Advanced Topics in Statistical Mechanics
Michaelmas Term, 2007 – Prof. J. Cardy
Homework Problems

These problems should be done and handed in for marking by all the first year condensed matter theory students, by Thursday Dec 6 at 10.00 am. They will be discussed and solutions distributed at a class early in the following week, TBA.

1. Calculate the low-temperature heat conductivity of a degenerate electron gas, using the Boltzmann equation in the same approximation as was used in the lecture to compute the electrical conductivity. Specifically, introduce a small uniform temperature gradient $\partial T/\partial x$ and replace the collision term by $-(f-f_{\rm eq})/\tau$. Also note that in the drift term $\mathbf{v}\cdot\partial f/\partial\mathbf{x}$ you can replace f by $f_{\rm eq}$, where $f_{\rm eq}$ is the Fermi distribution corresponding to the local value of $T(x)$. [**Warning:** A gradient in T will also produce a non-zero electric current. To compute properly the heat conductivity you need also to introduce a gradient in the chemical potential μ so as to cancel this effect.]

2. Generalise the discussion in the lecture of the S-K model to the case when there is a uniform magnetic field term $-h\sum_j S_j$ in the hamiltonian, both by using replicas (assuming replica symmetry) and by simply assuming that the $m_j = \langle S_j\rangle$ are statistically independent. (These two approaches should give the same equation.) Analyse how the uniform magnetisation $M \equiv N^{-1}\sum_j m_j$ behaves for small fields near T_c and show that while the susceptibility $\chi = \partial M/\partial h$ is finite at T_c, the non-linear susceptibility $\chi_{\rm nl} = \partial^3 M/\partial h^3$ diverges there.

3. This problem is about quenched disorder in the XY-model, and combines topics (2) and (3). Consider a nearest-neighbour XY model with p-fold anisotropy, with hamiltonian

$$H = -J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j) - h_p\sum_j \cos(p\theta_j-\phi_j)$$

Assume that the temperature is less than $T_{\rm KT}$, so that vortices can be ignored and the first term in H can be treated in the spin-wave approximation. In the second term p is an integer ≥ 1.

(a) first consider the homogeneous case when ϕ_j is independent of j. By calculating the 2-point correlation function of $\cos(p\theta_j - \phi)$, determine its scaling dimension x_p, and hence the RG eigenvalue y_p of h_p.

(b) For what values of the temperature can we simultaneously ignore both h_p and the vortices? What do you think happens in the others cases when one or both are relevant?

(c) Now suppose that the ϕ_j are quenched random variables, independently and uniformly distributed in the interval $[0, 2\pi]$. Describe qualitatively the physics of the model at very low temperatures, in the two cases $h \ll J$ and $h \gg J$.

(d) If we now introduce replicas θ_j^α and perform the quenched average over the ϕ_j, what is the form of the quenched hamiltonian to order $O(h^2)$?

(e) repeat the above analysis to work out the RG eigenvalue of $\Delta_p \equiv h_p^2$. For what values of T can we now ignore both the random anisotropy and the vortices? What do you think happens in the other cases?