Problem Set 2

Although everyone in the class is welcome to try these problems, only the papers of students in Theoretical Physics will be marked. They should be put in my box in Theoretical Physics by **Monday Nov 23 at noon.** They will be reviewed at a Problems Class on **Monday Nov 30, 2-4 pm**, in the **DWB Conference Room** (note change of venue), which everyone in the class is welcome to attend. Any questions before the due date should be directed to me at `j.cardy1@physics.ox.ac.uk`
Parts of questions marked † may be found slightly harder. However anyone seriously interested in learning QFT should try to tackle them.

1. (a) Consider a theory with two scalar fields $\phi^{(1)}$ and $\phi^{(2)}$ and a lagrangian density

$$\tfrac{1}{2}\sum_{j=1}^{2}((\partial\phi_0^{(j)})^2+m_0^2(\phi_0^{(j)})^2)+\frac{\lambda_0}{4!}((\phi_0^{(1)})^4+(\phi_0^{(2)})^4)+\frac{\sigma_0}{4}((\phi_0^{(1)})^2(\phi_0^{(2)})^2)\,.$$

   Assume that the bare mass m_0 is adjusted so that both fields have zero renormalised mass. Calculate, within minimal subtraction or any other suitable scheme, the renormalised couplings λ and σ to 1-loop order in λ_0 and σ_0. [Note that the Feynman integral is the same as that computed in the notes.]

   (b) Hence work out the beta-functions

$$\beta_\lambda(\lambda,\sigma)\equiv(\mu\partial/\partial\mu)\lambda|_{\lambda_0,\sigma_0} \quad \text{and} \quad \beta_\sigma(\lambda,\sigma)\equiv(\mu\partial/\partial\mu)\sigma|_{\lambda_0,\sigma_0}$$

   to 1-loop order in $d=4-\epsilon$ dimensions.

   (c) †Find the simultaneous zeroes of the beta-functions and analyse their relative IR stability in $4-\epsilon$ dimensions. Sketch the RG flows in the (λ,σ) plane.

2. In QCD, the renormalisation group functions have the form $\beta(g)=-bg^3+O(g^4)$ and $\gamma(g)=cg^2+O(g^3)$, where b and c are positive constants.

   (a) QCD is believed to exhibit dynamical mass generation: there is no mass term in the lagrangian, nevertheless the physical particles have mass. By observing that the mass of, say, the proton must have the form $M = \mu f(g)$, but that M cannot in fact depend on the renormalisation scale, deduce how M must depend on g for small g.

(b) †What is the asymptotic behaviour of $\Gamma^{(2)}(p)$ in this theory for $p \to \infty$? [By this I mean not just 0 or ∞ but how it gets there. You will need to use the full solution of the C-S equation.]

3. In the notes we already briefly discussed the renormalisation of ϕ^6 theory, with lagrangian density

$$\mathcal{L} = \tfrac{1}{2}((\partial\phi_0)^2 + m_0^2\phi_0^2) + \frac{\lambda_0}{4!}\phi_0^4 + \frac{\kappa_0}{6!}\phi^6$$

(a) Assume that m_0 and λ_0 are adjusted so as to make the renormalised m and λ vanish. Calculate the beta-function to lowest non-trivial order, and show that it has a non-trivial IR stable zero in $d_c - \epsilon$ dimensions. [You will meet one Feynman integral. †if you manage to evaluate it, otherwise assume it has the form C/ϵ where C is a constant.]

(b) †Calculate the anomalous dimension of $\phi^4(x)$ to $O(\epsilon)$ in this theory.

4. Consider the theory we discussed in Q.5 of the first problem set, with lagrangian density (now in Minkowski space)

$$\tfrac{1}{2}((\partial_\mu\phi)(\partial^\mu\phi) - m^2\phi^2) + \tfrac{1}{2}((\partial_\mu\phi)(\partial^\mu\phi) - M^2\Phi^2) - \tfrac{1}{2}\lambda\phi^2\Phi$$

Draw the diagrams contributing to the $\mathcal{T}$-matrix for $\phi\phi \to \phi\phi$ scattering to lowest non-trivial order in λ, and hence (using the results in the notes) calculate the differential cross-section for this process in the CM frame, showing explicitly how it depends on energy E and scattering angle θ.