Notes on SLE

These notes are an addendum to my 2008 les Houches lecture notes, where some familiarity with SLE was assumed. For further details, see, eg,
`http://www-thphys.physics.ox.ac.uk/people/JohnCardy/slereview.pdf`
Schramm-Loewner evolution is a probabilistic theory of the scaling limit of the planar curves which arise in 2d lattice models, e.g. those in the ADE and $O(n)$ models we have already met.
There are two different lines of argument: the first assumed a couple of simple axioms about the conformal invariance of this assumed scaling limit. From these many rigorous results may be derived as well as the relation to (boundary) CFT. The second, which is harder, actually proves that these axioms hold for the scaling limit of patriculat lattice models. So far this has been carried out in only a few cases.
Consider a simple curve γ in the upper half plane $\mathbf{H}$ from the point a_0 on the real axis to ∞. In mathematical terms such a curve comes along with a continuous parametrisation $0 \leq t < \infty$. Later we will choose a particular one. Let γ_t be the part of the curve up to 'time' t. Then $\mathbf{H} \setminus \gamma_t$ is a simply connected region, which, by the Riemann mapping theorem. may be cocformally mapped back to $\mathbf{H}$ by $z \to g_t(z)$. We can think of $\mathbf{H}$ as being opened up by a slit along γ_t. The L and R sides of this slit are mapped to the real axis. The function $g_t(z)$ is not uniqie but we can make it so by demanding that

$$g_t(z) = z + O(1/z)$$

as $z \to \infty$ (that is, no constant term.) It can also be shown that as the curve grows the coefficient of the $O(1/z)$ term always increases, so we can define t so that $g_t(z) = z + 2t/z + O(1/z^2)$. In that case the growing tip gets mapped to a point a_t on the real axis. a_t is a continuous function of t.
The simplest example is when γ is a straight line perpendicular to the real axis. In that case

$$g_t(z) = ((z - a_0)^2 + 4t)^{1/2} + a_0$$

Even though we expect the curve in a critical model to be fractal and therefore non-differentiable, the map $g_t(z)$ should evolve smoothly as long as z is not on the curve. We can get an equation for this by noting that

$$g_{t+\delta t}(z) \approx ((g_t(z) - a_t)^2 + 4\delta t)^{1/2} + a_t$$

or

$$dg_t(z) = \frac{2dt}{g_t(z) - a_t}$$

This is (ordinary) Lowener evolution. It shows that the study of simple curves γ_t growing into $\mathbf{H}$ is equivalent to looking at continuous functions a_t. If a_t is differentiable the curve

is smooth and we may write as an ordinary differential equation point-wise in z. Given γ_t we may infer a_t and vice versa. In our case this is a stochastic differential equation, and it is not obvious for which functions a_t we will get a continuous curve (but it is OK for SLE).
In SLE we are concerned with defining a measure on curves γ and hence a measure on functions a_t. This may be determined by assuming two basic axioms:

1. Domain Markov property: the measure on the rest of the curve $\gamma \setminus \gamma_t$, given γ_t, in $\mathbf{H}$, is the same as the measure on curves from the tip of the slit to ∞ in the domain $\mathbf{H} \setminus \gamma_\mathbf{t}$. This is obviously true, even on the lattice, for a domain wall in the ADE models with neighbouring fixed heights on the negative and positive real axes.

2. Conformal Invariance: if we map $\mathbf{H} \setminus \gamma_t$ back to $\mathbf{H}$ using g_t, the induced measure on $g_t(\gamma \setminus \gamma_t)$ is the same as that on the original γ (shifted by $a_t - a_0$).

Thena theorem due to Schramm states that the only possibility for a_t is a Brownian motion starting at a_0. The only free parameter is the diffusion constant:

$$\langle a_t - a_0 \rangle = 0, \quad \langle (a_t - a_{t'})^2 \rangle = \kappa |t - t'|$$

The proof goes by using (1,2) to show that a_t must obey a law of independent increments, and have zero drift. As we will see shortly, κ is related to the Coulomb gas coupling by $g = 4/\kappa$, and hence to the central charge c. The dilute critical point correspoinds to $\kappa \leq 4$, and for this it has been proved that γ is indeed a simple curve. In the dense phase γ continually self-intersects (but does not cross itself). In that case the original Loewner argument still applies, replacing γ_t by its 'hull' K_t, the region enclosed between γ_t and the real axis.