The 2-dimensional XY model

The $O(n)$ models with $n \geq 2$ are examples of models with a continuous (rather than a discrete - eg Ising) symmetry.
For these models, the Mermin-Wagner-Hohenberg-Coleman thm. states that there can be no ordered state $\langle \vec{s} \rangle = 0$
If we write $\vec{s}(r) = (s_1(r), \ldots, s_n(r))$ with $\vec{s}^2 = 1$

Suppose that $\langle \vec{s} \rangle \neq 0$ & points in the 1-direction

Write $\vec{s} = (\sqrt{1-\sigma^2}, \vec{\sigma})$ where $\vec{\sigma}$ has $n-1$ components

$$\mathcal{H} = -\tfrac{1}{2}\beta \sum_{r,r'} J(r-r')\, \sigma(r)\cdot\sigma(r') \approx \text{const} + \underbrace{\tfrac{1}{2}K \int (\nabla\vec{\sigma})^2 d^2r}$$

The effective action for the $\vec{\sigma}$-field (spin waves) is

with $$\langle \sigma_i(r)\,\sigma_j(r')\rangle = \frac{\delta_{ij}}{K} \int \frac{e^{ik\cdot(r-r')}}{k^2} \frac{d^dk}{(2\pi)^d}$$

Brillouin zone

To next order $\langle s_1(r)\rangle = 1 - \frac{1}{2}\langle \vec{\sigma}^2(r)\rangle$

$$= 1 - \frac{n-1}{2K} \int_{BZ} \frac{d^dk}{k^2 (2\pi)^d}$$

For $d>2$ this is finite, but for $d \leq 2$ it diverges $\Rightarrow$ the fluctuations destroy the ordered state.
$d=2$ is called the <u>lower critical dimension</u> -

Although this theorem means there is no spontaneous magnetisation, for $d=2$ it turns out that for $n=2$ there is still a special kind of a phase transition.

$n=2, d=2$ is important because it describes thin film superconductors, superfluid He, as well as being mapped to other important problems. It also can be solved 'exactly' using the RG.

For $n=2$ it is convenient to write $\vec{S}(r) = (\cos\theta(r), \sin\theta(r))$

$$\mathcal{H} = -\frac{1}{2}\beta\sum_{rr'} J(r-r')\cos(\theta(r)-\theta(r'))$$

At low T, we expect the fluctuations to be small, so expand the cosine. The result, in the naive continuum limit, is

$$\mathcal{H} = \text{const} + K\int\left[\frac{1}{2}a^2(\nabla\theta)^2 - u_4 a^4(\nabla\theta)^4 + \ldots\right]\frac{d^2r}{a^2}$$

where $K \propto \dfrac{\beta R^2 J}{a^2}$

Note that, by power counting, u_4 etc are irrelevant, so

that $\mathcal{H}$ flows into $\mathcal{H}^* = \frac{1}{2}K\int(\nabla\theta)^2 d^2r$

EN. This is exactly scale invariant : $dK/dl = 0$. Note that unlike ϕ^4 theory we cannot rescale θ so that $K=1$, since θ has period 2π : K parametrises a line of fixed points

Let us compute the properties of this fixed point theory

$$\langle \theta(r_1)\,\theta(r_2)\rangle \equiv G(r_1-r_2) = \frac{1}{K}\int_{BZ} \frac{e^{ik(r_1-r_2)}}{k^2}\frac{d^2k}{(2\pi)^2}$$

This is divergent at $k=0$. However it turns out that physical quantities involve only

$$G(r_{12}) - G(0) = -\frac{1}{K}\int \frac{1-e^{ik(r_1-r_2)}}{k^2}\frac{d^2k}{(2\pi)^2}$$

$$\sim -\frac{1}{2\pi K}\ln\left(r/a\right) + c/K + \dots$$

More interesting are the spin-spin correlators

$$\langle \vec{s}(r_1)\cdot\vec{s}(r_2)\rangle = \mathrm{Re}\,\langle e^{i(\theta(r_1)-\theta(r_2))}\rangle$$

In a Gaussian theory

$$\langle e^{i(\theta(r_1)-\theta(r_2))}\rangle = \langle 1 + \cancel{i(\theta(r_1)-\theta(r_2))} - \tfrac{1}{2}(\theta(r_1)-\theta(r_2))^2 + \dots\rangle$$

$$= e^{-\frac{1}{2}(\theta(r_1)-\theta(r_2))^2} + \text{nothing else}$$

$$= e^{-(G(0)-G(r_{12}))} \sim \frac{\text{const.}}{r_{12}^{1/2\pi K}}$$

so it decays with a continuous varying exponent $\eta = \frac{1}{2\pi K} \propto T$

However, this cannot be valid a high enough T, where $\langle \vec{s}\cdot\vec{s}\rangle$ should decay exponentially.

The reason is that vortices have been neglected - configs where the $\nabla\theta$ is small almost everywhere, but θ changes by $2\pi n$ on going round a given point.

The energy of a vertex with $\theta \approx n\phi$ is

$$E = \frac{1}{2}K \int_a^L \left(\frac{n}{r}\right)^2 d^2r \sim \pi n^2 K \ln\left(L/a\right)$$ so

a single vortex has infinite energy as $L \to \infty$.

The Kosterlitz - Thouless criterion balances this against its entropy:

$$F = E - (T)S = \pi K n^2 \ln L/a - \ln\left(L/a\right)^2$$

so for $K > 2/\pi$ $\left(T < T_{KT} = \frac{\pi J R^2}{4 k_B a^2} \right)$.

vortices are suppressed.
For $T > T_{KT}$ they proliferate.

A better way of doing this is through the RG:

let us introduce a fugacity y_0 for vortices: N vortices have a coefficient y_0^N.

The terms of $O(y_0^2)$ has energy given by substituting

$$\theta(r) = \overset{v}{\phi}(r - r_1) - \overset{\bar{v}}{\phi}(r - r_2)$$ into the action

$$E(r_1, r_2) \sim 2\pi K \ln(r_{12}/a) + 2\pi K \tilde{C}$$

so the vortex - antivortex correlator is $\propto \left(\frac{a}{r_{12}}\right)^{2\pi K}$

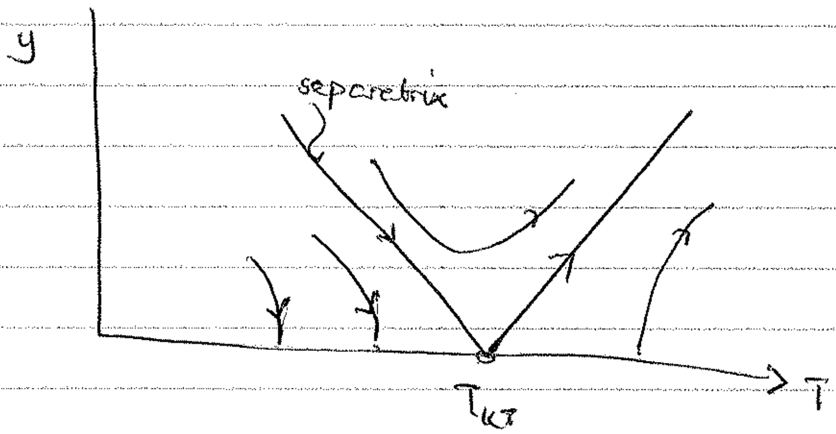
This means that $x_v = \pi K$, $y_v = 2 - \pi K$

so $[y = y_0 e^{-\pi K \tilde{C}}]$ $\quad \frac{dy}{dl} = (2 - \pi K) y + \dots$

If we let $x = 2\pi K$ then we also expect

$$\frac{dx}{dl} = Ay^2 \quad \text{where } A \text{ is the OPE coeff.}$$

RG flows.



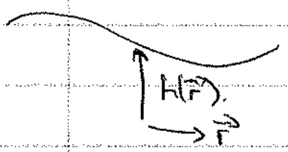
Features:

- low T phase : $y \to 0$ ie Gaussian theory works $\eta = \eta(T_\infty)$.
  At T_c $\quad \eta = \dfrac{1}{2\pi K (= \frac{2}{\pi})} = \dfrac{1}{4}$ exact

- high T phase : vortices proliferate.
  can show that $\quad \xi \propto e^{b/\sqrt{T - T_c}}$.

A related problem is the roughening transition

Imagine $h(\vec{r})$ $(\vec{r} \in \mathbb{R}^2)$ as giving the height of a crystalline surface (for simplicity we take this to be simple cubic).



The energy has 2 terms:

surface tension $\quad \sigma \times$ area

$$\simeq \sigma \int d^2 r \sqrt{1 + (\nabla h)^2}$$

$$\simeq \text{const} + \tfrac{1}{2}\sigma \int d^2 r \, (\nabla h)^2 .$$

and a term which says that h would like to be an integer multiple of a, lattice spacing.

We could for example choose $V(h) = -V_0 \cos(2\pi h/a)$

So $S = \frac{E}{kT} = \frac{1}{2}\frac{\sigma}{kT}\int d^2r\, (\nabla h)^2 - V_0 \int d^2r \cos\frac{2\pi h}{a}$

This is called a sine Gordon theory.

If we can neglect V_0, then $K = \frac{\sigma}{kT}$.

$$\langle (h(r)-h(0))^2\rangle \sim G(0) - G(r) \propto \frac{1}{2\pi K}\ln r/a .$$

– the surface is rough.

We can test for whether V_0 is relevant by working out

$$\left\langle \cos\frac{2\pi h(r)}{a}\cos\frac{2\pi h(0)}{a}\right\rangle \sim \mathrm{Re}\, e^{i\frac{2\pi}{a}(h(r)-h(0))}$$

$$\sim e^{-\left(\frac{2\pi}{a}\right)^2[G(0)-G(r)]} \sim \frac{1}{r^{\frac{1}{2\pi K}\cdot\left(\frac{2\pi}{a}\right)^2}}$$

So $x_{V_0} = \frac{2\pi k_B T}{\sigma a^2}$

$y_{V_0} = 2 - x_{V_0}$.

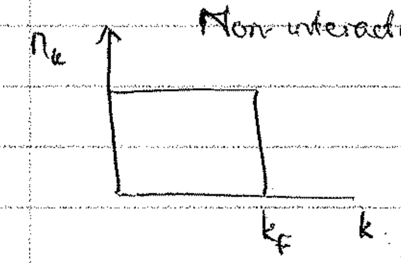
If $x_{V_0} > 2$ V_0 is irrelevant i.e. $T > T_R = \frac{\sigma a^2}{\pi k_B}$

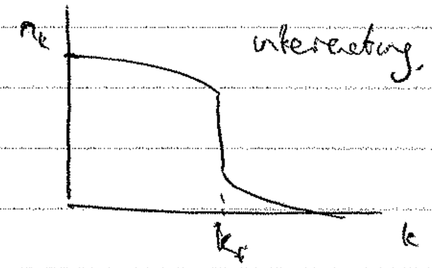
& the surface is smooth.

For $T < T_R$ it feels the crystalline structure and it is smooth or facetted.

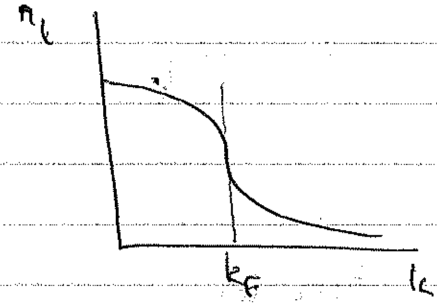
Bosonization in 1+1 dimensions

Interacting fermions are an obvious problem of importance in condensed matter theory: eg. electrons in a metal. Fortunately there is a simple qualitative (and quantitative) phenomelogical theory which works very well for $d \geq 2$: Fermi liquid theory. It turns out that, for reasons of phase space an interacting Fermi gas is not qualitatively that much different from a free gas. eg. the 1-particle momentum distribution at $T=0$:

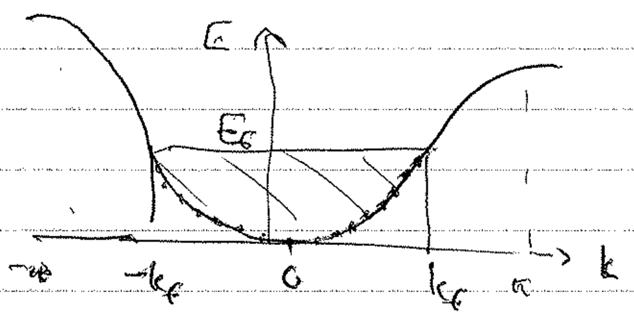




However, in $d=1$, things are very different: there is no discontinuity in n_k. Instead, there is a (non-universal) power law singularity at k_F:



If we think about a non-interacting Fermi gas in $d=1$, its dispersion relation looks like



Depending on the electron density, the Fermi sea at $T=0$ is filled to $E_F \leftrightarrow k_F$.

One of the important excitations at low T consists of moving a particle from $E \leq E_f$ to $E \geq E_f$ - exciting a particle-hole pair. This behaves like a <u>boson</u>. Bosonization is a way of deriving an effective low energy theory for these excitations

<u>Bosonization of a fermion with one chirality</u>

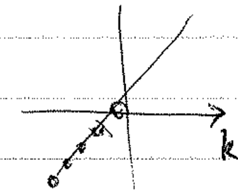
For simplicity consider to begin with the excitations around $+k_F$, and ignore spin. [These actually occur in the <u>edge states</u> of the quantum Hall effect].

We introduce a second-quantised Fermi field

$$\psi_R(x) = \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{\infty} c_{R,k}\, e^{ik\cdot x}\, \underset{\text{NB}}{e^{ik_F\cdot x}} \qquad k = \frac{2\pi}{L} \underset{\text{integer}}{n_k}$$

where $\{c_{R,k}, c_{R,k'}\} = 0 \qquad \{c_{Rk}, c_{Rk'}\} = \delta_{kk'}$

The Fermi sea is defined by
$$c_{R,k}|0\rangle = 0 \qquad k>0$$
$$c_{R,k}^{\dagger}|0\rangle = 0 \qquad k\leq 0$$



Fermion number operator $\hat{N}_R = \sum_{k=-\infty}^{\infty}{}' :c_{R,k}^{\dagger} c_{R,k}: \equiv \sum_{k>0} c_{R,k}^{\dagger} c_{R,k} - \sum_{k\leq 0} c_{R,k} c_{R,k}^{\dagger}$

so $\hat{N}_R|0\rangle = 0$.

Now define bosonic operators that create & destroy particle-hole pairs

$$b_{R,q}^{\dagger} = \frac{1}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} c_{R,k+q}^{\dagger} c_{R,k} \qquad (q>0)$$

$b_1 \equiv c_1^{\dagger} c_0 + \cancel{c_2^{\dagger} c_1} + c_0^{\dagger} c_{-1}$

$$\left(n_q = \frac{qL}{2\pi}\right)$$

$$b_{R,q} = \frac{1}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} c_{R,k}^{\dagger} c_{R,k+q}$$

Note that b & $b^\dagger$ commute with $\hat{N}_R$.

$$[b_q, b_{q'}] \propto \sum_{k,k'} \left[c^\dagger_k c_{k+q} c^\dagger_{k'} c_{k'+q'} - c^\dagger_{k'} c_{k'+q'} c^\dagger_k c_{k+q} \right]$$

$$= \sum_{kk'} \left[c^\dagger_k c^\dagger_{k'+q'} \delta_{k+q,k'} - c^\dagger_{k'} c_{k+q} \delta_{k'+q',k} \right]$$

$$= \sum_{k} \left[c^\dagger_k c^\dagger_{k+q+q'} \right] - \sum_{k'} \left[c^\dagger_{k'} c_{k'+q+q'} \right] = 0.$$

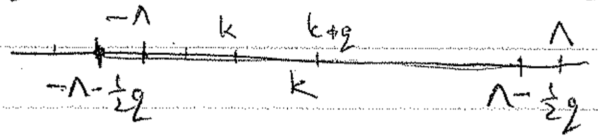
$$[b_q, b^\dagger_{q'}] = \frac{1}{\sqrt{n_q}} \frac{1}{\sqrt{n_{q'}}} \sum_{k,k'} \left[c^\dagger_k c_{k+q} c^\dagger_{k'+q'} c_{k'} - c^\dagger_{k'+q'} c_{k'} c^\dagger_k c_{k+q} \right]$$

$$= \frac{1}{\sqrt{n_q}} \frac{1}{\sqrt{n_{q'}}} \sum_{k,k'} \left[c^\dagger_k c_{k'} \delta_{k+q,k'+q'} - c^\dagger_{k'+q'} c_{k+q} \delta_{k',k} \right]$$

For $q = q'$ this looks like it is zero, but we have to be careful with the infinite sum over k.
The energy of one of these excitations is $v_F \, |2k+q|$
so impose a cut-off $|k + \frac{1}{2}q| < \Lambda$

Then eg. for $q = q'$ we get

$$\frac{1}{n_q} \sum_{|k+\frac{1}{2}q| < \Lambda} \left(c^\dagger_k c_k - c^\dagger_{k+q} c_{k+q} \right)$$



$$= \frac{1}{n_q} \left[\sum_{-\Lambda-\frac{1}{2}q < k < \Lambda - \frac{1}{2}q} c^\dagger_k c_k - \sum_{-\Lambda+\frac{1}{2}q < k < \Lambda + \frac{1}{2}q} c^\dagger_k c_k \right]$$

$$= \frac{1}{n_q} \left[\sum_{-\Lambda-\frac{1}{2}q < k < -\Lambda + \frac{1}{2}q} c^\dagger_k c_k - \sum_{\Lambda-\frac{1}{2}q < k < \Lambda+\frac{1}{2}q} c^\dagger_k c_k \right].$$

These are the numbers of particles in a range $|q|$ about $\pm\Lambda$. If we insist that these are not changed by the dynamics then we get 1, from the first term. Similarly if $q \neq q'$ we get $\sum_{|k-\frac{1}{2}q|<\Lambda} c^\dagger_k c_{k-q+q'} = 0$.

so $[b_q, b^\dagger_{q'}] = \delta_{qq'}$

Once we have bosonic creation & annihilation operators we can define fields

$$\chi_R(x) = \frac{i}{2\sqrt{\pi}} \sum_{q>0} \frac{1}{\sqrt{n_q}} b_{R,q} e^{iqx}$$

& $\chi_R^{\dagger}$.

$$[\chi_R(x), \chi_R^{\dagger}(x')] = +\frac{1}{4\pi} \sum_q \frac{1}{n_q} e^{iq(x-x')}$$

$$= -\frac{1}{4\pi}\sum' \ln\left(1 - e^{\frac{2\pi i}{L}(x-x')}\right) \qquad \text{recall } q = \frac{2\pi n_q}{L}$$

$$= -\frac{1}{4\pi} \ln\left[\frac{2\pi}{L} i(x-x')\right]$$

Defining $\phi_R(x) = \chi_R(x) + \chi_R^{\dagger}(x)$

$$[\phi_R(x), \phi_R(x')] = \frac{1}{4\pi} \ln\left[\frac{-i(x-x')}{+i(x-x')}\right] = -\frac{i}{4}\,\mathrm{sgn}(x-x')$$

We need formulas which connect the original fermion field $\psi_R(x)$ to ϕ_R. (or rather operators in terms of each of these).

The fermion density is

$$\rho_R(x) = :\psi_R^{\dagger}(x)\psi_R(x):$$

$$= \frac{1}{L}\sum_{k,k'} :c_{R,k}^{\dagger} c_{R,k'}: e^{-i(k-k')x} \qquad k' = k+q$$

$$= \frac{1}{L}\sum_{q>0}\underbrace{\sum_k c_{R,k}^{\dagger} c_{R,k+q}}_{\sqrt{n_q}\, b_{R,q}} e^{iqx} + \frac{1}{L}\sum_{q>0}\underbrace{\sum_k c_{R,k+q}^{\dagger} c_{R,k}}_{\sqrt{n_q}\, b_{R,q}^{\dagger}} e^{-iqx}$$

$$= -\frac{1}{\sqrt{\pi}} \frac{\partial \phi_R}{\partial x}$$

Similarly we can work out that

$$[b_{R_q}, \psi_R(x)] = -\frac{e^{-iqx}}{\sqrt{n_q}} \psi_R(x)$$

so $\quad b_{R_q} \psi_R(x) |0\rangle = -\frac{e^{-iqx}}{\sqrt{n_q}} \psi_R(x) |0\rangle$

$$\left[\frac{1}{\sqrt{n_q}\sqrt{L}} \sum_{k,k'} \underbrace{[c_k^\dagger c_{k+q}, c_{k'}]}_{-\delta_{k',k} c_{k+q}} e^{ik'x} \propto \sum_{k'} c_{k'+q} e^{ik'x} \propto e^{-iq \cdot x} \psi_R(x) \right]$$

These means that $\quad \psi_R(x)|0\rangle \quad$ is a coherent state of b_{R_q} for all q.

Therefore:

$$\psi_R(x)|0\rangle \propto e^{-\sum_{q>0} \frac{e^{-iqx}}{\sqrt{n_q}} b^\dagger_{R_q}} |0\rangle$$

$$= e^{-i2\sqrt{\pi}\, \chi_R^\dagger(x)} |0\rangle$$

$$= e^{-i2\sqrt{\pi}\, \phi_R(x)} |0\rangle .$$

We now make the stronger statement that this is true on all states, i.e.

$$\psi_R(x) \propto e^{-i2\sqrt{\pi}\, \phi_R(x)}$$

[check commutation relations]. Use $\quad e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$
if $[A,B]$ a c-number.

If $x<x'$ $\quad \psi_R(x)\psi_R^\dagger(x') \propto e^{-i2\sqrt{\pi}(\phi_R(x)-\phi_R(x')) + \frac{4\pi}{2}\cdot\frac{i}{4}}$

$\qquad\qquad\quad \psi_R(x')^\dagger \psi_R(x) \quad - - \quad . \quad . \quad . \quad -\frac{4\pi}{2}\frac{i}{4}$

$[\ ,\ \}$. sum $= 0$

Hamiltonian and interactions

First let us consider the ___free___ theory:

$$H_0 = v_F \sum_k k \, :c^\dagger_{R,k} c_{R,k}: = v_F\left(\sum_{k>0} c^\dagger_k c_k - \sum_{k\leq 0} c_k c^\dagger_k\right)$$

$$= -v_F \int dx \, :\psi^\dagger_R \, i\partial_x \psi_R: \qquad H_0|0\rangle = 0.$$

$b^\dagger_{R,q} \propto \sum_k c^\dagger_{k+q} c_k$ creates an excitation of energy $v_F q$,

and we can check that $[H_0, b^\dagger_{R,q}] = v_F q \cdot b^\dagger_{R,q}$.

Hence, in the boson language $H_0 = v_F \sum_{q>0} q \, b^\dagger_{R,q} b_{R,q}$

$$= v_F \int_0^L dx \, :(\partial_x \phi)^2:$$

because $\phi_R \propto \sum_{q>0} \frac{1}{\sqrt{q}} b_q e^{iqx}$

So H_0 is the hamiltonian for a free (chiral) boson.

The magic happens if we add interactions:

E.g. $V = \cancel{\frac{g_4}{2}\int \rho_R^2(x)\,dx \propto \int \psi^\dagger_R(x)}$

$$\int dx\,dx' \; V(x-x')\,\rho_R(x)\,\rho_R(x')$$ chiral Luttinger liquid.

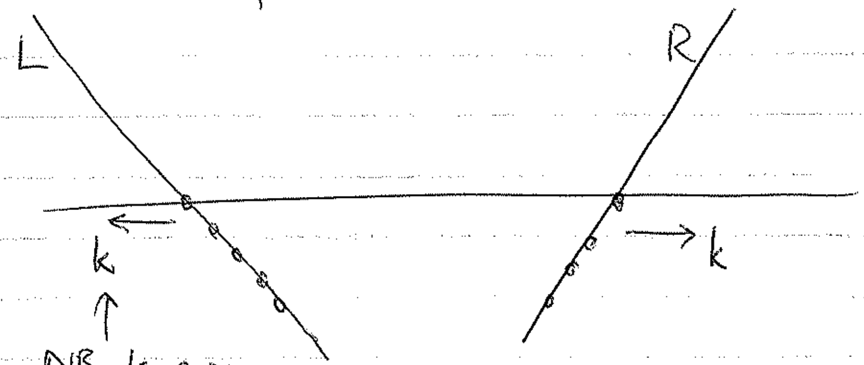
If $V(x-x')$ is short-ranged, this is equivalent in the low-energy approx. to

$$\frac{g_4}{2\pi}\int :(\partial_x\phi)^2:\,dx$$

i.e. all that happens is that v_F is renormalised!

Bosonization of a fermion with two chiralities

Before examining the consequences of this, let us immediately generalize to both chiralities, L & R.



NB k goes this way by convention.

Define $\quad b^{+}_{\nu,q} = \dfrac{1}{\sqrt{n_q}} \displaystyle\sum_{k=-\infty}^{\infty} c^{+}_{\nu,k+q}\, c_{\nu,k} \qquad \nu = L, R.$ or $\nu = -1, 1$

The corresponding fields

$$\phi_\nu(x) = -\frac{i\nu}{2\sqrt{\pi}} \sum_q \frac{1}{\sqrt{n_q}}\, b^{+}_{\nu,q}\, e^{i\nu q x} + \text{c.c.}$$

satisfy

$$[\phi_\nu(x), \phi_{\nu'}(x')] = -\frac{i\nu}{4}\,\delta_{\nu\nu'}\,\mathrm{sgn}(x-x').$$

Finally we can form non-chiral combinations

$$\left.\begin{aligned} \phi(x) &= \phi_R(x) + \phi_L(x) \\ \theta(x) &= -\phi_R(x) + \phi_L(x) \end{aligned}\right\} \quad \text{so that } [\phi(x), \theta(x')] = -\frac{i}{2}\,\mathrm{sgn}(x-x')$$

IMPORTANT

Recall that $\rho_R \propto \partial_x \phi_R$.

The total density is then $\rho(x) = \rho_R + \rho_L = -\dfrac{1}{\sqrt{\pi}}\,\partial_x \phi$

and the current is $j(x) = v_F(\rho_R - \rho_L) = \dfrac{v_F}{\sqrt{\pi}}\,\partial_x \theta$

The free Hamiltonian is

$$H_0 = v_F \int dx \left[(\partial_x \phi_L)^2 + (\partial_x \phi_R)^2 \right]$$

$$= \frac{1}{2} v_F \int dx \left[(\partial_x \phi) + (\partial_x \theta)^2 \right].$$

Note that ϕ & θ do not commute

$$[\phi(x), \theta(x')] = -\frac{i}{2} \operatorname{sgn}(x - x')$$

so $[\phi(x), \partial_{x'} \theta(x')] = i \delta(x - x')$

This means that $\partial_{x'} \theta(x') = \Pi(x')$, the canonical conjugate to $\phi(x')$!

Now add interaction: we can add terms $\propto \rho_L^2, \rho_R^2$ but once again they only renormalise v_F.

The interesting term is now $\propto g_2 \rho_L \rho_R$

$$= g_2 :\psi_L^\dagger(x) \psi_L(x): :\psi_R^\dagger(x) \psi_R(x):$$

This represents scattering of L-moving fermions against R-moving ones.

$$H = \int dx \left[v_F (\partial_x \phi_L)^2 + v_F (\partial_x \phi_R)^2 + 2 g_2 (\partial_x \phi_L)(\partial_x \phi_R) \right]$$

$$= \frac{1}{2} \int dx \left[(v_F + g_2)(\partial_x \phi)^2 + (v_F - g_2)(\partial_x \theta)^2 \right].$$

$$= \frac{1}{2} \int dx \left[(v_F + g_2)(\partial_x \phi)^2 + (v_F - g_2) \Pi(x)^2 \right].$$

This is more symmetric if we rescale

$$\Pi \Rightarrow \lambda^{-\frac{1}{2}} \tilde{\Pi} \qquad (\text{ie } \theta = \lambda^{-\frac{1}{2}} \tilde{\theta})$$

$$\phi = \lambda^{\frac{1}{2}} \tilde{\phi}$$

[thus preserving the commutation relations]

where $\lambda = \left(\frac{v_F - g_2}{v_F + g_2} \right)^{\frac{1}{2}}$

Then $H = \frac{1}{2} v \int ~~dt~~ dx \left[\tilde{\Pi}^2 + (\partial_x \tilde{\phi})^2 \right]$

where $v = \left[(v_F - g_2)(v_F + g_2) \right]^{\frac{1}{2}}$.

The action is $S = \frac{1}{2} \int dt\, dx \left[\frac{(\partial_t \tilde{\phi})^2}{v} - v (\partial_x \tilde{\phi})^2 \right]$

or, on letting $\tilde{t} = vt$:

$$S = \frac{1}{2} \int d\tilde{t}\, dx \left[(\partial_{\tilde{t}} \tilde{\phi})^2 - (\partial_x \tilde{\phi})^2 \right]$$

– the action for a (non-chiral) free field!

The 2-pt. function is

$$\langle 0 | T\, \tilde{\phi}(\tilde{t}, x)\, \tilde{\phi}^\dagger(0,0) | 0 \rangle = -\frac{1}{4\pi} \ln \left[\tilde{t}^2 - x^2 \right]$$

$$= -\frac{1}{2\pi} \ln (vt - x) - \frac{1}{2\pi} \ln (vt + x)$$

$$\equiv \langle \tilde{\phi}_R \tilde{\phi}_R^\dagger \rangle + \langle \tilde{\phi}_L \tilde{\phi}_L^\dagger \rangle \qquad \text{with } \langle \phi_R \phi_L^\dagger \rangle = 0.$$

Fermion Green's function

Consider the 1-particle propagator for the R-moving fermions:

$$\langle 0 | T\, \psi_R(t,x)\, \psi_R(0,0)^\dagger\, |0\rangle$$

$$\propto e^{ik_F x} \left\langle e^{-i2\sqrt{\pi}\,\phi_R(t,x)}\; e^{i2\sqrt{\pi}\,\phi_R(0,0)} \right\rangle$$

$$= e^{ik_F x} \left\langle e^{-i2\sqrt{\pi}\,(\phi-\theta)/2}\; e^{i2\sqrt{\pi}\,(\phi-\theta)/2} \right\rangle$$

$$= e^{ik_F x} \left\langle e^{-i2\sqrt{\pi}\,(\lambda^{\frac{1}{2}}\tilde{\phi} - \lambda^{-\frac{1}{2}}\tilde{\theta})/2}\; e^{i2\sqrt{\pi}\,(\lambda^{\frac{1}{2}}\tilde{\phi} - \lambda^{-\frac{1}{2}}\tilde{\theta})/2} \right\rangle$$

$$= e^{ik_F x} \left\langle e^{-i2\sqrt{\pi}\left[\frac{\lambda^{\frac{1}{2}}+\lambda^{-\frac{1}{2}}}{2}\tilde{\phi}_R + \frac{\lambda^{\frac{1}{2}}-\lambda^{-\frac{1}{2}}}{2}\tilde{\phi}_L\right]}\; e^{i2\sqrt{\pi}\left[\frac{\lambda^{\frac{1}{2}}+\lambda^{-\frac{1}{2}}}{2}\tilde{\phi}_R + \frac{\lambda^{\frac{1}{2}}-\lambda^{-\frac{1}{2}}}{2}\tilde{\phi}_L\right]} \right\rangle$$

$$= \frac{e^{ik_F x}}{(vt-x)^{((\lambda^{\frac{1}{2}}+\lambda^{-\frac{1}{2}})/2)^2}\,(vt+x)^{((\lambda^{\frac{1}{2}}-\lambda^{-\frac{1}{2}})/2)^2}}$$

From this we can work out various things.

If we set $x=0$ and Fourier transform wrt. t we get the 1-particle density of states:

$$\int dt\, \frac{e^{-i\omega t}}{t^{\frac{\lambda+\lambda^{-1}}{2}}} \sim |\omega|^{\beta} \qquad \text{where} \qquad \beta = \frac{\lambda+\lambda^{-1}}{2} - 1 = \frac{(1-\lambda)^2}{2\lambda}$$

Similarly, if we set $t=0$ and Fourier transform wrt. x, we get the momentum distribution

$$n(k) \sim n(k_F) + \text{const.}\ \mathrm{sgn}(k-k_F)\, |k-k_F|^{\beta}$$

Note that β depends on g_4. Luttinger liquid.

Other perturbations of a Luttinger liquid

There are many other interesting effects.
One occurs in a half-filled system, when $4k_F a = 2\pi$.

Then a term

$$\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R$$

can occur in the Hamiltonian, since it is consistent with lattice translations under which (by $2a$)

$$\psi_L \rightarrow \psi_L e^{-ik_F \cdot 2a}$$

$$\psi_R^\dagger \rightarrow \psi_R^\dagger e^{-ik_F \cdot 2a}$$

In bosonised language, such a term is proportional to

$$M = e^{i2\sqrt{\pi}\phi} + e^{-i2\sqrt{\pi}\phi}$$

We therefore have a Sine-Gordon theory!

The $\langle MM \rangle$ correlator is

$$\langle e^{-i2\sqrt{\pi}\phi} e^{i2\sqrt{\pi}\phi} \rangle = \langle e^{-i2\sqrt{\pi}\lambda^{\frac{1}{2}}\tilde{\phi}} e^{+i2\sqrt{\pi}\lambda^{\frac{1}{2}}\tilde{\phi}} \rangle$$

$$\sim \frac{1}{\left[(vt)^2 - x^2\right]^{\lambda}}$$

so its scaling dimension is λ.
If $\lambda > 2$, we still have a Luttinger liquid.

But if $\lambda < 2$ the cosine interaction opens up a gap,

called a dimerisation gap.