I ELECTROSTATICS

B Poisson and Laplace Equations

1. The equations and uniqueness.
2. Poisson 1D
3. Laplace 3D Cartesians : field in a slit
4. Laplace 3D spherical polars: spherical conductor, in a uniform field
5. Laplace 3D cylindrical polars: cylinder with a fixed surface charge density

B. Poisson and Laplace Equations

1. The equations and uniqueness

Poisson : $\nabla^2 V = -\frac{\rho}{\epsilon_0}$

Laplace $\nabla^2 V = 0$

Give suitable boundary conditions the solution to these equations is unique so if a solution obeys the boundary conditions it must be the right solution.

a simple example of suitable b.c's is to specify V everywhere on the boundary of a given region

to demonstrate the uniqueness thm. consider 2 sol$^{\underline{ns}}$ of Laplace eq$^{\underline{n}}$ V_1, V_2 each of which obey the given b.c.'s

$$\nabla^2 V_1 = 0 \qquad \nabla^2 V_2 = 0$$

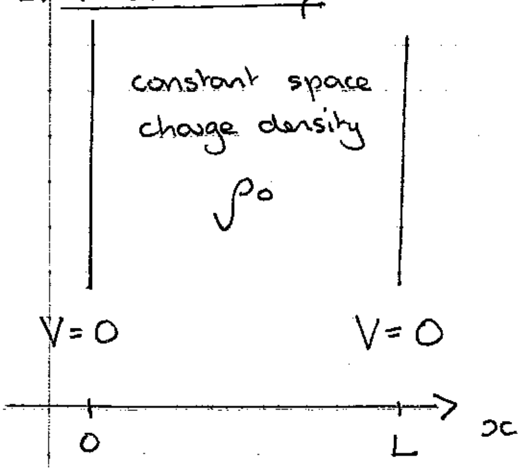
define $V_3 = V_1 - V_2$

$\therefore \nabla^2 V_3 = 0$ and $V_3 = 0$ on all boundaries

so it is physically sensible that $V_3 = 0$ everywhere (see eg Griffiths for a math. justification)

$$\therefore V_1 = V_2$$

2. Poisson 1D



equation $\frac{d^2 V}{dx^2} = -\frac{\rho_0}{\epsilon_0}$

boundary conditions

$V = 0$ at $x = 0$, $x = L$

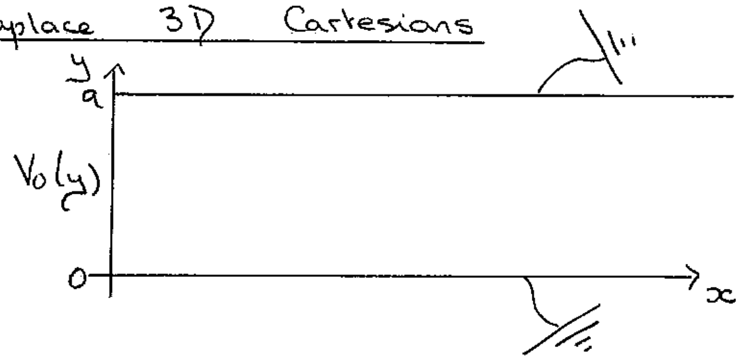
$$\therefore \frac{dV}{dx} = -\frac{\rho_0 x}{\epsilon_0} + C_1$$

$$V = -\frac{\rho_0 x^2}{2\epsilon_0} + C_1 x + C_2$$

$\Downarrow$ use boundary conditions to get C_1, C_2

$$V(x) = \frac{\rho_0}{2\epsilon_0} x (L - x)$$

3. Laplace 3D Cartesians



translationally invariant in z-direction $\therefore$ sol^n independent of z.

equation $\nabla^2 V = 0$

boundary conditions $V(x,0) \overset{①}{=} V(x,a) \overset{②}{=} 0$

as $x \to \infty$ $V \to 0$ ③

$V(0,y) = V_0(y)$ ④

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

assume $V = X(x)\,Y(y)$

$\therefore X''Y + XY'' = 0$ where $X'' = \frac{d^2X}{dx^2}$, etc.

$$\therefore \frac{X''}{X} + \frac{Y''}{Y} = 0$$

x, y are independent variables

$$\therefore \frac{X''}{X} = k^2 ; \quad \frac{Y''}{Y} = -k^2$$

constant, chosen to match boundary conditions as easily as possible

$$\therefore V(x,y) = \sum_k \left(A_k e^{kx} + B_k e^{-kx}\right)\left(C_k \sin ky + D_k \cos ky\right)$$
$$+ (A_0 x + B_0)(C_0 y + D_0)$$

term from constant of separation being zero

boundary conditions:

① $\Rightarrow D_k = 0$

② $\Rightarrow k = \frac{n\pi}{a}$

③ $\Rightarrow A_k = 0$

$$\therefore V(x,y) = \sum_n A_n \sin \frac{n\pi y}{a} e^{-kx}$$

④ $\Rightarrow$ $V(0,y) \equiv V_0(y) = \sum_n A_n \sin \frac{n\pi y}{a}$

$$\therefore A_n = \frac{2}{a}\int_0^a V\ \ y) \sin \frac{n\pi y}{a}\, dy$$

e.g. if $V_0(y) = V_0 y(a-y)$

check you can do this

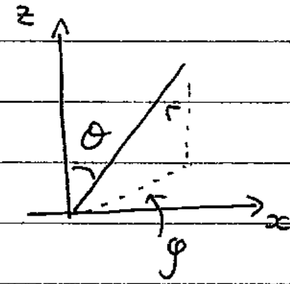
$$A_n = \frac{8 V_0 a^2}{n^3 \pi^3}, \quad n \text{ odd} \quad ; \quad 0, \quad n \text{ even}$$

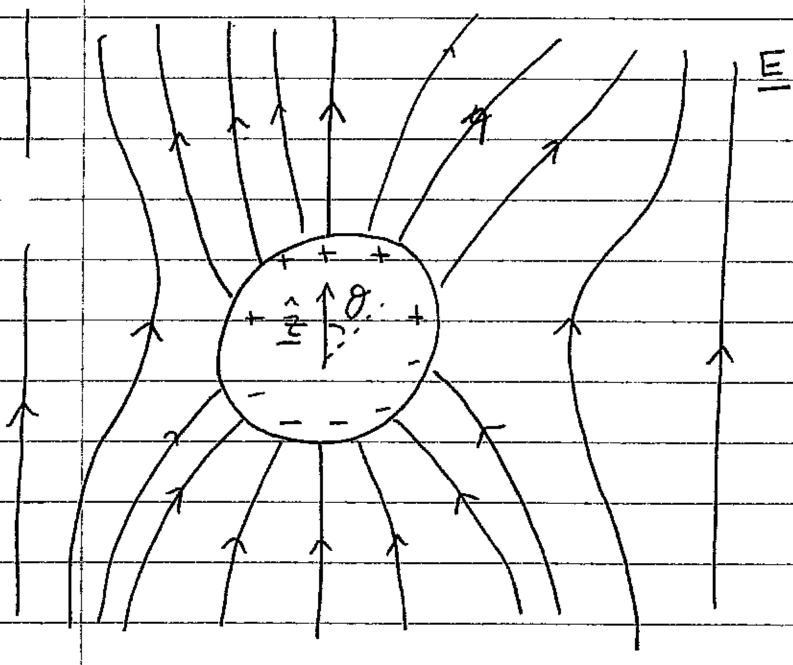
$$\therefore V(x,y) = \sum_{n \text{ odd}} \frac{8 V_0 a^2}{n^3 \pi^3} \sin \frac{n\pi y}{a} e^{-kx}$$

4. Laplace 3D spherical polars

spherical conductor, radius a, no net charge in a uniform $\underline{E}$-field. Find the potential everywhere

use spherical polars, take z-axis along $\underline{E}$ $\Rightarrow$ azimuthal symmetry (no φ dependence)





field lies $\perp^r$ to surface of a conductor because $E^{\parallel}=0$ inside and $E^{\parallel}$ continuous across surface

boundary conditions:

sphere equipotential ; choose $V=0$ on surface of sphere

as $r \to \infty$, $V \to -Er\cos\theta$

why?

$$V = -\int \underline{E}.\,d\underline{l} = -\int E\,dz = -Ez + C = -Er\cos\theta + C$$

when $\theta = \pi/2$ $V=0$ by symmetry $\therefore C=0$

summary: ① at $r=a$, $V=0$ $\forall\,\theta$

② as $r\to\infty$, $V\to -Er\cos\theta$

Equation:

Laplace, spherical polars

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\cancel{\frac{1}{r^2\sin\theta}\frac{\partial^2 V}{\partial\varphi^2}}=0$$

no φ-dependence

separate variables

see Math. Phys

$$V(r,\theta)=\sum_{l=0}^{\infty}\left\{A_l r^l+\frac{B_l}{r^{l+1}}\right\}P_l(\cos\theta) \qquad (3)$$

Legendre polynomials

$P_0(\cos\theta)=1$

$P_1(\cos\theta)=\cos\theta$

$P_2(\cos\theta)=\frac{1}{2}(3\cos^2\theta-1)$

$\vdots$

We need to find the A_l and B_l that match the b.c.'s (1) and (2)

we have to match a '$\cos\theta$' boundary condition

$\therefore$ we are likely to need the '$\cos\theta$' terms in (3) $(l=1)$

$\therefore$ try a solution (if it matches the boundary conditions it must be the correct solution due to uniqueness thm.)

$$V(r,\theta)=\left(A_1 r+\frac{B_1}{r^2}\right)\cos\theta$$

(2) $\Rightarrow$ $A_1=-E$

(1) $\Rightarrow$ $B_1=a^3E$

$$\therefore\ V(r,\theta) = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta$$

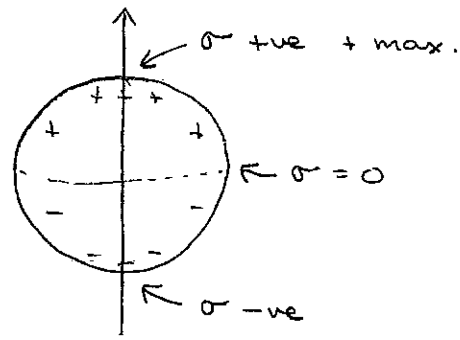
term due to external field

dipole term due to charge ~~induced~~ distribution on conductor

what is the induced surface charge density?

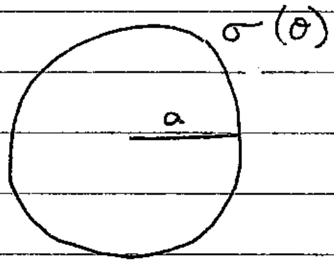
$$E^{\perp}\Big|_{r=a} = \frac{\sigma}{\epsilon_0} = -\frac{\partial V}{\partial r}\Big|_{r=a} = E_0\left(1 + \frac{2a^3}{r^3}\right)\cos\theta\Big|_{r=a}$$

$$\therefore\ \sigma = 3\epsilon_0 E_0 \cos\theta$$



5. Laplace 3D; cylindrical polars; specify $\sigma(\theta)$ (charge density) on surface of a cylinder of radius a.

What is the potential everywhere?



choose $\sigma(\theta) = \sigma_0 \sin\theta$

boundary conditions:

① at $r = a$ $\quad V$ continuous $\quad (E^{\parallel}$ continuous$)$

② $$E^{\perp}_{out} - E^{\perp}_{in} = \frac{\sigma(\theta)}{\varepsilon_0} \equiv \frac{\sigma_0 \sin\theta}{\varepsilon_0}$$

③ V_{in} must be finite at $r = 0$

④ $V_{out} \to 0$ as $r \to \infty$

equation:

Laplace, cylindrical polars, no z-dependence

$$V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \cancel{\frac{\partial^2 V}{\partial z^2}} = 0$$

solve by separation of variables

see math. phys. lectures

$$V = \sum_n \left(C_n r^n + D_n r^{-n}\right)\left(A_n \sin n\theta + B_n \cos n\theta\right) + C_0 \ln r + D_0$$

this term comes from putting the separation of variables constant to zero

it is the potential of a line charge

to fit b.c.'s ② $\sin\theta$ term looks most likely $\therefore$ try the $n = 1$ '$\sin\theta$' term zero from b.c. ③

$$V_{in} = \left(C^i r + \cancel{\frac{D^i}{r}} \right) \sin\theta$$

$$V_{out} = \left(\cancel{C^o r} + \frac{D^o}{r} \right) \sin\theta$$

zero from b.c. ④

(I'll suppress the ^{i} and ^{o} labels, just to make writing neater)

$$E_{in} = -\operatorname{grad} V_{in} = -\left(\underbrace{\frac{\partial V}{\partial r}}_{E^{\perp}}, \underbrace{\frac{1}{r}\frac{\partial V}{\partial \theta}}_{E^{\parallel}}, 0 \right)$$

$$= -\left(C\sin\theta, \; C\cos\theta, \; 0 \right)$$

$$E_{out} = -\operatorname{grad} V_{out} = -\left(-\frac{D}{r^2}\sin\theta, \; \frac{D}{r^2}\cos\theta, \; 0 \right)$$

$\therefore$ b.c. ① $\Rightarrow$ $\quad Ca = \dfrac{D}{a}$

b.c. ② $\Rightarrow$ $\quad \dfrac{D}{a^2}\sin\theta + C\sin\theta = \dfrac{\sigma_0}{\epsilon_0}\sin\theta$

$$\therefore \quad C = \frac{\sigma_0}{2\epsilon_0}, \quad D = \frac{\sigma_0 a^2}{2\epsilon_0}$$

$$\boxed{V_{in} = \frac{\sigma_0}{2\epsilon_0} r\sin\theta, \qquad V_{out} = \frac{\sigma_0 a^2}{2\epsilon_0 r}\sin\theta}$$