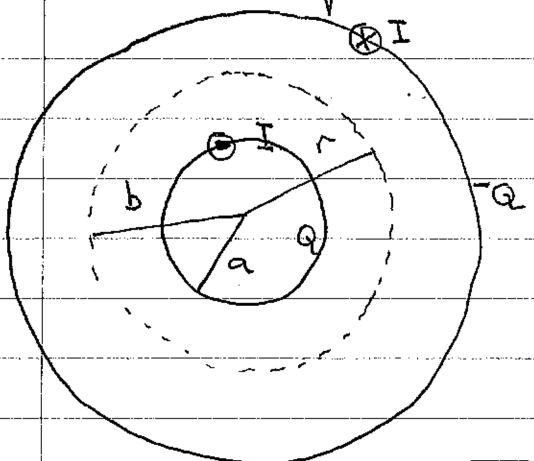
$$\therefore Z_{in} = \frac{Z^2}{Z_L}$$

(can be placed between a long line & a load to aid impedance matching)

N.B. in general $\quad Z_L$ and Z_{in} are complex
Z is real for a loss-free line

5. <u>Coaxial transmission line</u> (to calculate v, Z)

(i) <u>capacitance</u>
(all extensive quantities defined per unit length)



Gauss:

$$E \cdot 2\pi r = \frac{Q}{\epsilon\epsilon_0}$$

$$\therefore V = -\int_b^a \frac{Q \, dr}{2\pi\epsilon\epsilon_0 r} = \frac{Q}{2\pi\epsilon\epsilon_0} \ln\frac{b}{a}$$

$$\boxed{\therefore C = \frac{Q}{V} = \frac{2\pi\epsilon\epsilon_0}{\ln\frac{b}{a}}}$$

(ii) <u>inductance</u>

Ampère:

$$B \cdot 2\pi r = \mu\mu_0 I$$

$$\therefore \phi = \int_a^b \frac{\mu\mu_0 I}{2\pi r}\,dr = \frac{\mu\mu_0 I}{2\pi}\ln\frac{b}{a}$$

$$\boxed{\therefore L = \frac{\phi}{I} = \frac{\mu\mu_0}{2\pi}\ln\frac{b}{a}}$$

$$\therefore v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\mu_0\epsilon\epsilon_0}} \quad \text{as expected}$$

the waves are e.m. waves – can either view them in terms of V, I in wires or B, E in the space around the wires

6. em waves travelling along a coaxial line

need a solution which obeys Maxwell's equations and the boundary conditions on the surfaces of the conductors:

$B^{\perp} = 0$ (from div $\underline{B} = 0$)

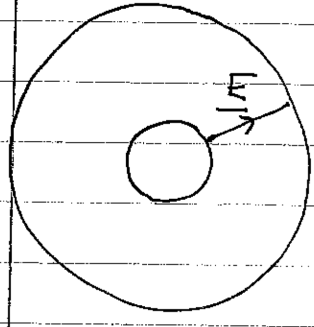
$E^{\parallel} = 0$ ($\underline{E} = 0$ inside conductors and $E^{\parallel}$ continuous)

check that a solution that works is

$$\underline{E} = \frac{A}{r} e^{j(\omega t - kz)}\,\hat{\underline{r}} \ ; \quad \underline{B} = \frac{A\sqrt{\epsilon\epsilon_0\mu\mu_0}}{r} e^{j(\omega t - kz)}\,\hat{\underline{\theta}}$$

amplitude

can fix A by looking at the line voltage



$$V = -\int_b^a \frac{A}{r} e^{j(\omega t - kz)}\,dr$$

$$= A\ln\frac{b}{a}\, e^{j(\omega t - kz)}$$

$$\equiv V_0 e^{j(\omega t - kz)}$$

$$\therefore \underline{E} = \frac{V_0}{\ln b/a} \cdot \frac{1}{r} e^{j(\omega t - kz)} \hat{\underline{r}}$$

$$\underline{B} = \frac{V_0}{\ln b/a} \sqrt{\mu\mu_0 \epsilon\epsilon_0} \frac{1}{r} e^{j(\omega t - kz)} \hat{\underline{\theta}}$$

we have two other boundary conditions:

on $E^{\perp}$ (ie. Gauss' law) can be used to give the charge on the plates

on $B^{\parallel}$ (ie Ampère's law)
assuming $B^{\parallel}$ is small inside the conductor and can be ignored this can be used to give the current

$$2\pi a B(r=a) = \mu\mu_0 I$$

$$\therefore I_0 = \frac{2\pi}{\mu\mu_0} \frac{V_0}{\ln b/a} \sqrt{\epsilon\epsilon_0 \mu\mu_0}$$

$$\therefore Z \equiv \frac{V_0}{I_0} = \sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}} \frac{\ln(b/a)}{2\pi} \equiv \sqrt{\frac{L}{C}}$$ as before

2. Waveguides

- hollow metal conductor with cross section of constant shape
- em waves reflect off walls to set up an e.m. wave which transports energy along guide.
- if walls perfectly conducting, there are no losses
- most efficient method of power transport for frequencies $\gtrsim 10^3$ MHz

Wave equation for lossless waveguides

assume:

1. wave propagates in a linear dielectric (metallic walls are O.K: they just provide boundary conditions)
2. zero attenuation
3. propagation is in a straight line along $\hat{z}$

(N.B. differs from Sec. 3B (plane waves in dielectric) because waves are not plane $\therefore \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \neq 0$)

Maxwell's equations $\Rightarrow$

$$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{v}\right)^2 - k^2 \right\} E_z = 0 \qquad (1)$$

$$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{v}\right)^2 - k^2 \right\} B_z = 0 \qquad (2)$$

T.E. modes $\quad E_z = 0$

solve (2) using separation of variables, and boundary condition that $B^{\perp} = 0$ to give

$$B_z = \sum_{m,n} B_{m,n} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \, e^{j(\omega t - kz)}$$

and a dispersion relation

$$k^2 = \left(\frac{\omega}{v}\right)^2 - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}$$

for a given mode to propagate

$$\omega > \omega_{m,n} = v\pi\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{1/2}$$