

I. Relation between Euclidean QFT and classical critical behaviour

Euclidean path integral for a scalar field theory in d dimensions:

Generating function: $Z[J] = \int \mathcal{D}\phi e^{-\frac{1}{\hbar} \int [\frac{1}{2}(\nabla\phi)^2 + V(\phi)] d^d x + \int J\phi}$

Source $J(x)$,
Functional derivatives

$$\frac{1}{Z[0]} \frac{\delta^n}{\delta J(x_1) \dots \delta J(x_n)} Z[J] \Big|_{J=0}$$

give correlation functions $\langle \phi(x_1) \dots \phi(x_n) \rangle$

$$\frac{\delta^n}{\delta J(x_1) \dots \delta J(x_n)} \ln Z[J] \Big|_{J=0} \quad \text{give connected correlation functions}$$

$$\langle \phi(x_1) \dots \phi(x_n) \rangle_c$$

Thus all looks very much like classical statistical mechanics:

$$Z = \text{Tr} e^{-\frac{1}{k_B T} E}$$

E.g. Ising model on a lattice : sites labelled by r
'spins' $s(r) = \pm 1$

$$E = -\frac{1}{2} \sum_{r,r'} J(r,r') s(r) s(r') + \sum_r \mu H(r) s(r)$$

external varying
mag. field

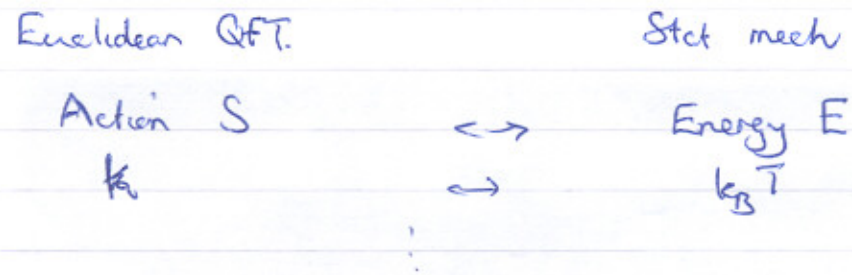
Absorb $\frac{\mu}{k_B T}$ into H : $\frac{1}{k_B T}$ into J

$$Z[H] = \text{Tr}_s e^{-\frac{1}{k_B T} \left(\frac{1}{2} \sum_{r,r'} J(r,r') s(r) s(r') + \sum_r H(r) s(r) \right)}$$

exchange interaction

$$\frac{\delta^n}{\delta H(r_1) \dots \delta H(r_n)} \ln Z[H] \quad \text{give} \quad \langle s(r_1) \dots s(r_n) \rangle_c$$

Formally these look quite similar, ~~except~~: with



but there are differences, mainly:

- 1) continuous space r \neq lattice r
- 2) continuous field $\phi(r)$ \neq discrete spins $s(r)$.

To make the mapping more explicit, we can use the

Hubbard-Stratonovich transformation (which has many other uses)

This generalises the simple Gaussian integral

$$\begin{aligned}
 & \int_{-\infty}^{\infty} d\phi \ e^{-\frac{1}{2} \phi K^{-1} \phi + \phi s} \\
 &= \int d\phi \ e^{-\frac{1}{2} (\phi - sK) K^{-1} (\phi - sK) + \frac{1}{2} sKs} \\
 &= \int d\phi' \ e^{-\frac{1}{2} \phi' K^{-1} \phi' + \frac{1}{2} sKs} \\
 &= \sqrt{2\pi K} \ e^{\frac{1}{2} sKs}
 \end{aligned}$$

to the case where K is a matrix, whose rows & columns are labelled by r, r' :

$$\begin{aligned}
 & \int \prod_r d\phi(r) \ e^{-\frac{1}{2} \sum_{r,r'} \phi(r) K^{-1}(r,r') \phi(r') + \sum_r \phi(r) s(r)} \\
 & \propto \sqrt{\det K} \ e^{\frac{1}{2} \sum_{r,r'} s(r) K(r,r') s(r')}
 \end{aligned}$$

Thus, apart from an unimportant constant,

$$Z_I[H] = \prod_s \int \prod_r d\phi(r) e^{-\frac{1}{2} \sum_{r,r'} \phi(r) K^{-1}(r,r') \phi(r') + \sum_r (\phi(r) + H(r)) s(r)}$$

The \sum_s can be done for each r :

$$\sum_{s(r)=\pm 1} e^{(\phi(r) + H(r)) s(r)} \propto \cosh[\phi(r) + H(r)].$$

Thus

$$Z_I[H] = \int \prod_r d\phi(r) e^{-\frac{1}{2} \sum_{r,r'} \phi(r) K^{-1}(r,r') \phi(r') + \sum_r \ln \cosh[\phi(r) + H(r)]}$$

This is a Euclidean lattice field theory.

Usually $K(r,r') = K(r-r')$ (translational invariance) ^{lattice}

so if we write $K(r-r') = \int_{\text{BZ}} \frac{d^d k}{(2\pi)^d} e^{ik(r-r')} \tilde{K}(k)$
BZ \leftarrow first Brillouin zone

Then $K^{-1}(r-r') = \dots = \frac{1}{\tilde{K}(k)}$

and $\frac{1}{2} \sum_{r,r'} \phi(r) K^{-1}(r-r') \phi(r') = \frac{1}{2} \int_{\text{BZ}} \frac{d^d k}{(2\pi)^d} \tilde{\phi}(k)^* \frac{1}{\tilde{K}(k)} \tilde{\phi}(k)$.

Now suppose that $K(r-r')$ is short-ranged

This means that $\tilde{K}(k) = \tilde{K}(0) [1 + \underset{\substack{\uparrow \\ \text{range of interaction}}}{R^2} k^2 + O(k^4)]$

$$\tilde{K}(k)^{-1} = \frac{1}{\tilde{K}(0)} [1 + R^2 k^2 + \dots]$$

Going back to real space $\sum_r \rightarrow \int \frac{d^d r}{a^d}$ a : lattice spacing.
 & take "naive continuum limit"
 $k^2 \rightarrow -\nabla^2$

$$Z_I[H] \approx \int \mathcal{D}\phi \ e^{-\int \frac{d^d r}{a^d} \left[\frac{1}{2\tilde{K}(0)} \phi (1 - R^2 \nabla^2) \phi \right] + \int \frac{d^d r}{a^d} \ln \cosh[\phi + H]}$$

↑
Functional

[NB: $R \rightarrow \infty \Rightarrow t$: const. \Rightarrow mean field theory.]

Rescale $\phi^2 \rightarrow \tilde{K}(0) \phi^2 a^d / R^2$

$$Z_I[H] \propto \int \mathcal{D}\phi \ e^{-\int d^d r \left[\frac{1}{2} (\nabla\phi)^2 + V(\phi) \right]}$$

where $V(\phi) = \frac{1}{2R^2} \phi^2 - \frac{1}{a^d} \ln \cosh \left[\sqrt{\frac{\tilde{K}(0) a^d}{R^2}} \phi + H \right]$.

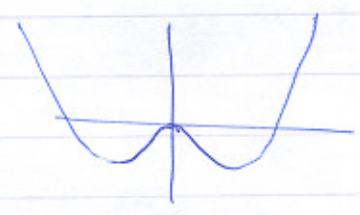
$$V(\phi) \Big|_{H=0} = \frac{1}{2R^2} (1 - \tilde{K}(0)) \phi^2 + O\left(\frac{\tilde{K}(0)^2 a^d}{R^4}\right) \phi^4 + \dots$$

Recall $\tilde{K}(0) = \frac{1}{k_B T} \sum_r \mathcal{J}(r)$

so for $T > T_{CMF} = \frac{1}{k_B} \sum_r \mathcal{J}(r)$



$T < T_{CMF}$



We have ~~the~~ cast the statistical mechanics model in the form of a Euclidean field theory.

In the naive continuum limit $a \rightarrow 0$ we simply replace sums by integrals. But we then recall that the Feynman diagram expansion has integrals which are ultraviolet divergent: this can be regularised with a cut-off $|k| < \Lambda$ on all loop integrals. The natural thing is to take $\Lambda \sim a^{-1} \rightarrow$ thus the lattice spacing creeps back in.

In order to take the continuum limit properly, we need to renormalise the theory.

Renormalised field theory and critical behaviour

Suppose we expand $V(\phi) = \frac{1}{2} m_0^2 \phi^2 + \frac{\lambda_0}{4!} \phi^4 + \dots$ and we drop the higher-order terms, for the time being.

We can imagine computing the ^{bare} Green's functions (correlation functions) $G_N = \langle \phi(x_1) \dots \phi(x_N) \rangle$ in the cut-off theory, with $|k| < \Lambda$. [These are the quantities most closely related to ~~stat.~~ original stat. mech. problem]

G_N depends on $m_0^2, \lambda_0,$ and Λ (as well as (x_1, \dots, x_N))

~~The statement~~

The statement of renormalisability (which varies slightly in detail depending on the theory and the renormalisation scheme) (for $d \leq 4$)

says that if we do field renormalisation: $\phi = Z_\phi^{1/2} \phi_R$

and mass renormalisation $G_2 \sim_{r \rightarrow d} e^{-m_R \cdot r}$

and coupling constant renormalisation: $\lambda_R = \dots$

so that Z_ϕ, m_R, λ_R are functions of m_0, λ_0, Λ

then $(G_N)_R \equiv \langle \phi_R(x_1) \dots \phi_R(x_N) \rangle = Z_\phi^{-N/2} \langle \phi(x_1) \dots \phi(x_N) \rangle$

when considered as a function of m_R, λ_R , has a finite limit as $\Lambda \rightarrow \infty$.

ie. all the UV divergences have been absorbed into λ_R & m_R .

Historically in particle physics, this property of renormalisability

was taken as essential: the cut-off Λ is introduced to define the theory, then removed: the renormalised Green's functions are the physical ones.

More recently, this point of view has changed slightly: all field theories are thought of as effective; valid only at energy (wavenumber) scales \ll some cut-off Λ (e.g. the Planck scale). However since we would like the predictions of the theory to be insensitive to the particular value of Λ , ~~the~~ this is mathematically equivalent to the statement that the limit $\Lambda \rightarrow \infty$ is finite.

In statistical mechanics however, the physical correlation functions are the bare ones. What, then, is the role of the renormalised field theory? Its existence tells us that, when $m_R \ll \Lambda$, that apart from constant ~~per~~ factors of Z_ϕ , the correlation functions are universal = those of the renormalised theory, or the way they depend on (x_1, \dots, x_N) .

m_R is just ~~the~~ inverse of ξ , the correlation length, so $m_R \ll \Lambda$ means $\xi \gg a$.

This happens close to a second order phase transition, and is called the scaling ~~at~~ region.

From the existence of the limit $\Lambda \rightarrow \infty$, we can derive the Gellman-Symanzik equation. For the renormalised Green functions do not know about Λ in this limit.

$$\text{Thus } \Lambda \frac{\partial}{\partial \Lambda} G_{RNR} \Big|_{\lambda, m_R} = \Lambda \frac{\partial}{\partial \Lambda} \left(Z_\phi^{-N/2} G_N \right) \Big|_{\lambda, m_R} = 0$$

G_N depends on λ_0, m_0 and Λ explicitly.

For simplicity let us work at the critical point where $m_R = 0$ ($\beta = \infty$)
Then m_0 is fixed in terms of (λ_0, Λ) .

It is also useful to (recalling that $\lambda_0 \sim \frac{\tilde{K}(e)^2 a^d}{R^4}$)

so to define $g_0 = \lambda_0 a^{4-d} = \lambda_0 \Lambda^{d-4}$ as dimensionless coupling.

so G_N depends on (g_0, Λ) as well as (x_1, \dots, x_N)

Then
$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g_0) \frac{\partial}{\partial g_0} - \frac{N}{2} \gamma(g_0) \right] G_N = 0$$

↑
explicit

where
$$\beta(g_0) \equiv \Lambda \frac{\partial}{\partial \Lambda} g_0 \Big|_{\lambda_R \text{ fixed}}$$

$$\gamma(g_0) = \Lambda \frac{\partial}{\partial \Lambda} \ln Z_\phi \Big|_{\lambda_R \text{ fixed}}$$

This is an example of a renormalisation group equation :

if we change Λ (or equivalently a), it tells us how we have to change g_0 in order that G_N behaves simply.

If we change $a \rightarrow ae^l$ ($l > 0 \rightarrow$ coarse-graining)

then
$$\frac{d}{dl} = a \frac{\partial}{\partial a} = -\Lambda \frac{\partial}{\partial \Lambda}$$

Thus
$$\frac{dg_0}{dl} = -\beta(g_0)$$

What is the form of $\beta(g_0)$? We know that
 $\lambda_e = \lambda_0 + O(\lambda_0^2)$

In fact the second term is $O\left(\frac{1}{4-d}\right)$ reflecting a UV divergence for $d \geq 4$. Then

$$\lambda_e = \Lambda^{4-d} \left[g_0 + \frac{b}{4-d} g_0^2 + \dots \right] \quad b = O(1)$$

Take $\Lambda \frac{\partial}{\partial \Lambda} \Big|_{g_0}$:

$$0 = (4-d) \left[g_0 + \frac{b}{4-d} g_0^2 + \dots \right] + \left(1 + \frac{2b}{4-d} g_0 + \dots \right) \beta(g_0) \quad \text{CHECK}$$

so $\beta(g_0) = -(4-d)g_0 + b g_0^2 + O(g_0^3)$, $\Lambda \frac{\partial g_0}{\partial \Lambda}$

or $\frac{dg_0}{d\ell} = + (4-d)g_0 - b g_0^2 + \dots$

For $d \geq 4$ we see that $g_0 \rightarrow 0$: it is irrelevant

For $d < 4$ ^{thus} $g_0 \rightarrow g^* = O(4-d)$ (if $b > 0$).

[NB. At this point we can see that the coeffs. of the terms we have dropped, like ϕ^6 , $(\nabla^2 \phi)^2$, etc are all irrelevant].

What does this mean for physics? (In this approach) we must also use dimensional analysis:

eg $G_2(m_R^2=0, g_0, \Lambda, x_1, x_2) = \langle \phi(x_1) \phi(x_2) \rangle$

$$= |x_1 - x_2|^{-\uparrow (d-2)} F(|x_1 - x_2| \Lambda)$$

from rescaling $\phi \rightarrow \phi \sqrt{\frac{a^d}{R^2}}$

If we write $r = |\mathbf{x}_1 - \mathbf{x}_2|$ then

$$\left[r \frac{\partial}{\partial r} - \Lambda \frac{\partial}{\partial \Lambda} - (d-2) \right] G_2 = 0.$$

(Euler's eqn). So we can trade varying Λ for varying r , which is physically what we are interested in. We also see that ~~being~~ increasing $a \Leftrightarrow$ decreasing $\Lambda \Leftrightarrow$ increasing r

so by taking $g \rightarrow g^*$ we explore the infrared ($r \rightarrow \infty$) behaviour, appropriate for critical phenomena.

If we eliminate $\Lambda \frac{\partial}{\partial \Lambda}$ and take $g = g^*$, we get

$$\left[r \frac{\partial}{\partial r} - \delta^* - (d-2) \right] G_2 = 0$$

where $\delta^* = \delta(g^*)$. The solution is $G_2 \propto \frac{1}{r^{d-2+\delta^*}}$

δ^* is an example of a critical exponent (conventionally called η).

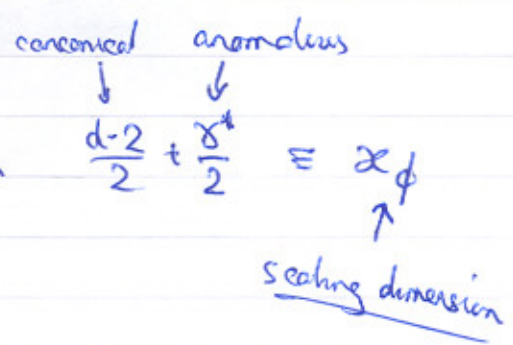
In field theory it is also called an anomalous dimension :

what is really happening is that

$$\langle \phi(\mathbf{x}_1) \phi(\mathbf{x}_2) \rangle \sim \frac{1}{r^{d-2}} \left(\frac{a}{r} \right)^{\delta^*}$$

$$\langle \phi_R(\mathbf{x}_1) \phi_R(\mathbf{x}_2) \rangle \sim \frac{1}{r^{d-2+\delta^*}}$$

so it looks like ϕ_R has a dimension



General RG theory

We have seen in the above example of the Ising model (ϕ^4 theory in $d=4-\epsilon$ dimensions) some of the essential features of an RG calculation in classical stat mech. (= euclidean QFT),

1). We vary the cut-off $\Lambda \rightarrow \Lambda e^{-l}$ (or $a \rightarrow ae^l$) (corresponding to coarse-graining - eliminating short wavelengths) and attempt to vary the dimensionless couplings (like g_i) so as to keep the long-wavelength physics 'the same' [this might require eg field renormalisation]

2). RG equations $\frac{dg_i}{dl} = -\beta_i(\{g_j\})$

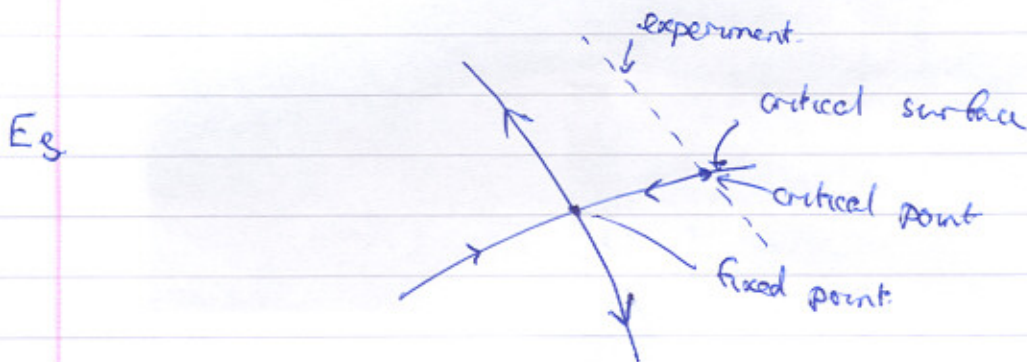
3). look for fixed points where $\beta_i(\{g_j^*\}) = 0$.

4). Linearise around these $g'_i = g_i - g_i^*$ (might require a rotation)

such that $\frac{dg'_i}{dl} = y_i g'_i + O(g'^2)$.

5) RG eigenvalues: $y_i > 0$ g'_i relevant
 $y_i < 0$ g'_i irrelevant
($y_i = 0$ marginal)

6). Assume local RG flows extend to some finite region:



This accounts for universality: many different systems described by same fixed point.

7). Behaviour of correlation functions ~~near~~ at fixed point

$$S = S_{\text{fixed pt.}} + \sum_i g'_i \int \frac{d^d r}{a^d} \phi_i(r)$$

↑
scaling operators
(e.g. $g = t \quad \phi \propto s$),

If under $a \rightarrow ae^l$ $g'_i \rightarrow e^{y_i l} g'_i$
 then $\phi_i \rightarrow e^{(d-y_i)l} \phi_i$

But $\langle \phi_i(r_1) \phi_i(r_2) \rangle = \left(\frac{a}{r}\right)^{2x_i} \Rightarrow$

so $\boxed{x_i = d - y_i}$

This relates correlation functions at the fixed pt. to critical exponents away from the fixed pt.

E.g. suppose that for $g' \neq 0$ the theory develops a finite correlation length ξ .

Dimensional analysis: $\xi = a f(g')$.

Under ~~renorm~~ RG we keep ξ (long distance physics) the same:

so $ae^l f(g'e^{y_l}) = a f(g')$

$$\Rightarrow f(g') \propto g'^{-\nu} \Rightarrow \xi = a g'^{-\nu}$$

$\nu = \frac{1}{y}$