# Electromagnetic Waves

## C. ENERGY IN ELECTROMAGNETIC FIELDS

1. conservation of energy ... from which we get

(a) the Poynting vector

(b) stored energy in em field

2. check that $\frac{1}{2}\int \vec{D}.\vec{E}d\tau = \frac{1}{2}\int \rho V d\tau$

3. example: charging a capacitor

4. radiation pressure

## D. REFLECTON AND REFRACTION OF ELECTROMAGNETIC WAVES

1. normal incidence

2. general incidence

(a) laws of reflection and refraction

(b) Fresnel equations: $\vec{E}$ in plane of incidence

(c) Fresnel equations: $\vec{E}$ perpendicular to plane of incidence

(d) check conservation of energy

3. physical consequences of the Fresnel equations

(a) Brewster angle

(b) total internal reflection

(c) reflection from a metal

Maxwell:

$$\text{div } \underline{D} = \rho \quad (1)$$

$$\text{div } \underline{B} = 0 \quad (2)$$

$$\text{curl } \underline{E} = -\frac{\partial \underline{B}}{\partial t} \quad (3)$$

$$\text{curl } \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t} \quad (4)$$

$$\text{div } (\underline{E} \wedge \underline{H}) = \underline{H}.\text{curl } \underline{E} - \underline{E}.\text{curl } \underline{H} \quad (V1)$$

$$\text{div } (V\underline{D}) = V \text{ div } \underline{D} + \underline{D}.\text{grad } V \quad (V2)$$

# C. Energy in EM Fields

1. conservation of energy $\Rightarrow$ energy stored in EM fields
the Poynting vector, $\underline{P}$

rate of doing work by fields in volume $v$ = rate of decrease of energy stored in fields − rate at which energy is flowing out across surface $S$ of $v$

work done by fields when a volume $d\tau$ containing charge density $\rho$ is moved through $d\underline{l}$ is

$$dW = \underline{F} \cdot d\underline{l} = \rho \, d\tau \, (\underline{E} + \underline{v} \wedge \underline{B}) \cdot d\underline{l}$$

$$= \rho \, d\tau \, (\underline{E} + \cancel{\underline{v} \wedge \underline{B}}) \cdot \underline{v} \, dt$$

$$\therefore \quad = \underline{E} \cdot \underline{J} \quad d\tau \, dt \qquad (\text{using } \underline{J} = \rho \underline{v})$$

$\therefore$ rate of doing work by fields on a volume $v$ is

$$\frac{dW}{dt} = \int_v \underline{E} \cdot \underline{J}_f \, d\tau \qquad \ast$$

$$= \int_v \left\{ \underline{E} \cdot \text{curl } \underline{H} - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t} \right\} d\tau \qquad (\text{using Maxwell } ④)$$

$$= \int_v \left\{ \underline{H} \cdot \text{curl } \underline{E} - \text{div}(\underline{E} \wedge \underline{H}) - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t} \right\} d\tau \qquad (\text{using V1})$$

$$= \int_v \left\{ -\underline{H} \cdot \frac{\partial \underline{B}}{\partial t} - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t} - \text{div}(\underline{E} \wedge \underline{H}) \right\} d\tau \qquad (\text{Maxwell } ③)$$

$\downarrow$ if medium linear
$\underline{E} \cdot \frac{\partial \underline{D}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\underline{E} \cdot \underline{D})$
+ sim. for $\underline{B}, \underline{H}$

$\downarrow$ divergence thm

$$= -\frac{\partial}{\partial t}\int_v \underbrace{\left\{\frac{1}{2}\underline{E}.\underline{D} + \frac{1}{2}\underline{B}.\underline{H}\right\}}_{\substack{\downarrow \\ \text{energy stored in} \\ \text{field per unit volume}}} d\tau \quad - \underbrace{\int_S (\underline{E}\wedge\underline{H}).d\underline{S}}_{\substack{\downarrow \\ \text{rate at which energy} \\ \text{flows out across surface} \\ S \text{ of } V}}$$

the Poynting vector $\boxed{\underline{P} = \underline{E}\wedge\underline{H}}$ is the rate of flow of energy per unit area

N.B. formula is not the same as Griffiths. Here we have calculated the rate of doing work on the free currents (see $*$). The Griffiths version includes the work done on bound currents.

(2) stored energy : equivalence of formulae in terms of fields and charges :

does $\frac{1}{2}\int_v \underline{E}.\underline{D}\, d\tau = \frac{1}{2}\int_v V\rho\, d\tau$ ? (see section IA6)

$$\text{lhs.} = \frac{1}{2}\int_v \underline{E}.\underline{D}\, d\tau = -\frac{1}{2}\int_v \text{grad } V.\underline{D}\, d\tau$$

$$= \frac{1}{2}\int_v V \underbrace{\text{div }\underline{D}}_{\downarrow \text{ Maxwell } 1}\, d\tau - \frac{1}{2}\underbrace{\int_v \text{div }(V\underline{D})\, d\tau}_{\downarrow \text{ divergence thm.}} \quad (\text{using } V8)$$

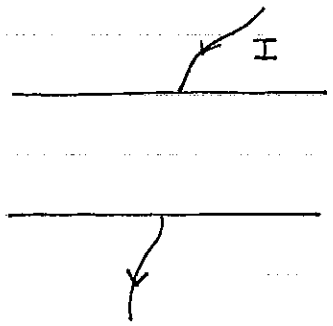
$$= \frac{1}{2}\int_v V\rho\, d\tau - \frac{1}{2}\int_S V\underline{D}.d\underline{S}$$

far from the charge distribution $V \sim \frac{1}{r}$, $D \sim \frac{1}{r^2}$, $S \sim r^2$

$\therefore$ this term $\to 0$ as $r \to \infty$

$\therefore$ equivalence works as long as we integrate over all space i.e. include all points where $\underline{E}, \underline{D} \neq 0$

(3) <u>an example: charging a parallel plate capacitor</u>



plates area $A$; spacing $d$ $\qquad d^2 \ll A$
(ie assume no fringing fields)

circumference $\tilde{C}$

filled with linear dielectric of permittivity $\epsilon$

charging with current $I$

charge on plates $Q$ ; $\quad I = \dfrac{dQ}{dt}$

capacitance $\quad C = \dfrac{\epsilon_0 A}{d}$

from Gauss' law $\quad DA = Q \quad \therefore D = \dfrac{Q}{A}, \quad E = \dfrac{Q}{\epsilon\epsilon_0 A}$

from Ampère's law $\quad H\tilde{C} = I$
($H$ at boundary)

<u>What is the rate of increase of stored energy?</u>

(a) <u>from stored energy formula</u>

$$U = \frac{Q^2}{2C} \qquad \therefore \frac{dU}{dt} = \frac{Q}{C}\frac{dQ}{dt}$$

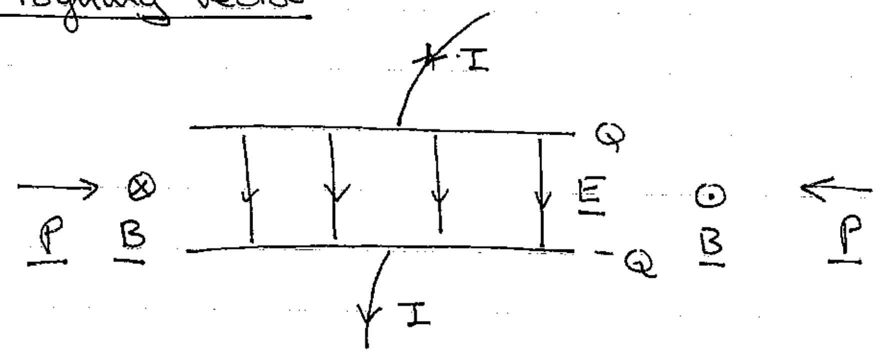
(b) <u>from energy stored in fields formula</u>

$$U = \frac{1}{2} E D A d = \frac{1}{2}\frac{Q}{\epsilon\epsilon_0 A} \cdot \frac{Q}{A} \cdot Ad = \frac{Q^2}{2C}$$

$$\therefore \frac{dU}{dt} = \frac{Q}{C}\frac{dQ}{dt}$$

N.B. For this problem energy stored in the magnetic fields can be ignored because $\underline{B}.\underline{H} \ll \underline{E}.\underline{D}$.

(c) from Poynting vector




$$U = P\tilde{C}d = E H \tilde{C} d$$

$$= \frac{Q}{\epsilon\epsilon_0 A}\frac{I}{\tilde{c}}\tilde{C}\, d = \frac{Q}{C}\frac{dQ}{dt}$$

(a) – current / charge picture

(b), (c) – field picture

different ways of describing the same thing

4. Radiation Pressure

When e.m. radiation falls on a surface it exerts a radiation pressure because it transfers momentum to the surface

For a wave at normal incidence, energy hitting the surface per unit time per unit area is the Poynting vector $P$

for photons $E = pc$

$$\underset{\text{energy}}{E} = \underset{\text{momentum}}{p}c$$

$\therefore$ momentum hitting surface per unit time per unit area is $\frac{P}{c}$

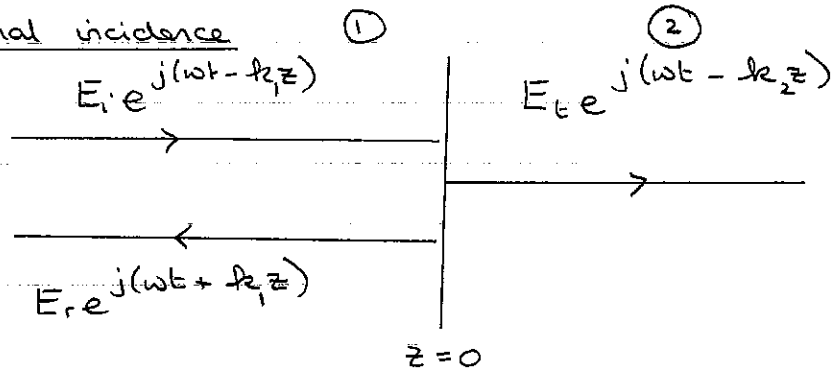
pressure is rate of change of momentum per unit area

$\therefore$ for a perfect absorber radiation pressure is $\frac{P}{c}$

for a perfect reflector radiation pressure is $\frac{2P}{c}$

# D. Reflection and Refraction of EM Waves

1. normal incidence




boundary conditions:

$E^{\parallel}$ continuous

$$E_i + E_r = E_t$$

$H^{\parallel}$ continuous

$$\therefore \frac{E_i}{Z_1} - \frac{E_r}{Z_1} = \frac{E_t}{Z_2} \qquad \text{where } Z_1 = \sqrt{\frac{\cancel{\mu}\mu_0}{\epsilon_1\epsilon_0}}, \text{ etc.}$$

$$\therefore \frac{E_r}{E_i} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \qquad \frac{E_t}{E_i} = \frac{2Z_2}{Z_2 + Z_1}$$

Poynting vector $\underline{P} = \underline{E} \wedge \underline{H}$

expect

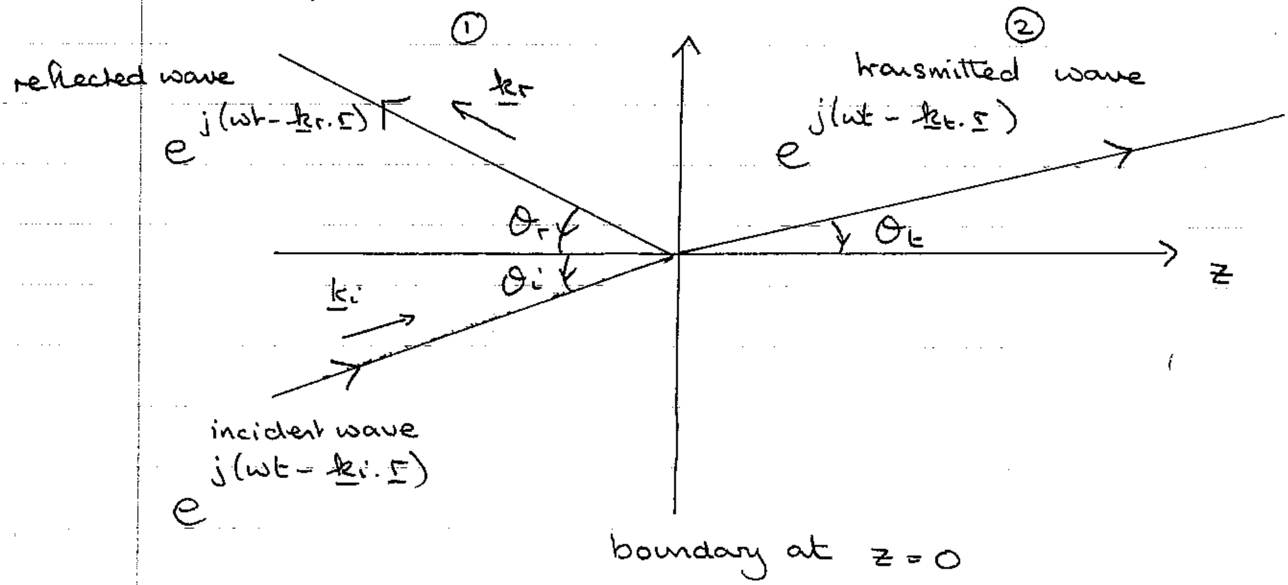
$$P_{incident} = P_{reflected} + P_{transmitted}$$

$$\frac{E_i^2}{Z_1} = \frac{E_r^2}{Z_1} + \frac{E_t^2}{Z_2}$$

which it does.

## 2. general angle of incidence

### (a) laws of reflection and refraction




At $z=0$ the boundary conditions must hold $\forall\ x, y, t$

$\therefore\ \omega$ must be same for $i, r, t$ waves

and

$$\underline{k}_i \cdot \underline{r} = \underline{k}_r \cdot \underline{r} = \underline{k}_t \cdot \underline{r} \quad \forall\ x, y \text{ when } z=0 \qquad ①$$

$$\therefore (k_i)_x x + (k_i)_y y = (k_r)_x x + (k_r)_y y = (k_t)_x x + (k_t)_y y \quad \forall\ x, y$$

$$\therefore\ (k_i)_x = (k_r)_x = (k_t)_x \qquad ②$$

$$(k_i)_y = (k_r)_y = (k_t)_y \qquad ③$$

choose $\underline{k}_i$ to lie in the $x-z$ plane $\therefore$ ③ $= 0$ &
$\underline{k}_r, \underline{k}_t$ will also lie in the $x-z$ plane

① The incident, transmitted and reflected wavevectors (rays) lie in the same plane; called the plane of incidence

$$|\underline{k}_i| = |\underline{k}_r| \equiv k_1 \text{, say} \qquad (k \text{ is a material property})$$
$$|\underline{k}_t| = k_2 \text{, say}$$

$\therefore$ (2) $\Rightarrow$ $\hat{x}$-component of $\underline{k}_i$, etc.

$$k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$$

$\therefore$

(2') $\boxed{\theta_i = \theta_r}$ law of reflection

(3) $$\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \frac{k_1}{k_2} = \frac{n_1}{n_2}}$$ Snell's law (recall $n = \frac{ck}{\omega}$)

generally true for wave motion – we have not specified e + m boundary conditions yet ...

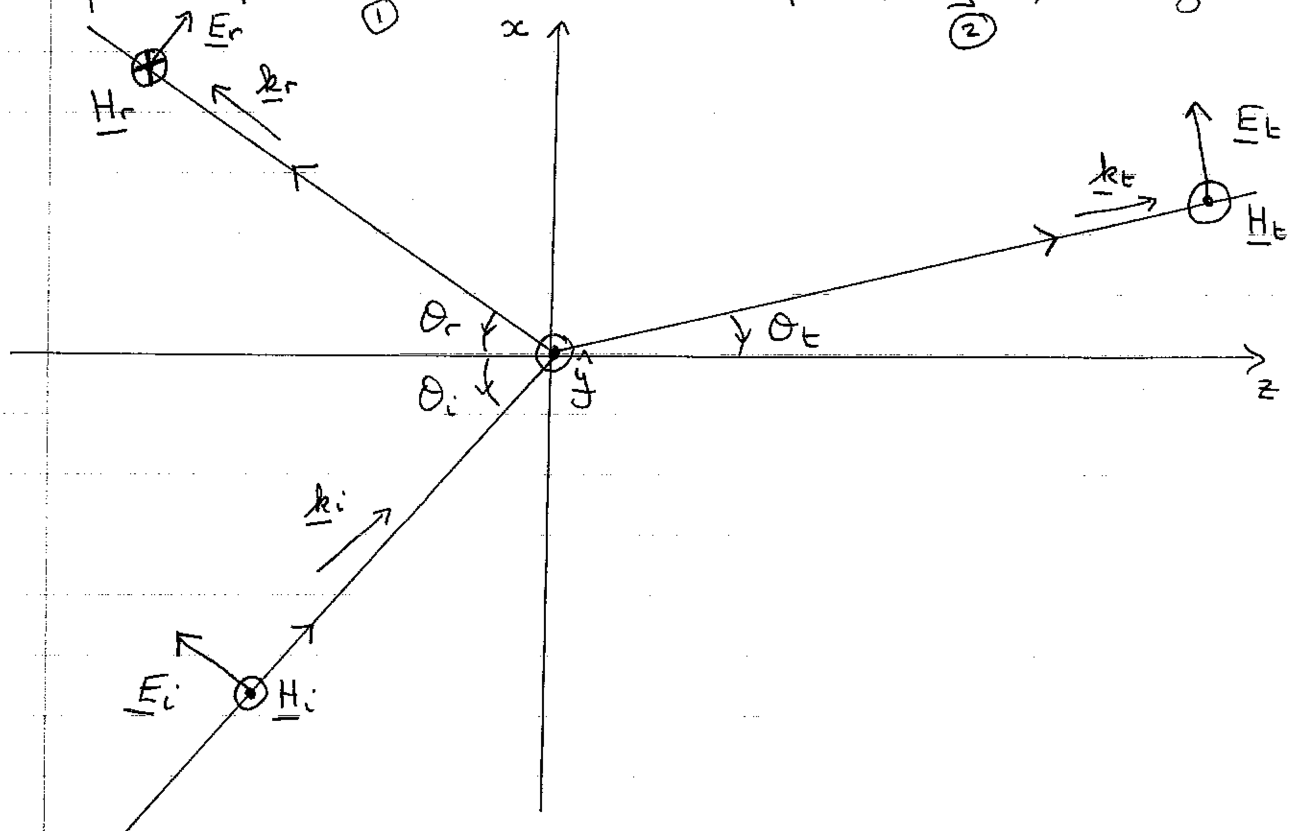
(b) Fresnel equations: $\underline{E}$ in plane of incidence

( $\therefore$ $\underline{H}$ $\perp^r$ to plane of incidence)

$\odot$ = out of page

$\otimes$ = into page

plane of incidence is the $x$-$z$ plane, say ; boundary is at $z=0$




choose $\underline{E}$ as shown

the choose $\underline{H}$ : $\underline{E}, \underline{H}, \underline{k}$ form a r.h. set

then $\underline{E} = ^{+}\underline{H}\, Z$

(can check using curl $\underline{E} = -\mu\mu_0 \frac{\partial \underline{H}}{\partial t}$ that for a r.h. set & a dielectric $\frac{E}{H} = \sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}}$ ; for a l.h. set $\frac{E}{H} = -\sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}}$ )

incident wave

$$\left.\begin{aligned} E_x &= E_i \cos\theta_i \\ E_z &= -E_i \sin\theta_i \\ H_y \overset{= H_i}{=} & \frac{E_i}{Z_1} \end{aligned}\right\} e^{j\{\omega t - \overbrace{k_1(\sin\theta_i x + \cos\theta_i z)}^{\text{ie } k_x x + k_z z}\}}$$

reflected wave

$$\left.\begin{aligned} E_x &= E_r \cos\theta_r \\ E_z &= E_r \sin\theta_r \\ H_y \overset{= -H_i}{=} & -\frac{E_r}{Z_1} \end{aligned}\right\} e^{j\{\omega t - k_1(\sin\theta_r x - \cos\theta_r z)\}}$$

transmitted wave

$$\left.\begin{aligned} E_x &= E_t \cos\theta_t \\ E_z &= -E_t \sin\theta_t \\ H_y &= \frac{E_t}{Z_2} \end{aligned}\right\} e^{j\{\omega t - k_2(\sin\theta_t x + \cos\theta_t z)\}}$$

boundary conditions

$E^{\parallel}$ continuous ie $E_x$ continuous

$$E_i \cos\theta_i + E_r \cos\theta_r = E_t \cos\theta_t \qquad (4)$$

$H^{\parallel}$ continuous ie $H_y$ continuous

$$\frac{E_i}{Z_1} - \frac{E_r}{Z_1} = \frac{E_t}{Z_2} \qquad (5)$$

solving (4) and (5) gives (& using $\theta_i = \theta_r$)

$$\boxed{\begin{aligned} \frac{E_r}{E_i} &= \frac{Z_2 \cos\theta_t - Z_1 \cos\theta_i}{Z_2 \cos\theta_t + Z_1 \cos\theta_i} \\ \frac{E_t}{E_i} &= \frac{2 Z_2 \cos\theta_i}{Z_2 \cos\theta_t + Z_1 \cos\theta_i} \end{aligned}}$$

Fresnel equations

$\underline{E}$ in plane of incidence

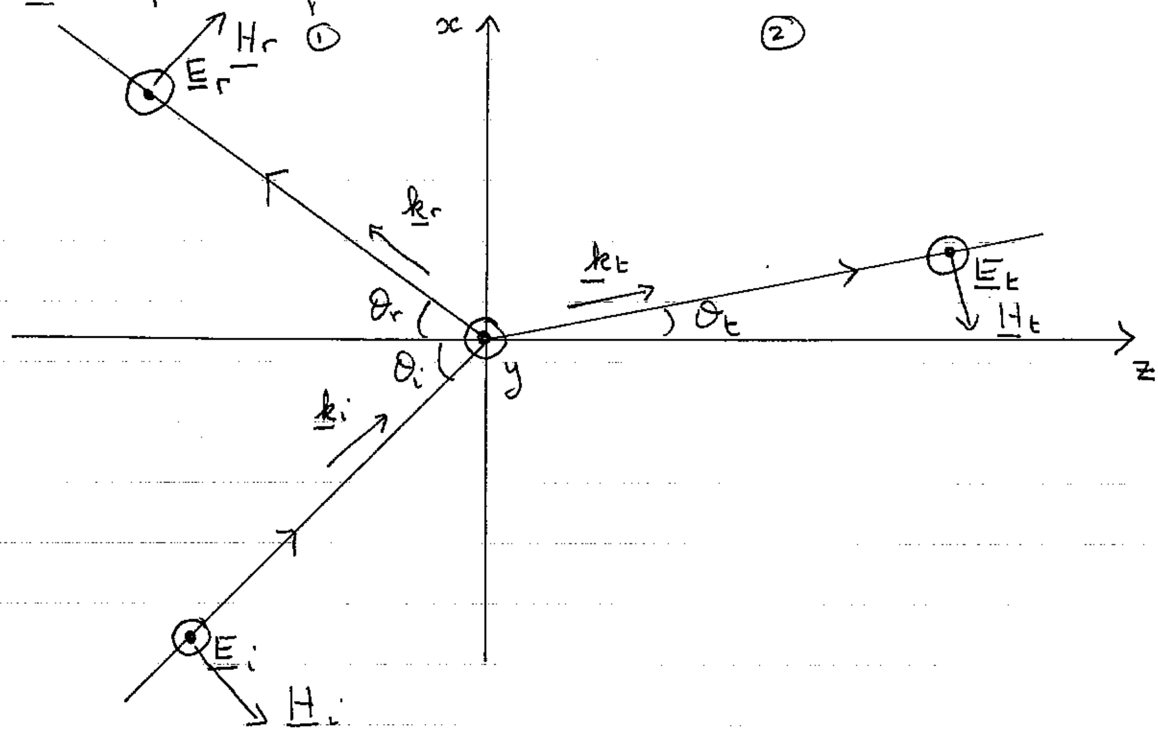
if $\mu_1 = \mu_2 = 1$ $\quad Z \propto \frac{1}{n}$ $\quad \left( \text{because } Z = \sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}},\ n = \sqrt{\mu\epsilon} \right)$

& these become (using Snell's law)

$$\boxed{\begin{aligned} \frac{E_r}{E_i} &= \frac{\sin 2\theta_t - \sin 2\theta_i}{\sin 2\theta_t + \sin 2\theta_i} \\ \frac{E_t}{E_i} &= \frac{4 \sin\theta_t \cos\theta_i}{\sin 2\theta_t + \sin 2\theta_i} \end{aligned}}$$

(ii) $\underline{E}$ $\perp^r$ to plane of incidence

$\therefore \underline{H}$ in plane of incidence




choose $\underline{E}$ out of page $\forall$ rays

then choose $\underline{H}$ : $\underline{E}, \underline{H}, \underline{k}$ form a r.h. set

<u>incident wave</u>

$$\left.\begin{aligned} E_y &= E_i \\ H_x &= -H_i \cos\theta_i = -\frac{E_i}{Z_1}\cos\theta_i \\ H_z &= +\frac{E_i}{Z_1}\sin\theta_i \end{aligned}\right\} e^{j\{\omega t - k_1(\sin\theta_i x + \cos\theta_i z)\}}$$

<u>reflected wave</u>

$$\left.\begin{aligned} E_y &= E_r \\ H_x &= \frac{E_r}{Z_1}\cos\theta_r \\ H_z &= \frac{E_r}{Z_1}\sin\theta_r \end{aligned}\right\} e^{j\{\omega t - k_1(\sin\theta_r x - \cos\theta_r z)\}}$$

<u>transmitted wave</u>

$$\left.\begin{aligned} E_y &= E_t \\ H_x &= -\frac{E_t}{Z_2}\cos\theta_t \\ H_z &= \frac{E_t}{Z_2}\sin\theta_t \end{aligned}\right\} e^{j\{\omega t - k_2(\sin\theta_t x + \cos\theta_t z)\}}$$

$E^{\parallel}$ continuous $\Rightarrow$ $E_y$ continuous

$H^{\parallel}$ " $\Rightarrow$ $H_x$ "

+ solve to give

$$\therefore \boxed{\begin{aligned} \frac{E_r}{E_i} &= \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t} \\ \frac{E_t}{E_i} &= \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_i + Z_1\cos\theta_t} \end{aligned}}$$

for $\mu_1 = \mu_2 = 1$ $\quad Z \propto \frac{1}{n}$

$$\boxed{\begin{aligned} \frac{E_r}{E_i} &= \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)} \\ \frac{E_t}{E_i} &= \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_t + \theta_i)}. \end{aligned}}$$

Fresnel's equations
($\underline{E}$ $\perp^r$ plane of incidence)

(d) <u>Poynting vector and the conservation of energy</u>

$$P^{\perp}_{incident} = P^{\perp}_{reflected} + P^{\perp}_{transmitted}$$

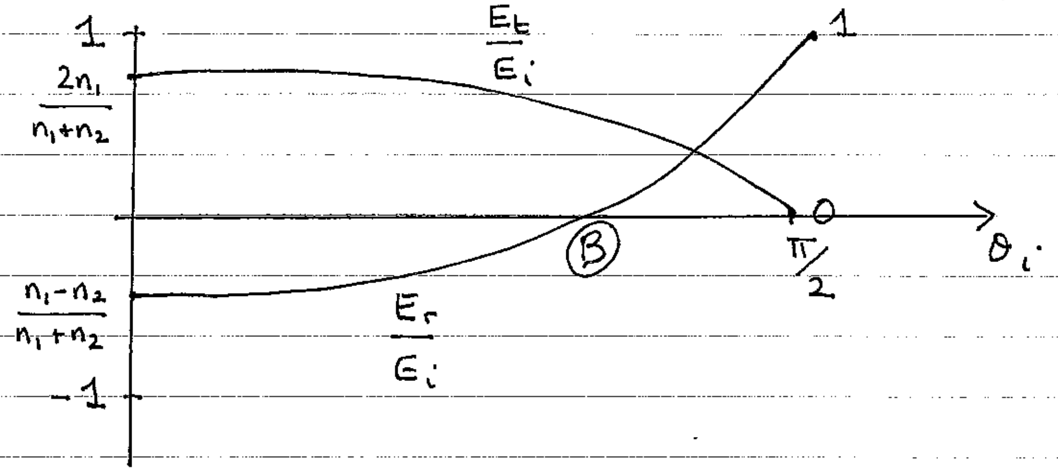
component perpendicular to surface

$$\therefore \quad \frac{E_i^2}{Z_1}\cos\theta_i = \frac{E_r^2}{Z_1}\cos\theta_r + \frac{E_t^2}{Z_2}\cos\theta_t$$

.... check that this holds

## 4. Physical consequences of the Fresnel Equations

**E in plane of incidence** $\boxed{n_2 > n_1}$ (eg ray from air incident on glass or water)

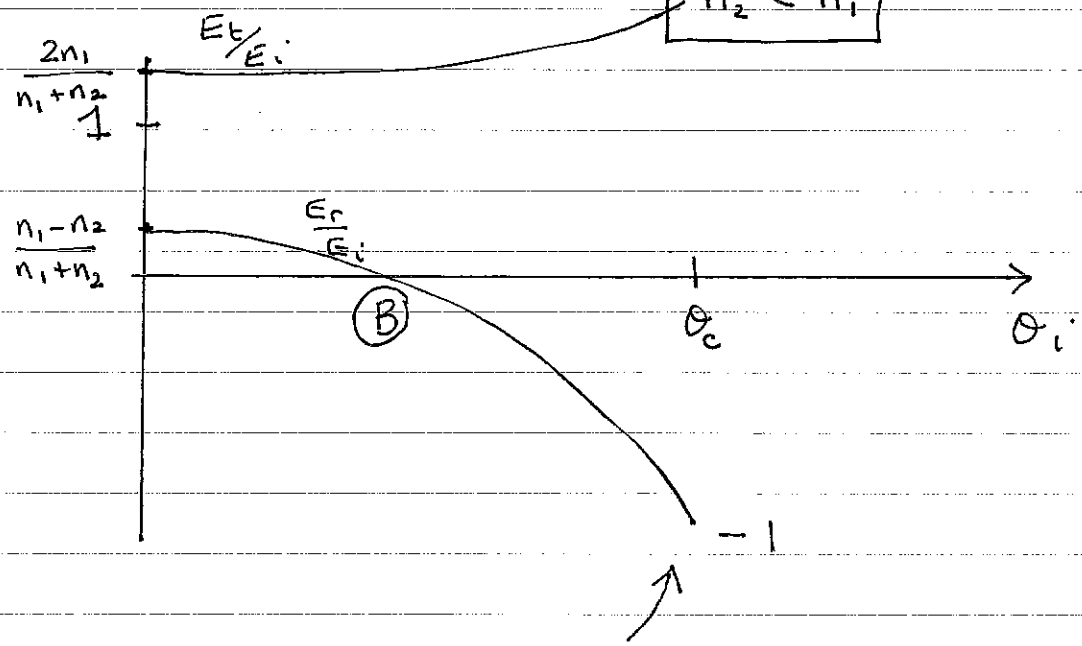



↑ using $\theta_i, \theta_t$ small and $\frac{\sin\theta_i}{\sin\theta_t} = \frac{n_2}{n_1} \approx \frac{\theta_i}{\theta_t}$

↑ putting $\theta_i = \pi/2$

$\frac{2n_1}{n_2}$

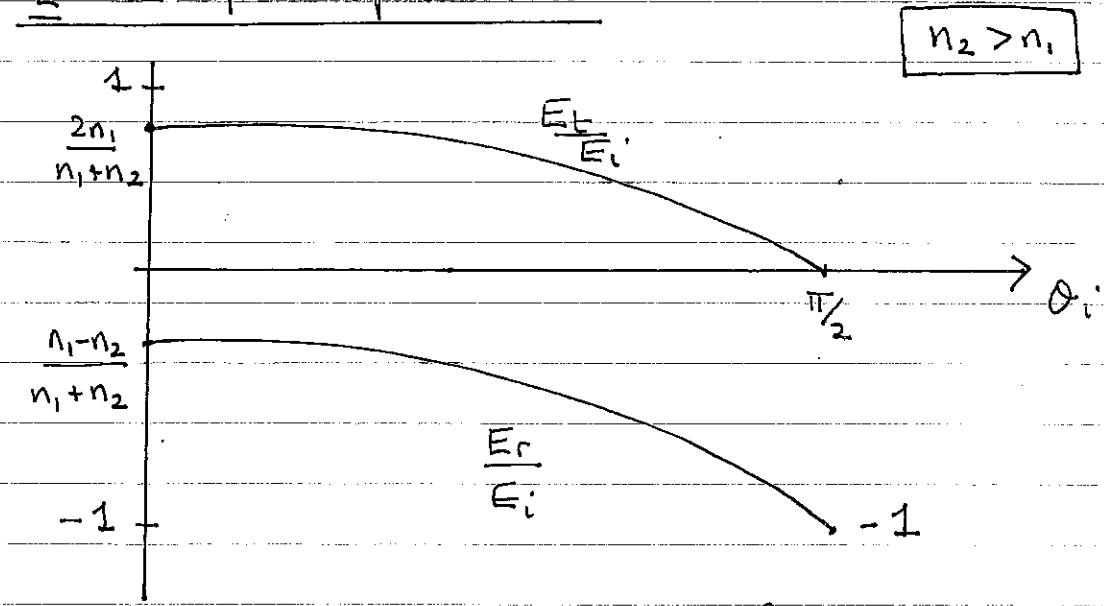
$\boxed{n_2 < n_1}$

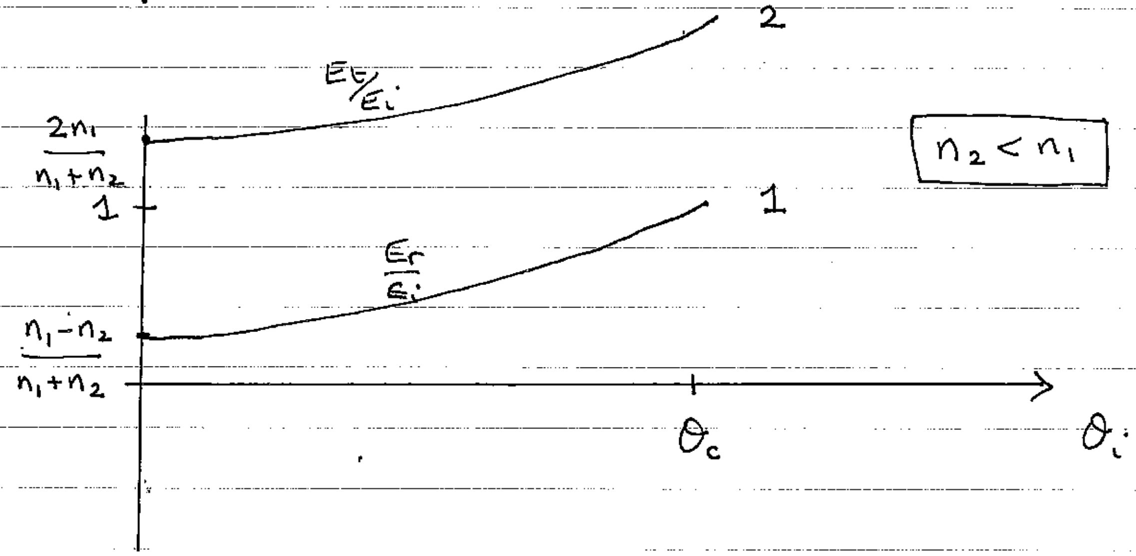



at $\theta_c$ ; $\theta_t = \pi/2$ (i.e. onset of total internal reflection)

$\sin\theta_i = \frac{n_2}{n_1}$

$\underline{E \perp^r \text{ plane of incidence}}$



(1) Brewster angle - no reflection if $\underline{E}$ in plane of incidence
(Ⓑ on diagram)

at the Brewster angle

$$\sin 2\theta_t^B = \sin 2\theta_i^B$$

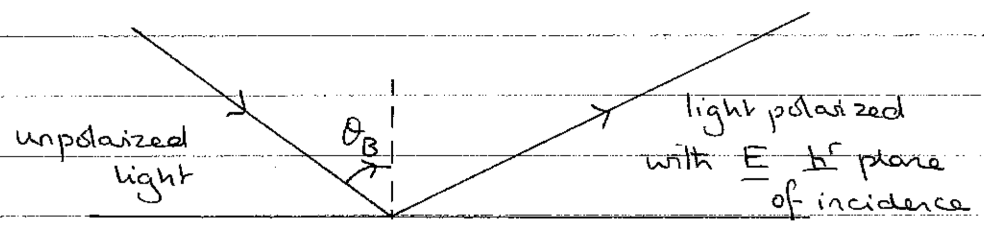
$$\pi - 2\theta_t^B = 2\theta_i^B$$

$$\theta_i^B + \theta_t^B = \pi/2$$

or, equivalently, from Snell's law:

$$\frac{\sin\theta_i^B}{\sin\theta_t^B} = \frac{\sin\theta_i^B}{\sin(\pi/2 - \theta_i^B)} = \tan\theta_i^B = \frac{n_2}{n_1}$$

way of producing plane polarised light




- reflected light is usually at least partially polarized
- ∴ polarizing sunglasses let through a larger fraction of direct light than reflected light thus reducing glare
- radiofreq. antennae can be designed to preferentially pick up directly transmitted waves over those reflected from the surface or ionosphere.

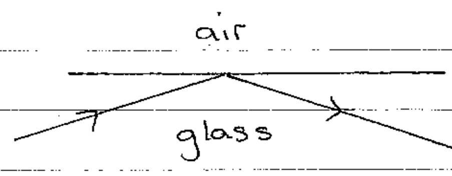
② total internal reflection

we know:

Snell's law: $\dfrac{\sin\theta_t}{\sin\theta_i} = \dfrac{n_1}{n_2}$

if $n_1 \sin\theta_i > n_2$ ; $\sin\theta_t > 1$ ∴ no refracted wave can exist

occurs if $n_2 < n_1$ e.g.



how do we see this in the Fresnel equations?

(a) reflection coefficient?

$\sin\theta_t > 1$ $\qquad \therefore \cos\theta_t = \left(1-\sin^2\theta_t\right)^{\frac{1}{2}} = \pm j\left(\sin^2\theta_t - 1\right)^{\frac{1}{2}}$

for $\underline{E}$ in plane of incidence

$$\frac{E_r}{E_i} = \frac{\sin 2\theta_t - \sin 2\theta_i}{\sin 2\theta_t + \sin 2\theta_i}$$

$$= \frac{2\sin\theta_t \, j\left(\sin^2\theta_t - 1\right) - \sin 2\theta_i}{2\sin\theta_t \, j\left(\sin^2\theta_t - 1\right) + \sin 2\theta_i}$$

$\therefore \left|\frac{E_r}{E_i}\right| = 1$ total reflection

+ similarly for $\underline{E}$ $\perp^r$ plane of incidence

(N.B. $\left|\frac{E_t}{E_i}\right|$ gives rubbish because not a travelling wave)

(b) disturbance in the second medium?

transmitted wave: $\sim e^{j\{\omega t - k_2(\sin\theta_t x + \cos\theta_t z)\}}$

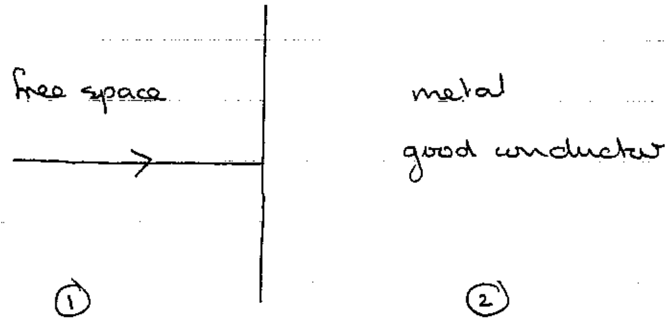
$$\sim e^{j\{\omega t - k_2 \sin\theta_t x\}} \; e^{-(\sin^2\theta_t - 1)^{\frac{1}{2}} k_2 z}$$

there is a wave

evanescent wave

decay length $\sim \dfrac{\lambda_2}{2\pi\left(\sin^2\theta_t - 1\right)^{\frac{1}{2}}}$

(3) <u>reflection at a metal surface</u>




$$Z_1 = \sqrt{\frac{\mu_0}{\epsilon_0}} \qquad Z_2 = \sqrt{\frac{\mu_0 \omega}{2\sigma}}\,(1+j) \equiv (1+j)t$$

$$\frac{t^2}{Z_1^2} = \frac{\mu_0 \omega\, \epsilon_0}{2\sigma\, \mu_0} \sim \underbrace{\frac{\epsilon\epsilon_0\omega}{\sigma}}_{\ll 1} \cdot \underbrace{\frac{1}{\epsilon}}_{O(1)} \ll 1$$

|impedance| of a good conductor $\ll$ impedance of free space

reflection coefficient for normal incidence : $\dfrac{E_r}{E_i} = \dfrac{Z_2 - Z_1}{Z_2 + Z_1}$

$$\therefore \left|\frac{E_r}{E_i}\right|^2 = \left|\frac{(1+j)t - Z_1}{(1+j)t + Z_1}\right|^2$$

$$= \frac{(t-Z_1)^2 + t^2}{(t+Z_1)^2 + t^2} \approx \frac{Z_1^2 - 2tZ_1}{Z_1^2 + 2tZ_1}$$

$$= \left(1 - {}^{2t}/_{Z_1}\right)\left(1 + {}^{2t}/_{Z_1}\right)^{-1} \approx 1 - \underbrace{{}^{4t}/_{Z_1}}_{\text{small}}$$

$\therefore$ <u>most of incident radiation reflected</u> (metals shiny)

rest of power dissipated within ~ skin depth