Solutions Problem Set 2

1. *Show that if the Hamiltonian is independent of a generalized coordinate q_0, then the conjugate momentum p_0 is a constant of motion. Such coordinates are called* **cyclic coordinates**. *Give two examples of physical systems that have a cyclic coordinate.*

Clearly

$$\dot{p}_0 = -\frac{\partial H}{\partial q_0} \quad \text{if } H \text{ is not a function of } q_0$$

e.g. 1) An axisymmetric potential does not depend on ϕ so p_ϕ is a constant of motion.
e.g. 2) In a magnetic field $\mathbf{B} = B\hat{\mathbf{k}}$, the Hamiltonian is independent of z, so $p_z = $ constant

2. *A dynamical system has generalized co-ordinates q_i and generalized momenta p_i.*
Verify the following properties of the Poisson brackets:

$$[q_i, q_j] = [p_i, p_j] = 0; \qquad [q_i, p_j] = \delta_{ij}.$$

If $\mathbf{p}$ is the momentum conjugate to a position vector $\mathbf{r}$, and $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, evaluate the Poisson brackets $[L_x, L_y]$, $[L_y, L_x]$ and $[L_x, L_x]$. Comment on their significance.

The Lagrangian of a particle of mass m and charge e in a uniform magnetic field $\mathbf{B}$ and an electrostatic potential ϕ is

$$L = \tfrac{1}{2}m\dot{\mathbf{r}}^2 + \tfrac{1}{2}e\dot{\mathbf{r}} \cdot (\mathbf{B} \times \mathbf{r}) - e\phi.$$

Derive the corresponding Hamiltonian and verify that the rate of change of $m\dot{\mathbf{r}}$ equals the Lorentz force. Show that the momentum component along $\mathbf{B}$ and the sum of the squares of the two other momentum components are all constants of motion. Find another constant of motion associated with time translation symmetry.

$$[q_i, q_j] = \sum_k \frac{\partial q_i}{\partial q_k}\frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial q_j}{\partial q_k} = 0$$

trivially, etc.

$$[q_i, p_j] = \sum_k \frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k} = 0 \text{ unless } i = j$$

The Poisson bracket is antisymmetric, so $[L_x, L_x] = 0$ trivially and $[L_y, L_x] = -[L_x, L_y]$ so we simply have to evaluate $[L_x, L_y]$

$L_x = yp_z - zp_y$ and $L_y = zp_x - xp_z$. Thus L_x is independent of x and p_x while L_y is independent of y and p_y so we can collapse the sum over k in the definition of the Poisson bracket to just the term with $k = z$:

$$\begin{aligned}[L_x, L_y] &= \frac{\partial}{\partial z}(yp_z - zp_y)\frac{\partial}{\partial p_z}(zp_x - xp_z) - \frac{\partial}{\partial p_z}(yp_z - zp_y)\frac{\partial}{\partial z}(zp_x - xp_z) \\ &= (-p_y)(-x) - yp_x = L_z\end{aligned}$$

In QM we have $[L_x, L_y] = \mathrm{i}\hbar L_z$, so here's a close connection between clasical and quantum mechanics – which turns out to arise because both theories have to reflect the group of three-dimensional rotations.

From L we find the momenta are

$$\mathbf{p} = m\dot{\mathbf{r}} + \tfrac{1}{2}e(\mathbf{B} \times \mathbf{r})$$

Thus

$$\begin{aligned}H = \mathbf{p} \cdot \dot{\mathbf{r}} - L &= m\dot{\mathbf{r}}^2 + \tfrac{1}{2}e\dot{\mathbf{r}} \cdot (\mathbf{B} \times \mathbf{r}) - \tfrac{1}{2}m\dot{\mathbf{r}}^2 - \tfrac{1}{2}e\dot{\mathbf{r}} \cdot (\mathbf{B} \times \mathbf{r}) + e\phi \\ &= \tfrac{1}{2}m\dot{\mathbf{r}}^2 + e\phi = \frac{1}{2m}\left|\mathbf{p} - \tfrac{1}{2}e\mathbf{B} \times \mathbf{r}\right|^2 + e\phi\end{aligned}$$

From Hamilton's equations

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{r}} = -\frac{1}{m}(\mathbf{p} - \tfrac{1}{2}e\mathbf{B}\times\mathbf{r})\cdot\frac{\partial}{\partial \mathbf{r}}(-\tfrac{1}{2}e\mathbf{B}\times\mathbf{r}) - e\frac{\partial \phi}{\partial \mathbf{r}} \qquad (\dagger)$$

Now

$$\frac{\partial}{\partial \mathbf{r}}(\mathbf{p}\cdot\mathbf{B}\times\mathbf{r}) = \frac{\partial}{\partial \mathbf{r}}(\mathbf{r}\cdot\mathbf{p}\times\mathbf{B}) = \mathbf{p}\times\mathbf{B}$$

$$\mathbf{B}\times\mathbf{r}\cdot\frac{\partial}{\partial \mathbf{r}}(\mathbf{B}\times\mathbf{r}) = \tfrac{1}{2}\frac{\partial}{\partial \mathbf{r}}(\mathbf{B}\times\mathbf{r}\cdot\mathbf{B}\times\mathbf{r}) = \tfrac{1}{2}\frac{\partial}{\partial \mathbf{r}}(B^2r^2 - (\mathbf{B}\cdot\mathbf{r})^2) = B^2\mathbf{r} - (\mathbf{B}\cdot\mathbf{r})\mathbf{B}$$

Substituting back into (†) we have

$$\dot{\mathbf{p}} = \frac{e}{2m}\mathbf{p}\times\mathbf{B} - \frac{e^2}{4m}(B^2\mathbf{r} - (\mathbf{B}\cdot\mathbf{r})\mathbf{B}) - e\frac{\partial \phi}{\partial \mathbf{r}} \qquad (\ddagger)$$

Replacing $\mathbf{p}$ with $\dot{\mathbf{r}}$ we find

$$m\ddot{\mathbf{r}} + \tfrac{1}{2}e\mathbf{B}\times\dot{\mathbf{r}} = \frac{e}{2m}(m\dot{\mathbf{r}} + \tfrac{1}{2}e\mathbf{B}\times\mathbf{r})\times\mathbf{B} - \frac{e^2}{4m}(B^2\mathbf{r} - (\mathbf{B}\cdot\mathbf{r})\mathbf{B}) - -e\frac{\partial \phi}{\partial \mathbf{r}}$$

The triple vector product cancels on the term with $(B^2\mathbf{r} - (\mathbf{B}\cdot\mathbf{r})\mathbf{B})$, so we get at the end the usual equation of motion with the Lorentz force.

$$m\ddot{\mathbf{r}} = e\dot{\mathbf{r}}\times\mathbf{B} - e\frac{\partial \phi}{\partial \mathbf{r}}.$$

Dotting (‡) through by $\mathbf{B}$, we find

$$\frac{\mathrm{d}\mathbf{B}\cdot\mathbf{p}}{\mathrm{d}t} = -e\mathbf{B}\cdot\frac{\partial \phi}{\partial \mathbf{r}}$$

Thus the next part of the question holds only if the term with the electrostatic potential vanishes, for example because $\phi =$ constant.

The final proposition is false, as you can convince yourself by dotting (‡) through by $\mathbf{p}$: on the left you then have $\mathrm{d}p^2/\mathrm{d}t$ and on the right an expression that does not vanish even when $\phi =$ constant. But we already know that the component of $\mathbf{p}$ along $\mathbf{B}$ is constant, so the sum of the squares of the other two components can be constant only if p^2 is. What we *is* true is that when $\phi =$ constant, the sum of the squares of the components of $\dot{\mathbf{r}}$ perpendicular to $\mathbf{B}$ is constant.

Time-translation symmetry causes H to be constant.

3. *Let p and q be canonically conjugate coordinates, and let $f(p,q)$ and $g(p,q)$ be functions on phase space. Define the Poisson bracket $[f,g]$. Let $H(p,q)$ be the Hamiltonian that governs the system's dynamics. Write down the equations of motion of p and q in terms of H and the Poisson bracket.*

In a galaxy, the density of stars in phase space is $f(\mathbf{p},\mathbf{q},t)$, where $\mathbf{p}$ and $\mathbf{q}$ each have three components. When evaluated at the location $\big(\mathbf{p}(t),\mathbf{q}(t)\big)$ of any given star, f is time-independent. Show that f consequently satisfies

$$\frac{\partial f}{\partial t} = [H,f],$$

where H is the Hamiltonian that governs the motion of every star.

Consider motion in a circular, razor-thin galaxy in which the potential energy of any star is given by the function $V(R)$, where R is a radial coordinate. Express H in terms of plane polar coordinates R, ϕ and their conjugate momenta, with the origin coinciding with the galaxy's centre. Hence, or otherwise, show that in this system f satisfies the equation

$$\frac{\partial f}{\partial t} + \frac{p_R}{m}\frac{\partial f}{\partial R} + \frac{p_\phi}{mR^2}\frac{\partial f}{\partial \phi} - \left(\frac{\partial V}{\partial R} - \frac{p_\phi^{\,2}}{mR^3}\right)\frac{\partial f}{\partial p_R} = 0,$$

where m is are the mass of the star.

$$\dot{q} = [q, H] \qquad \dot{p} = [p, H]$$

From the constancy of f we have with the chain rule

$$\begin{aligned} 0 = \frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \dot{\mathbf{q}} \cdot \frac{\partial f}{\partial \mathbf{q}} + \dot{\mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{p}} \\ &= \frac{\partial f}{\partial t} + \frac{\partial H}{\partial \mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{q}} - \frac{\partial H}{\partial \mathbf{q}} \cdot \frac{\partial f}{\partial \mathbf{p}} \\ &= \frac{\partial f}{\partial t} + [f, H] \end{aligned}$$

from which the required result follows immediately.

$$L = \tfrac{1}{2} m(\dot{R}^2 + (R\dot{\phi})^2) - V(R)$$

so $p_R = m\dot{R}$, $p_\phi = mR^2\dot{\phi}$ and

$$\begin{aligned} H &= p_R \dot{R} + p_\phi \dot{\phi} - \tfrac{1}{2} m(\dot{R}^2 + (R\dot{\phi})^2) + V(R) \\ &= \frac{p_R^2}{2m} + \frac{p_\phi^2}{2mR^2} + V \end{aligned}$$

Thus

$$\begin{aligned} [H, f] &= \frac{\partial}{\partial R}\left(\frac{p_\phi^2}{2mR^2} + V\right)\frac{\partial f}{\partial p_R} - \frac{\partial}{\partial p_R}\left(\frac{p_R^2}{2m}\right)\frac{\partial f}{\partial R} - \frac{\partial}{\partial p_\phi}\left(\frac{p_\phi^2}{2mR^2}\right)\frac{\partial f}{\partial \phi} \\ &= \left(-\frac{p_\phi^2}{mR^r} + \frac{\partial V}{\partial R}\right)\frac{\partial f}{\partial p_R} - \frac{p_R}{m}\frac{\partial f}{\partial R} - \frac{p_\phi}{mR^2}\frac{\partial f}{\partial \phi} \end{aligned}$$

from which the required result follows.

4. *Show that in spherical polar coordinates the Hamiltonian of a particle of mass m moving in a potential $V(\mathbf{x})$ is*

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + V(\mathbf{x}).$$

Show that $p_\phi =$ constant when $\partial V/\partial\phi \equiv 0$ and interpret this result physically.

Given that V depends only on r, show that $[H, K] = 0$ where $K \equiv p_\theta^2 + \dfrac{p_\phi^2}{\sin^2\theta}$. By expressing K as a function of $\dot{\theta}$ and $\dot{\phi}$ interpret this result physically.

Consider circular motion with angular momentum h in a spherical potential $V(r)$. Evaluate $p_\theta(\theta)$ when the orbit's plane is inclined by ψ to the equatorial plane. Show that $p_\theta = 0$ when $\sin\theta = \pm\cos\psi$ and iterpret this result physically.

K.E. is $\frac{1}{2}m[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2]$ and P.E. is V

$L = \frac{1}{2}m[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2] - V$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad p_\theta = mr^2\dot{\theta} \quad p_\phi = mr^2\sin^2\theta\dot{\phi}$$

$$\begin{aligned} H = \mathbf{p}\cdot\dot{\mathbf{q}} - L &= m\dot{r}^2 + mr^2\dot{\theta}^2 + mr^2\sin^2\theta\dot{\phi}^2 - \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) + V \\ &= \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) + V \\ &= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + V \end{aligned}$$

As in Q1,

$$\frac{\partial H}{\partial \phi} = \frac{\partial V}{\partial \phi} = 0 \quad \Rightarrow \quad \dot{p}_\phi = 0$$

i.e., angular momentum about the symmetry axis is conserved. Introducing $K \equiv p_\theta^2 + p_\phi^2/\sin^2\theta$

$$H = \frac{p_r^2}{2m} + \frac{K}{2mr^2} + V$$

$$[H, K] = \frac{1}{2m}[p_r^2, K] + \frac{1}{m}\left[\frac{K}{r^2}, K\right] + [V, K].$$

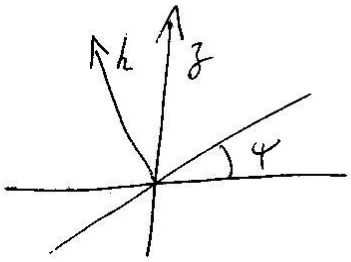
Now $[., K]$ is a first-order differential operator, so if A, B are any two functions on phase space, $[AB, K] = A[B, K] + [A, K]B$. Applying this result with $A = K$ and $B = r^{-2}$ we have

$$[H, K] = \frac{1}{2m}[p_r^2, K] + \frac{1}{m}K\left[\frac{1}{r^2}, K\right] + [V, K].$$

In these Poisson brackets the only terms are those in which the object on the left is differentiated w.r.t θ, ϕ or p_θ since K is a function of p_θ, p_ϕ and θ only. But none of p_r^2, r^{-2} and V depends on θ, ϕ or p_θ. So $[H, K] = 0$.

$$K = m^2r^2[(r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2] = m(rv_t)^2.$$

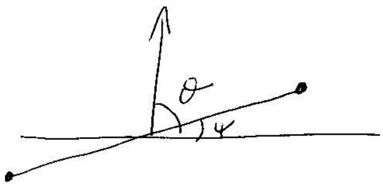
K is the total squared angular momentum and this is constant because the potential is spherically symmetric.



From the figure $p_\phi = h\cos\psi$, $K = h^2$, so

$$p_\theta^2 = h^2\left(1 - \frac{\cos^2\psi}{\sin^2\theta}\right) \quad \to 0 \quad \text{as} \quad \sin\theta \to \pm\cos\psi$$

When the particle is at the furthest point above the plane, it is moving into the paper when we view the orbit edge-on like this:



5. *Oblate spheroidal coordinates* (u, v, ϕ) *are related to regular cylindrical polars* (R, z, ϕ) *by*

$$R = \Delta\cosh u\cos v \quad ; \quad z = \Delta\sinh u\sin v.$$

Show that in these coordinates momenta of a particle of mass m *are*

$$\begin{aligned} p_u &= m\Delta^2(\cosh^2 u - \cos^2 v)\dot{u}, \\ p_v &= m\Delta^2(\cosh^2 u - \cos^2 v)\dot{v}, \\ p_\phi &= m\Delta^2\cosh^2 u\cos^2 v\dot{\phi}. \end{aligned}$$

Hence show that the Hamiltonian for motion in a potential $\Phi(u,v)$ is

$$H = \frac{p_u^2 + p_v^2}{2m\Delta^2(\cosh^2 u - \cos^2 v)} + \frac{p_\phi^2}{2m\Delta^2\cosh^2 u\cos^2 v} + \Phi.$$

Show that $[H, p_\phi] = 0$ and hence that p_ϕ is a constant of motion. Identify it physically.

From the lecture notes

$$L = \tfrac{1}{2}m\Delta^2\left[(\cosh^2 u - \cos^2 v)(\dot u^2 + \dot v^2) + \cosh^2 u\cos^2 v\dot\phi^2\right] - \Phi$$

Hence

$$\begin{aligned}
p_u &= \frac{\partial L}{\partial \dot u} = m\Delta^2(\cosh^2 u - \cos^2 v)\dot u \\
p_v &= m\Delta^2(\cosh^2 u - \cos^2 v)\dot v \\
p_\phi &= m\Delta^2\cosh^2 u\cos^2 v\,\dot\phi
\end{aligned}$$

Now

$$\begin{aligned}
H &= \frac{p_u^2}{m\Delta^2(\cosh^2 u - \cos^2 v)} + \frac{p_v^2}{m\Delta^2(\cosh^2 u - \cos^2 v)} + \frac{p_\phi^2}{m\Delta^2\cosh^2 u\cos^2 v} \\
&\quad - \tfrac{1}{2}m\Delta^2\left[\frac{p_u^2 + p_v^2}{(m\Delta^2)^2(\cosh^2 u - \cos^2 v)} + \left(\frac{p_\phi}{m\Delta^2\cosh u\cos v}\right)^2\right] + \Phi \\
&= \frac{p_u^2 + p_v^2}{2m\Delta^2(\cosh^2 u - \cos^2 v)} + \frac{p_\phi^2}{2m\Delta^2\cosh^2 u\cos^2 v} + \Phi
\end{aligned}$$

If $\Phi(u,v)$ only, $\partial H/\partial\phi = 0$ so $\dot p_\phi = 0$ and p_ϕ (angular momentum about the symmetry axis) is constant.

6. *A particle of mass m and charge Q moves in the equatorial plane $\theta = \pi/2$ of a magnetic dipole. Given that the dipole has vector potential*

$$\mathbf{A} = \frac{\mu\sin\theta}{4\pi r^2}\,\mathbf{e}_\phi,$$

evaluate the Hamiltonian $H(p_r, p_\phi, r, \phi)$ of the system.

The particle approaches the dipole from infinity at speed v and impact parameter b. Show that p_ϕ and the particle's speed are constants of motion.

Show further that for $Q\mu > 0$ the distance of closest approach to the dipole is

$$D = \tfrac{1}{2}\begin{cases} b - \sqrt{b^2 - a^2} & \text{for } \dot\phi > 0 \\ b + \sqrt{b^2 + a^2} & \text{for } \dot\phi < 0 \end{cases} \quad \text{where} \quad a^2 \equiv \frac{\mu Q}{\pi m v}.$$

Motion is in the equatorial plane, so we can set $\sin\theta = 1$ and $\dot\theta = 0$. Then

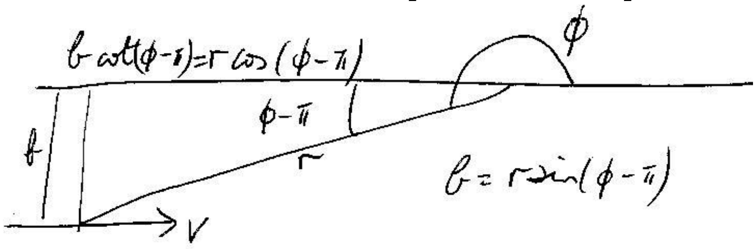
$$L = \tfrac{1}{2}m(\dot r^2 + (r\dot\phi)^2) + Qr\dot\phi\frac{\mu}{4\pi r^2}$$

$$\begin{aligned}
p_r &= m\dot r \\
p_\phi &= mr^2\dot\phi + \frac{Q\mu}{4\pi r} \quad \Rightarrow \quad \dot\phi = \left(p_\phi - \frac{Q\mu}{4\pi r}\right)\Big/ mr^2
\end{aligned}$$

Hence

$$\begin{aligned}
H &= p_r\dot r + p_\phi\dot\phi - \tfrac{1}{2}m(\dot r^2 + (r\dot\phi)^2) - Qr\dot\phi\frac{\mu}{4\pi r^2} \\
&= \tfrac{1}{2}m(\dot r^2 + r^2\dot\phi^2) \\
&= \frac{1}{2m}\left[p_r^2 + \frac{1}{r^2}\left(p_\phi - \frac{Q\mu}{4\pi r}\right)^2\right]
\end{aligned}$$

Since $\partial H/\partial\phi = 0$, $p_\phi =$ constant. Also constant $= H = \frac{1}{2}m(\dot r^2 + (r\dot\phi)^2) = \frac{1}{2}mv^2$, so the speed is constant.



There are 2 cases to consider: initially either $\dot\phi > 0$ or $\dot\phi < 0$. In the figure $\dot\phi > 0$ and

$$
\begin{aligned}
v &= -\frac{\mathrm{d}}{\mathrm{d}t}(b\cot(\phi-\pi)) = -b\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\tan(\phi-\pi)} = b\frac{\sec^2(\phi-\pi)}{\tan^2(\phi-\pi)}\dot\phi \\
&= b\dot\phi\frac{1}{\sin^2(\phi-\pi)} = \frac{r^2\dot\phi}{b}
\end{aligned}
$$

or similarly, $v = -r^2\dot\phi/b$ if $\dot\phi < 0$.

At ∞ $p_\phi = mr^2\dot\phi = \pm mb\dot r$. At closest approach $r\dot\phi = \pm v$, so

$$\pm mbv = p_\phi = mr^2\dot\phi + \frac{Q\mu}{4\pi r} = \pm mrv + \frac{Q\mu}{4\pi r}.$$

Hence

$$r^2 - br \mp \frac{Q\mu}{4\pi mv} = 0 \quad ; \quad r = b \pm \sqrt{b^2 \pm a_0^2}$$

We require $r > 0$, so if we take the plus sign in the root, we must take the plus sign before the root. That is

$$r = \begin{cases} b + \sqrt{b^2 + a_0^2} & \text{if } \dot\phi < 0 \\ b - \sqrt{b^2 - a_0^2} & \text{if } \dot\phi > 0 \end{cases}$$

7. *A point charge q is placed at the origin in the magnetic field generated by a spatially confined current distribution. Given that*

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3}$$

and $\mathbf{B} = \nabla \times \mathbf{A}$ with $\nabla\cdot\mathbf{A} = 0$, show that the field's momentum

$$\mathbf{P} \equiv \epsilon_0 \int \mathbf{E}\times\mathbf{B}\,\mathrm{d}^3\mathbf{x} = q\mathbf{A}(0).$$

Use this result to interpret the formula for the canonical momentum of a charged particle in an e.m. field.

$$
\begin{aligned}
P &= \epsilon_0 \int \mathrm{d}^3\mathbf{x}\,\frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3} \times (\nabla\times\mathbf{A}) \\
&= -\frac{q}{4\pi}\int \mathrm{d}^3\,\mathbf{x}\left(\nabla\frac{1}{r}\right) \times (\nabla\times\mathbf{A})
\end{aligned}
$$

In tensor notation

$$
\begin{aligned}
P_i &= -\frac{q}{4\pi}\int \mathrm{d}^3\mathbf{x}\,\epsilon_{ijk}\left(\nabla_j\frac{1}{r}\right)\epsilon_{klm}\nabla_l A_m \\
&= -\frac{q}{4\pi}\int \mathrm{d}^3\mathbf{x}\,(\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im})\left(\nabla_j\frac{1}{r}\right)\nabla_l A_m \\
&= -\frac{q}{4\pi}\int \mathrm{d}^3\mathbf{x}\left[\left(\nabla_j\frac{1}{r}\right)\nabla_i A_j - \left(\nabla_j\frac{1}{r}\right)\nabla_j A_i\right].
\end{aligned}
$$

Now

$$\int \mathrm{d}^3\mathbf{x}\,\left(\nabla_j\frac{1}{r}\right)\nabla_i A_j = \oint \mathrm{d}^2 S_j\,\left(\frac{1}{r}\nabla_i A_j\right) - \int \mathrm{d}^3\mathbf{r}\,\frac{1}{r}\nabla_i\nabla_j A_j = 0$$

$$\int \mathrm{d}^3\mathbf{x}\,\left(\nabla_j\frac{1}{r}\right)\nabla_j A_i = \oint \mathrm{d}^2 S_j\,\left(\nabla_j\frac{1}{r}\right)A_i - \int \mathrm{d}^3\mathbf{r}\,\left(\nabla^2\frac{1}{r}\right)A_i$$

We can discard the surface terms, which vanish provided $\mathbf{A}\to 0$ at ∞, no matter how slowly. Also $\nabla^2 r^{-1} = -4\pi\delta^3(\mathbf{r})$, so

$$P_i = q\int \mathrm{d}^3\mathbf{r}\,\delta^3(\mathbf{r})A_i = A_i(0).$$

8. *For each convex function $f(x)$, i.e. for each $f(x)$ for which $f''(x)>0$, define $F(x,p)$ to be the function of two variables*

$$F(x,p)\equiv xp - f(x).$$

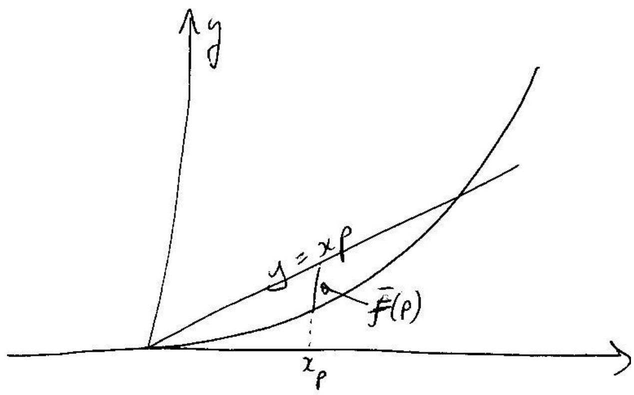
Show that for each fixed p, $F(x,p)$ has a unique maximum with respect to x when $f'(x)=p$. Let this maximum occur at x_p. We define the Legendre transform of f to be

$$\overline{f}(p)\equiv F(x_p,p).$$

Show that the Legendre transform $\overline{\overline{f}}(q)$ of $\overline{f}(p)$ is $\overline{\overline{f}}(q)=f(q)$. (In other words on applying the transform twice you recover your original function.)

$0=\partial F/\partial x = p - f'(x)$ so x_p is the root of $p=f'(x_p)$.

$$\overline{f}(p)\equiv F(x_p,p) = x_p p - f(x_p)$$



Now introduce $G(p,q)\equiv pq-\overline{f}(p)$. p_q is the root of

$$q=\overline{f}'(p_q) = \frac{\partial x_p}{\partial p}p + x_p - \frac{\partial f}{\partial x_p}\frac{\partial x_p}{\partial p} = x_p \equiv x_{p_q}.$$

$$\begin{aligned}\overline{\overline{f}}(q) &= G(p_q,q) = p_q q - \overline{f}(p_q)\\ &= p_q q - [x_{p_q}p_q - f(x_{p_q})]\\ &= f(x_{p_q}) - p_q(q - x_{p_q}).\end{aligned}$$

But we have shown that $x_{p_q}=q$, so $\overline{\overline{f}}(q)=f(q)$ as required.

9. *Show that the generating function of the form $S(\mathbf{P}, \mathbf{x})$ which generates the Gallilean transformation between frames in relative motion at velocity $\mathbf{V}$ is*

$$S = \mathbf{P} \cdot \mathbf{x} + \mathbf{V} \cdot (m\mathbf{x} - t\mathbf{P}).$$

Given $S(P, x) = Px + V(mx - tP)$

$$X = \frac{\partial S}{\partial P} = x - Vt \quad ; \quad p = \frac{\partial S}{\partial x} = P + mV$$

Thus $P = p - mV$.

10. *A point transformation is specified by n functions $Q_j(\mathbf{q})$ of the old coordinates $\mathbf{q}$. Show that any point transformation is canonical by evaluating $[Q_i, Q_j]$, $[P_i, P_j]$, etc., where $\mathbf{P} \equiv \partial L/\partial \dot{\mathbf{Q}}$, with L the Lagrangian.*

By the chain rule

$$\dot{q}_i = \sum_j \frac{\partial q_i}{\partial Q_j} \dot{Q}_j \quad \Rightarrow \quad \frac{\partial \dot{q}_i}{\partial \dot{Q}_j} = \frac{\partial q_i}{\partial Q_j}.$$

Hence

$$P_i = \frac{\partial L}{\partial \dot{Q}_i} = \sum_j \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\dot{Q}_i} = \sum_j p_j \frac{\partial q_j}{\partial Q_i}.$$

Now

$$[Q_i, Q_j] = \sum_k \frac{\partial Q_i}{\partial q_k} \frac{\partial Q_j}{\partial p_k} - \frac{\partial Q_i}{\partial p_k} \frac{\partial Q_j}{\partial q_k} = 0$$

because when all the q_l are held constant, all the Q_l have to be constant also. Also

$$\begin{aligned}
[Q_i, P_j] &= \sum_k \frac{\partial Q_i}{\partial q_k} \frac{\partial P_j}{\partial p_k} - \frac{\partial Q_i}{\partial p_k} \frac{\partial P_j}{\partial q_k} \\
&= \sum_{kl} \frac{\partial Q_i}{\partial q_k} \frac{\partial}{\partial p_k} \left(p_l \frac{\partial q_l}{\partial Q_j} \right) \\
&= \sum_k \frac{\partial Q_i}{\partial q_k} \frac{\partial q_k}{\partial Q_j} = \frac{\partial Q_i}{\partial Q_j} = \delta_{ij}.
\end{aligned}$$

Finally,

$$\begin{aligned}
[P_i, P_j] &= \sum_k \frac{\partial P_i}{\partial q_k} \frac{\partial P_j}{\partial p_k} - \frac{\partial P_i}{\partial p_k} \frac{\partial P_j}{\partial q_k} \\
&= \sum_{kl} p_l \frac{\partial^2 q_l}{\partial q_k \partial Q_i} \frac{\partial q_k}{\partial Q_j} - \frac{\partial q_k}{\partial Q_i} p_l \frac{\partial^2 q_l}{\partial q_k \partial Q_j} \\
&= \sum_{kl} p_l \left(\frac{\partial^2 q_l}{\partial Q_j \partial Q_i} - \frac{\partial^2 q_l}{\partial Q_i \partial Q_j} \right) = 0.
\end{aligned}$$