Solutions 1

1. Consider an inversion $r \to r/|r|^2$. The Jacobian is $|r|^{-2}$, so

$$|r_1 - r_2|^{-x_1-x_2} = |r_1|^{-2x_1}|r_2|^{-2x_2}\left|\frac{r_1}{|r_1|^2} - \frac{r_2}{|r_2|^2}\right|^{-x_1-x_2}$$

This is an identity if $x_1 = x_2$ but cannot be true otherwise: for example by taking $r_1 \to 0$ we see that it is false. Note that this uses only the special conformal group so is true for primary fields in higher dimensions.

2. Choose spherical polar coordinates (R, θ, ϕ) on the sphere, with metric $ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2)$. The plane is tangent to the sphere at the N pole $\theta = 0$, and the projection P' of a point P is where a line from the S pole through P intersects this plane. This gives P' to be at (ρ, ϕ) in polar coordinates, where $\rho = 2R\sin\frac{\theta}{2}$. This is conformal because $ds^2 = \cos^2\frac{\theta}{2}(d\rho^2 + \rho^2 d\phi^2)$. Hence

$$\begin{aligned}\langle\Phi(\theta_1,\phi_1)\Phi(\theta_2,\phi_2)\rangle_{S^2} &= \frac{[\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}]^x}{[4R^2\sin^2\frac{\theta_1}{2} + 4R^2\sin^2\frac{\theta_2}{2} - 8R^2\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos(\phi_1 - \phi_2)]^x} \\ &= R^{-2x}[\sin^2\theta_1 + \sin^2\theta_2 - 2\sin\theta_1\sin\theta_2\cos(\phi_1 - \phi_2)]^{-x}\end{aligned}$$

Note that the expression in brackets is the (square of) the chordal distance between the two points on the sphere.

3.

$$\langle\phi(w_1,\bar{w}_1)\phi(w_2,\bar{w}_2)\rangle_{\text{cyl}} = \left|\frac{dz_1}{dw_1}\right|^x\left|\frac{dz_2}{dw_2}\right|^x\frac{1}{|z_1 - z_2|^{2x}}$$

After some algebra this gives, with $w = r + i\tau$,

$$\langle\phi(w_1,\bar{w}_1)\phi(w_2,\bar{w}_2)\rangle_{\text{cyl}} = \left(\frac{2\pi}{\beta}\right)^{2x}\frac{1}{[2\cosh(\frac{2\pi(r_1-r_2)}{\beta}) - 2\cos(\frac{2\pi(\tau_1-\tau_2)}{\beta})]^x}$$

Note this decays exponentially as $|r_1 - r_2| \to \infty$, with the thermal correlation length $\xi \sim 2\pi x/\beta$. Continuing to real time we find (setting $r_2 = t_2 = 0$ for convenience)

$$\langle\phi(r,t)\phi(0,0)\rangle_\beta = \frac{(2\pi/\beta)^{2x}}{[2\cosh(2\pi r/\beta) - 2\cosh(2\pi t/\beta)]^x}$$

However, this is valid only outside the light cone $t < |r|$. A naive continuation beyond this suggests that it gains an imaginary part $\propto \sin(\pi x)$. In some lattice models this agrees with explicit results, taking first $t > |r|$ and then the continuum limit.

4. (a) use the conformal mapping

$$w = iR\left(\frac{z-i}{z+i}\right)$$

which takes $\mathrm{Im}\, z > 0$ into $|w| < R$. Hence

$$\langle\phi(w,\bar{w})\rangle_{\text{disc}} = \left|\frac{dz}{dw}\right|^x \frac{1}{(\mathrm{Im}\, z)^x} = \left(\frac{2R}{R^2-|w|^2}\right)^x$$

(b) use $w = (L/\pi)\log z$. The same sort of argument gives

$$\langle\phi(w=iy)\rangle_{\text{strip}} = \left(\frac{(\pi/L)}{\sin(\pi y/L)}\right)^x$$

5. The normalisation constant in front of the action is somewhat arbitrary. We can choose it so the 2-point functions are

$$\langle\psi(z_1)\psi(z_2)\rangle = \frac{1}{z_1-z_2}, \qquad \langle\bar{\psi}(\bar{z}_1)\bar{\psi}(\bar{z}_2)\rangle = \frac{1}{\bar{z}_1-\bar{z}_2}$$

The stress tensor T may be found using Noether's theorem, of simply by observing that $T = \alpha\psi\partial_z\psi$ is the only bilinear in the fields which has the correct scaling dimensions $(2,0)$. The constant α may be fixing by demanding the T satisfy the correct OPE with ψ. By Wick's theorem (note that the contraction between the $\psi(z)$ at the same point is removed by point-splitting and subtracting this contribution)

$$\begin{aligned}
\psi(z)\partial_z\psi(z)\cdot\psi(z_1) &= \psi(z)\partial_z\left(\frac{1}{z-z_1}\right) - \partial_z\psi(z)\frac{1}{z-z_1} + \cdots \\
&= \frac{-1}{(z-z_1)^2}(\psi(z_1) + (z-z_1)\partial_{z_1}\psi(z_1)) - \frac{1}{z-z_1}\partial_{z_1}\psi(z_1) + \cdots \\
&= \frac{-1}{(z-z_1)^2}\psi(z_1) - \frac{2}{z-z_1}\partial_{z_1}\psi(z_1) + \cdots
\end{aligned}$$

so we should take $\alpha = -\frac{1}{2}$ to get the last term right. This also gives $\Delta_\psi = \frac{1}{2}$ as expected. The 2-point function is now

$$\begin{aligned}
\langle T(z_1)T(z_2)\rangle &= \frac{1}{2^2}\langle\psi(z_1)\partial_{z_1}\psi(z_1)\,\psi(z_2)\partial_{z_2}\psi(z_2)\rangle \\
&= \frac{1}{4}\left[-\frac{1}{z_1-z_2}\partial_{z_1}\partial_{z_2}\frac{1}{z_1-z_2} + \partial_{z_1}\frac{1}{z_1-z_2}\partial_{z_2}\frac{1}{z_1-z_2}\right] \\
&= \frac{1}{4}\frac{2-1}{(z-1-z_2)^4} \qquad \text{so that } c = \tfrac{1}{2}
\end{aligned}$$