Problem Set 1

Although everyone in the class is welcome to try these problems, only the papers of students in Theoretical Physics will be marked. They should be handed in to Dr. Stefan Zohren in Theoretical Physics (either to his room 4.5 or his pigeon hole) by **Thursday November 4 at 2pm.** They will be reviewed at a Problems Class on **Monday November 15** from 2–4pm in the Seminar Room, DWB, which everyone in the class is welcome to attend. Any questions before the due date should be directed to me at `j.cardy1@physics.ox.ac.uk`
Parts of questions marked † may be found slightly harder. However anyone seriously interested in learning QFT should try to tackle them.

1. A one-dimensional quantum simple harmonic oscillator with hamiltonian $\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2$ is in contact with a heat bath at temperature T.

   (a) write down an expression for the partition function $Z = \operatorname{Tr} e^{-\beta\hat{H}}$ as a path integral over functions $q(\tau)$ in imaginary time τ.

   (b) by expanding $q(\tau)$ in a suitable basis of functions, use the path integral to get an expression for $\langle q(\tau_1)q(\tau_2)\rangle$ at finite temperature T. Show that for large T your result agrees with what you get from classical statistical mechanics.

   (c)† What about the partition function Z itself? Is your answer finite? If not, how can you make it so by adjusting the normalisation of the path integral? ††Show the result is then the same as what you would get by computing Z as $\sum_n e^{-\beta E_n}$ where E_n is the energy of the nth eigenstate.

2. The displacement of an infinitely long stretched polymer is described by a function $q(\tau)$. It is in contact with a heat bath at temperature T and is held in place by forces $\pm f$ acting at 0 and ℓ respectively, so the total energy is

$$E[q] = \int_{-\infty}^{\infty} [\tfrac{1}{2}\sigma(dq/d\tau)^2]d\tau - f(q(\ell) - q(0))$$

   (a) evaluate $\langle (q(\tau_1) - q(\tau_2))^2 \rangle$ when $f = 0$.

(b) observe that when $f \neq 0$ the partition function $Z = \int[dq]e^{-\beta E[q]}$ has the form of a euclidean path integral with delta-function sources at 0 and ℓ. Use the result for the 2-point correlation function obtained above to get an expression for how the free energy $F = -kT \log Z$ depends on f and ℓ.

3. What are the propagators (in momentum space) of the euclidean field theories with the following free actions? (You should be able to write these down more or less by inspection. There may be more than one propagator if there is more than one type of field.)

   (a) $S = \frac{1}{2}\int[(\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_z^2\phi)^2]dxdydz$ (note the ∂^2 in the last term)

   (b) $S = \frac{1}{2}\int[(\partial\phi_1)^2 + (\partial\phi_2)^2 + m^2(\phi_1^2 + \phi_2^2) + 2\mu\phi_1\phi_2]d^dx$

   (c)† $S = \int[\frac{1}{2}(\phi^*\partial_\tau\phi - \phi\partial_\tau\phi^*) + (\nabla\phi^*)\cdot(\nabla\phi)]d\tau d^Dx$ (ϕ is complex)

4. What are the vertices, in momentum space, corresponding to the following interaction terms in a scalar field theory? Draw the way they would appear at lowest order in a tree diagram, give the numerical factor, and label any lines as appropriate.

   a) $\lambda(\phi^{*4} + \phi^4)$ (ϕ is complex and the propagator is $\langle\phi\phi^*\rangle$); (b) $\lambda\, e^{\phi}$; (c)† $\lambda\phi(\partial_\mu\phi)(\partial^\mu\phi)$.

5. Consider a euclidean QFT with two real scalar fields ϕ and Φ and a lagrangian density

$$\tfrac{1}{2}((\partial\phi)^2 + m^2\phi^2) + \tfrac{1}{2}((\partial\Phi)^2 + M^2\Phi^2) + \tfrac{1}{4}\lambda\phi^2\Phi^2$$

   Write down the Feynman rules in momentum space for this theory (be careful to use a different sort of line for the propagators of different fields). Draw the tree and one loop diagrams that contribute to the correlation functions $\langle\phi\phi\rangle$, $\langle\Phi\Phi\rangle$, $\langle\phi\phi\phi\phi\rangle$, $\langle\phi\phi\Phi\Phi\rangle$, and $\langle\Phi\Phi\Phi\Phi\rangle$, and write down explicit expressions for the 1-loop diagrams, including the correct symmetry factors.