Exercises 2

These problems will not be marked. Solutions will be posted on my web page in due course. Any questions or clarifications before then can be addressed to me at
`j.cardy1@physics.ox.ac.uk`

1. There is a 1-1 correspondence between scaling operators and eigenstates of $\hat{H}$ and $\hat{P}$ on the cylinder. If the three operators in question are all primary, compute the matrix element $\langle\phi_i|\hat{\phi}_j|\phi_k\rangle$ in terms of CFT data in the plane.

2. Suppose we add to the action of the CFT the irrelevant term $\lambda\int T\overline{T}d^2r$. Compute, to first order in λ, the shift in the energy on the cylinder $E_j - E_0 = (2\pi/L)x_j$ of an eigenstate corresponding to a primary operator of scaling dimension x_j.

3. Consider a minimal model (not necessarily unitary) whose OPE is closed under the fusion rules and which has only two primary operators (including the identity). Show that there is only one such model, and give the value of the central charge and the scaling dimension of the non-trivial operator. Repeat the exercise for minimal models with 3 primary operators (including the identity.)

4. Suppose we have a torus with modular parameter $\tau = i\delta$ with δ real. Use the symmetry of the partition function under $\delta \to \delta^{-1}$ near $\delta = 1$ to obtain a sum rule for the central charge c in terms of the set of scaling dimensions x_j of the CFT. Show that this is in general rapidly convergent. Use it to obtain a simple upper bound on the lowest scaling dimension for a given c.

5. The modular **S**-matrix for the $c = \frac{1}{2}$ unitary minimal model has the form

$$\mathbf{S} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

   where the rows and columns are labelled by the scaling dimensions $(0, \frac{1}{2}, \frac{1}{16})$. The usual partition function is $Z = \operatorname{Tr} q^{L_0-c/24}\bar{q}^{\overline{L}_0-c/24} = |\chi_0|^2 + |\chi_{1/2}|^2 + |\chi_{1/16}|^2$. The CFT has a $\mathbf{Z}_2$ symmetry, call it Σ (corresponding to the spin symmetry of the Ising model) under which the $(0, \frac{1}{2})$ fields are even and $\frac{1}{16}$ is odd. What it the form of the modified partition function $Z = \operatorname{Tr}\Sigma\, q^{L_0-c/24}\bar{q}^{\overline{L}_0-c/24}$? How does it behave under the element S of the modular group? Hence work out what are the scaling dimensions of the operators in the sector of the theory where the $\frac{1}{16}$ field obeys anti-periodic boundary conditions around the cylinder.