

I ELECTROSTATICS

C. MULTIPOLES

1. Electric dipole

a. potential

b. electric field

c. energy in a constant field

d. force in a constant field

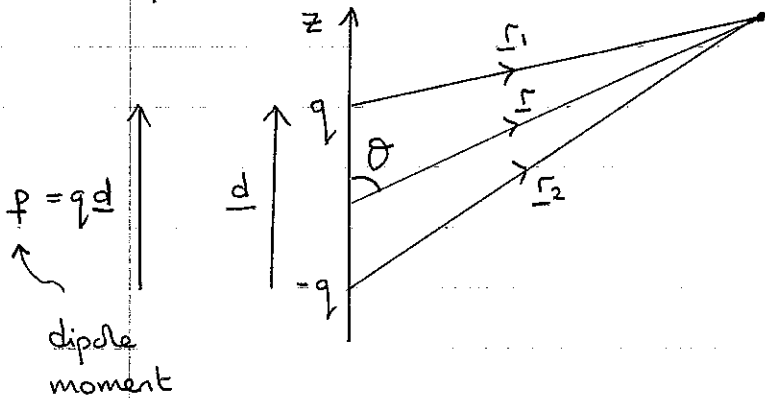
e. torque in a constant field

f. force in a field that varies

2. The multipole expansion

Electric Dipole

a. potential



N.B. \mathbf{p} is along the z -axis of a spherical polar co-ordinate system

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

$$\frac{1}{r_{1,2}} = \left\{ r^2 + \left(\frac{d}{2}\right)^2 \mp \frac{2rd \cos\theta}{2} \right\}^{-\frac{1}{2}}$$

$$= r^{-1} \left\{ 1 + \left(\frac{d}{2r}\right)^2 \mp \frac{d}{r} \cos\theta \right\}^{-\frac{1}{2}}$$

$$= r^{-1} \left(1 \pm \frac{d}{2r} \cos\theta + O\left(\left(\frac{d}{r}\right)^2\right) \right)$$

$$\therefore V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} r^{-1} \cdot \frac{2d \cos\theta}{2r}$$

$$V(\mathbf{r}) = \frac{p \cos\theta}{4\pi\epsilon_0 r^2} \equiv \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}$$

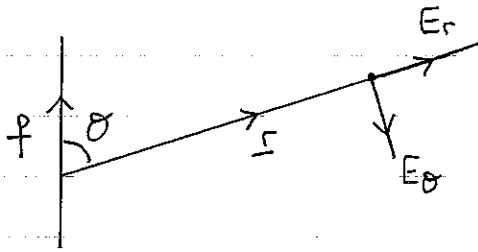
N.B.

b. electric field

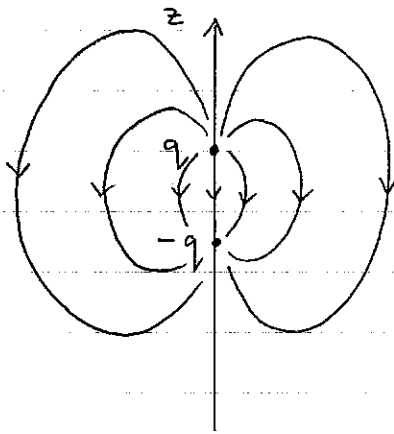
$$\underline{\mathbf{E}} = -\text{grad } V = \left(-\frac{\partial V}{\partial r}, -\frac{1}{r} \frac{\partial V}{\partial \theta}, -\frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi} \right)$$

$$= \left(\frac{2p \cos \theta}{4\pi \epsilon_0 r^3}, \frac{p \sin \theta}{4\pi \epsilon_0 r^3}, 0 \right)$$

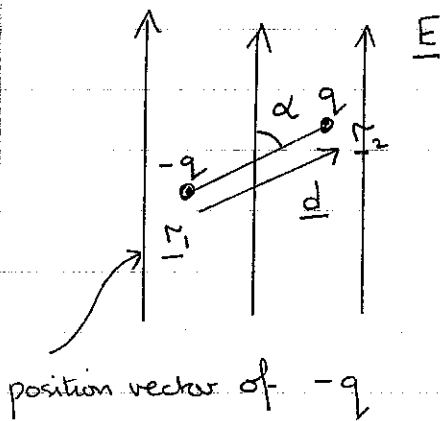
↑
N.B.



careful with co-ordinates



energy in a constant field



α is the angle between \underline{d} and \underline{E}

$$U = q V(\underline{r}_2) - q V(\underline{r}_1)$$

$$V(\underline{r}_2) - V(\underline{r}_1) = - \int_{\underline{r}_1}^{\underline{r}_2} \underline{E} \cdot d\underline{l}$$

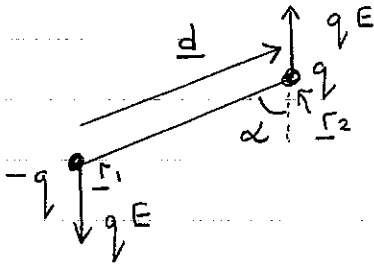
$$= - \underline{E} \cdot (\underline{r}_2 - \underline{r}_1)$$

$$= - \underline{E} \cdot \underline{d}$$

$$\therefore \underline{U} = -q \underline{d} \cdot \underline{E} = -p \cdot \underline{E}$$

d force on a dipole in a constant field is zero

e torque on a dipole in a constant field



$$M = qE d \sin \alpha = pE \sin \alpha$$

$$\underline{M} = \underline{p} \wedge \underline{E}$$

↑ out of paper
 ↑ right magnitude and out of paper

f force on dipole in field that is not constant (on the scale of d)

$$\underline{F} = q (\underline{E}(r_2) - \underline{E}(r_1))$$

$$\underline{E}(r_2) - \underline{E}(r_1) = \left(\frac{r_2 - r_1}{d} \cdot \text{grad} \right) \underline{E} = (\underline{d} \cdot \text{grad}) \underline{E}$$

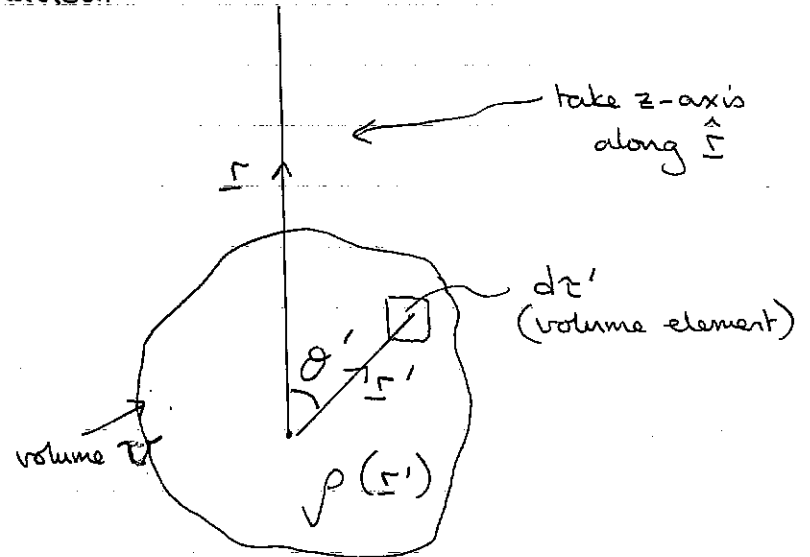
(e.g. x component

$$E_x(r_2) - E_x(r_1) = d_x \frac{\partial E_x}{\partial x} + d_y \frac{\partial E_x}{\partial y} + d_z \frac{\partial E_x}{\partial z})$$

$$\underline{F} = (\underline{p} \cdot \text{grad}) \underline{E}$$

2. Multipole Expansion

expansion of the potential of a localised charge distribution in powers of r^{-1} - so we can look at 'far field' behaviour



$$V(r) = \int_V \frac{\rho(r') d\tau'}{4\pi\epsilon_0 |\underline{r} - \underline{r}'|}$$

$$|\underline{r} - \underline{r}'|^{-1} = \{r^2 + r'^2 - 2rr'\cos\theta'\}^{-1/2}$$

$$= r^{-1} \left\{ 1 + \frac{r'^2}{r^2} - \frac{2r'\cos\theta'}{r} \right\}^{-1/2}$$

small \therefore \Downarrow binomial expansion

$$= \frac{1}{r} \left\{ 1 - \frac{1}{2} \left(\frac{r'^2}{r^2} - \frac{2r'\cos\theta'}{r} \right) + \frac{3}{8} \left(\frac{r'^2}{r^2} - \frac{2r'\cos\theta'}{r} \right)^2 - \frac{5}{16} \left(\frac{r'^2}{r^2} - \frac{2r'\cos\theta'}{r} \right)^3 + \dots \right\}$$

\Downarrow collect terms

$$= \frac{1}{r} \left\{ 1 + \frac{r'}{r} \cos\theta' + \frac{r'^2}{r^2} \frac{1}{2} (3\cos^2\theta' - 1) + \frac{r'^3}{r^3} \frac{1}{2} (5\cos^3\theta' - 3\cos\theta') + \dots \right\}$$

$$\equiv \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r} \right)^n P_n(\cos\theta')$$

$$\therefore V(\underline{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int_V (r')^n P_n(\cos\theta') \rho(\underline{r}') d\tau'$$

we have rewritten V as an expansion in powers of r^{-1}

monopole term $n=0$

$$V_0(\underline{r}) = \frac{1}{4\pi\epsilon_0 r} \underbrace{\int_V \rho(\underline{r}') d\tau'}_{\text{total charge}}$$

if total charge is zero the dominant term will be the

dipole term $n=1$

$$V_1(\underline{r}) = \frac{1}{4\pi\epsilon_0 r^2} \int_V r' \cos\theta' \rho(\underline{r}') d\tau'$$

$$r' \cos\theta' = \underline{r}' \cdot \hat{\underline{r}}$$

$$\therefore V_1(\underline{r}) = \frac{1}{4\pi\epsilon_0 r^2} \hat{\underline{r}} \cdot \underbrace{\int_V \underline{r}' \rho(\underline{r}') d\tau'}_{\text{dipole moment } \underline{p} \text{ (definition)}}$$

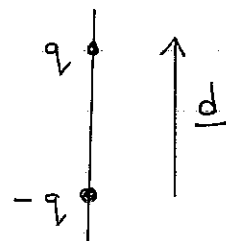
dipole moment \underline{p} (definition)

$$V_1(\underline{r}) = \frac{\underline{p} \cdot \hat{\underline{r}}}{4\pi\epsilon_0 r^2}, \text{ as before}$$

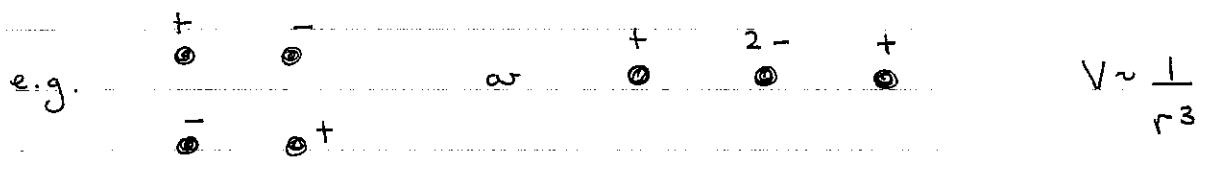
the dipole we are used to is a special case of a localized charge distribution with total charge zero and

$$\underline{p} = \int_V \underline{r}' \left\{ q \delta\left(\underline{r}' - \frac{\underline{d}}{2}\right) - q \delta\left(\underline{r}' + \frac{\underline{d}}{2}\right) \right\} d\tau'$$

$$= q \underline{d} \quad \text{as expected}$$



if total charge $Q = 0$ and $p = 0$ quadrupoles term dominates



next term is the octopole

