

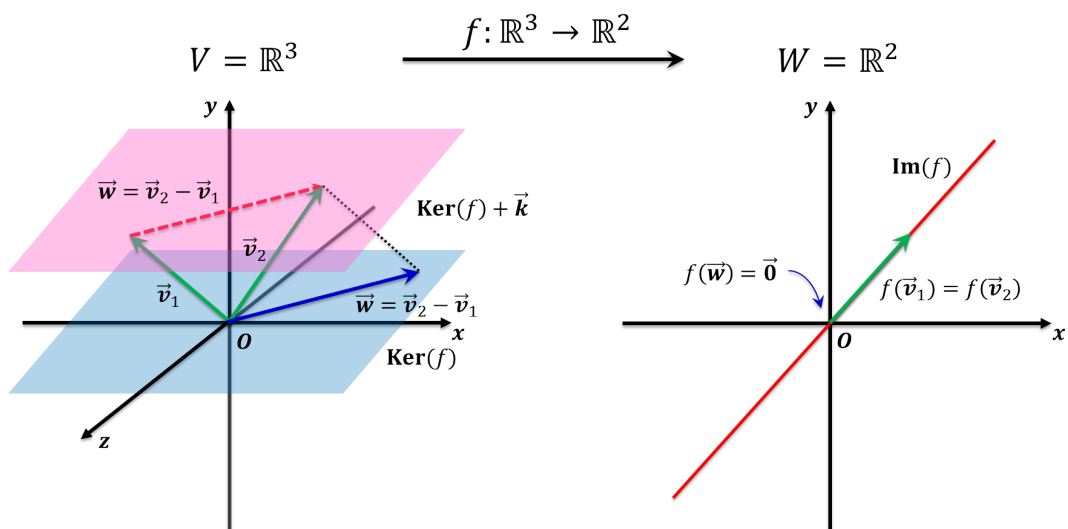
# Vectors and Matrices

aka Linear Algebra

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**Foreword:** The subject of “Vectors and Matrices”, more politely called *Linear Algebra*, is one of the basic disciplines of mathematics. It underlies many branches of more advanced mathematics, such as calculus of functions in many variables and differential geometry, and it has applications in practically all parts of physics. There are numerous textbooks on the subject ranging in style from low-level “how-to-do” guides, mainly teaching the mechanics of manipulating low-dimensional vectors and matrices, to hugely formalized treaties which barely ever write down a vector or a matrix explicitly. Naturally, a course for beginning physics students should stay away from either extreme.

In the present text we will follow the inherent logic of the subject, in line with how it is taught in research universities across the world. This will require some of the language of formal mathematics and the occasional proof but we will keep this as light as possible. We attempt to illustrate the material with many examples, both from physics and other areas, and teach the practical methods and algorithms required in the day-to-day work of a physicist.

Hopefully, a student will finish the course with a good working knowledge of “Vectors and Matrices” but also with an appreciation of the structure and beauty of the subject of Linear Algebra.

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Andre Lukas  
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## Notation

$\mathbb{R}$	the real numbers
$\mathbb{C}$	the complex numbers
$F$	a field, usually either the real or the complex numbers
$V, W, U$	vector spaces
$\mathbb{R}^n$	the vector space of $n$ -dimensional column vectors with real entries
$\mathbb{C}^n$	the vector space of $n$ -dimensional column vectors with complex entries
$\mathbf{v}, \mathbf{w}, \dots$	boldface lowercase letters are used for vectors
$\mathbf{0}$	the zero vector
$i, j, k, \dots$	indices to label vector components, usually in the range $1, \dots, n$
$v_i, w_i, \dots$	components of column vectors $\mathbf{v}, \mathbf{w}, \dots$
$\mathbf{e}_i$	the standard unit vectors in $\mathbb{R}^n$
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	another notation for the standard unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in $\mathbb{R}^3$
$\alpha, \beta, a, b, \dots$	lowercase letters are used for scalars
$A, B, \dots$	uppercase letters are used for matrices
$A_{ij}$	entry $(i, j)$ of a matrix $A$
$\mathbf{A}^i$	column vector $i$ of a matrix $A$
$\mathbf{A}_i$	row vector $i$ of a matrix $A$
$A^T, A^\dagger$	the transpose and hermitian conjugate of the matrix $A$
$(\mathbf{v}_1, \dots, \mathbf{v}_n)$	a matrix with column vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$
$\mathbb{1}_n$	the $n \times n$ identity matrix
$E_{ij}$	the standard matrices with $(i, j)$ entry 1 and zero otherwise
$\text{diag}(a_1, \dots, a_n)$	an $n \times n$ diagonal matrix with diagonal entries $a_1, \dots, a_n$
$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$	the span of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$
$\dim_F(V)$	the dimension of the vector space $V$ over $F$
$\mathbf{v} \cdot \mathbf{w}$	the dot product between two $n$ -dimensional column vectors
$ \mathbf{v} $	the length of a vector
$\sphericalangle(\mathbf{v}, \mathbf{w})$	the angle between two vectors $\mathbf{v}$ and $\mathbf{w}$
$\mathbf{v} \times \mathbf{w}$	the cross (vector) product of two three-dimensional column vectors
$\langle \mathbf{v}, \mathbf{w}, \mathbf{u} \rangle$	the triple product of three column vectors in three dimensions
$\delta_{ij}$	the Kronecker delta symbol in $n$ dimensions
$\epsilon_{ijk}$	the Levi-Civita tensor in three dimensions
$\epsilon_{i_1 \dots i_n}$	the Levi-Civita tensor in $n$ dimensions
$f$	a linear map, unless stated otherwise
$\text{id}_V$	the identity map on $V$
$\text{Im}(f)$	the image of a linear map $f$
$\text{Ker}(f)$	the kernel of a linear map $f$
$\text{rk}(f)$	the rank of a linear map $f$
$[A, B]$	the commutator of two matrices $A, B$
$(A \mathbf{b})$	the augmented matrix for a system of linear equations $A\mathbf{x} = \mathbf{b}$
$\det(\mathbf{v}_1, \dots, \mathbf{v}_n)$	the determinant of the column vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$
$\det(A)$	the determinant of the matrix $A$
$S_n$	the permutations of $1, \dots, n$
$\text{sgn}(\sigma)$	the sign of a permutation $\sigma$
$\langle \cdot, \cdot \rangle$	a real or hermitian scalar product (or a bi-linear form)
$f^\dagger$	the adjoint linear map of $f$
$\text{Eig}_A(\lambda)$	the eigenspace of $A$ for $\lambda$

$\chi_A(\lambda)$  the characteristic polynomial of  $A$  as a function of  $\lambda$   
 $\text{tr}(A)$  the trace of the matrix  $A$

# 1 Vector spaces and vectors

Linear algebra is foundational for mathematics and has applications in many parts of physics, including Classical Mechanics, Electromagnetism, Quantum Mechanics, General Relativity etc.

We would like to develop the subject, explaining both its mathematical structure and some of its physics applications. In this section, we introduce the “arena” for Linear Algebra: vector spaces. Vector spaces come in many disguises, sometimes containing objects which do not at all look like ”vectors”. Surprisingly, many of these “unexpected” vector spaces play a role in physics, particularly in quantum physics. After a brief review of “traditional” vectors we will, therefore, introduce the main ideas in some generality.

## 1.1 Vectors in $\mathbb{R}^n$

The set of real numbers is denoted by  $\mathbb{R}$  and by  $\mathbb{R}^n$  we mean the set of all *column vectors*

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \quad (1.1)$$

where  $v_1, \dots, v_n \in \mathbb{R}$  are the *components* of  $\mathbf{v}$ . We will often use *index notation* to refer to a vector and write the components collectively as  $v_i$ , where the index  $i$  takes the values  $i = 1, \dots, n$ . There are two basic operations for vectors, namely *vector addition* and *scalar multiplication* and for column vectors they are defined in the obvious way, that is “component by component”. For the vector addition of two vectors  $\mathbf{v}$  and  $\mathbf{w}$  this means

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \quad \mathbf{v} + \mathbf{w} := \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}, \quad (1.2)$$

where the vector sum  $\mathbf{v} + \mathbf{w}$  has the geometrical interpretation indicated in Fig. 1.

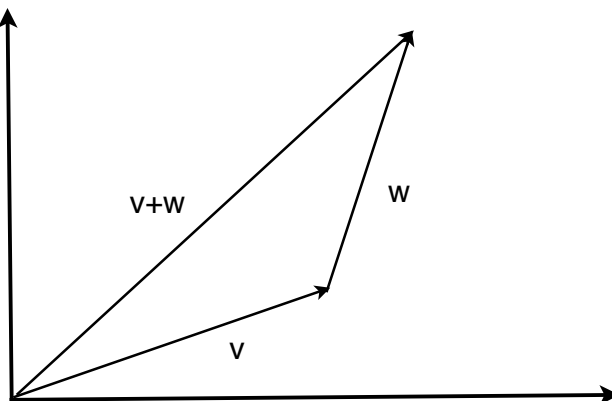


Figure 1: *Vector addition*

The scalar multiplication of a column vector  $\mathbf{v}$  with a scalar  $\alpha \in \mathbb{R}$  is defined as

$$\alpha \mathbf{v} := \begin{pmatrix} \alpha v_1 \\ \vdots \\ \alpha v_n \end{pmatrix}, \quad (1.3)$$

and the geometrical interpretation is indicated in Fig. 2.

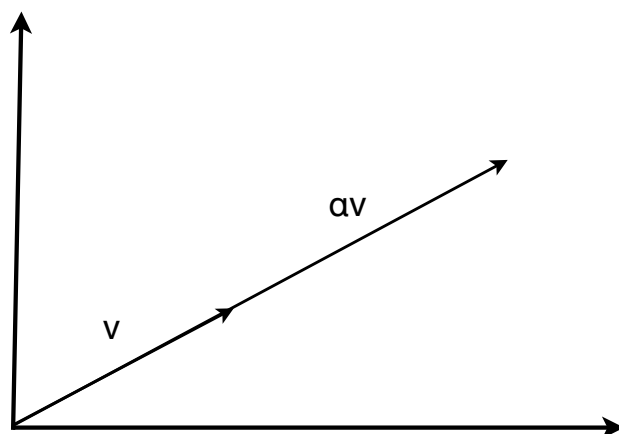


Figure 2: *Scalar multiplication of vectors*

**Example 1.1:** Vector addition and scalar multiplication in  $\mathbb{R}^3$

As an example in  $\mathbb{R}^3$  consider the two vectors

$$\mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix}. \quad (1.4)$$

Their vector sum is given by

$$\mathbf{v} + \mathbf{w} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + \begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}. \quad (1.5)$$

Further, scalar multiplication of  $\mathbf{v}$  by  $\alpha = 3$  gives

$$\alpha \mathbf{v} = 3 \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 15 \end{pmatrix}. \quad (1.6)$$

---

These two so-defined operations satisfy a number of obvious rules. The vector addition is *associative*,  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ , it is commutative,  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ , there is a neutral element, the *zero vector*

$$\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (1.7)$$



which satisfies  $\mathbf{v} + \mathbf{0} = \mathbf{v}$  and, finally, for each vector  $\mathbf{v}$  there is an *inverse*  $-\mathbf{v}$ , so that  $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ . The scalar multiplication satisfies three further rules, namely the distributive laws  $\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$ ,  $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$  and the associativity law  $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$ . These rules can be easily verified from the above definitions of vector addition and scalar multiplication and we will come back to this shortly. It is useful to introduce the *standard unit vectors*

$$\mathbf{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{\text{th}} \text{ position, for } i = 1, \dots, n, \quad (1.8)$$

in  $\mathbb{R}^n$  which are the  $n$  vectors obtained by setting the  $i^{\text{th}}$  component to one and all other components to zero. In terms of the standard unit vectors a vector  $\mathbf{v}$  with components  $v_i$  can be written as

$$\mathbf{v} = v_1\mathbf{e}_1 + \dots + v_n\mathbf{e}_n = \sum_{i=1}^n v_i\mathbf{e}_i. \quad (1.9)$$

The results of vector additions and scalar multiplications can also be expressed in terms of the standard unit vectors. With two vectors  $\mathbf{v} = \sum_{i=1}^n v_i\mathbf{e}_i$  and  $\mathbf{w} = \sum_{i=1}^n w_i\mathbf{e}_i$  we have

$$\mathbf{v} + \mathbf{w} = \sum_{i=1}^n v_i\mathbf{e}_i + \sum_{i=1}^n w_i\mathbf{e}_i = \sum_{i=1}^n (v_i + w_i)\mathbf{e}_i, \quad \alpha\mathbf{v} = \alpha \sum_{i=1}^n v_i\mathbf{e}_i = \sum_{i=1}^n (\alpha v_i)\mathbf{e}_i. \quad (1.10)$$

Here we have used some of the above general rules, notably the associativity and commutativity of vector addition as well as the associativity and distributivity of scalar multiplication.

In a physics context, the case  $n = 3$  is particularly important. In this case, sometimes the notation

$$\mathbf{i} = \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{j} = \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{k} = \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (1.11)$$

for the three standard unit vectors is used, so that a vector  $\mathbf{r}$  with components  $x, y, z$ , can be expressed as

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \quad (1.12)$$

**Example 1.2:** Vector addition and scalar multiplication with standard unit vectors

With the standard unit vectors  $\mathbf{i}, \mathbf{j}$  and  $\mathbf{k}$  in  $\mathbb{R}^3$  the vectors  $\mathbf{v}$  and  $\mathbf{w}$  from Eq. (1.4) can also be written as

$$\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k}, \quad \mathbf{w} = -4\mathbf{i} + \mathbf{j} - 3\mathbf{k}. \quad (1.13)$$

With this notation, the vector addition of  $\mathbf{v}$  and  $\mathbf{w}$  can be carried out as

$$\mathbf{v} + \mathbf{w} = (\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}) + (-4\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = -3\mathbf{i} - \mathbf{j} + 2\mathbf{k}. \quad (1.14)$$

For the scalar multiple of  $\mathbf{v}$  by  $\alpha = 3$  we have

$$\alpha\mathbf{v} = 3(\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}) = 3\mathbf{i} - 6\mathbf{j} + 15\mathbf{k}. \quad (1.15)$$

While the case  $n = 3$  is important for the description of physical space, other values are just as relevant in physics. For example, a system of  $k$  mass points moving in three-dimensional space can be described by a vector with  $n = 3k$  components. In Special Relativity, space and time are combined into a vector with four components. And finally, in Quantum Mechanics, the quantum states of physical systems are described by vectors which, depending on the system, can be basically any size. For this reason, we will keep  $n$  general, whenever possible.

You are probably not yet sufficiently familiar with the above physics examples. Let me discuss an example in a more familiar context which illustrates the need for vectors with an arbitrary number of components and also indicates some of the problems linear algebra should address.

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### Application: Google's search algorithm

Modern internet search engines order search results by assigning a page rank to each website. As we will see, this task can be formulated as a problem in linear algebra.

Consider an internet with  $n$  web sites labeled by an index  $k = 1, \dots, n$ . Each site  $k$  has  $n_k$  links to some of the other sites and is linked to by the sites  $L_k \subset \{1, \dots, n\}$ . We would like to assign a page rank  $x_k$  to each site  $k$ . A first attempt might be to define the page rank of a page  $k$  as the number of pages linking to it. However, is it desirable that a page linked to by high-ranked pages has itself a higher rank than a page linked to by low-ranked pages, even if the number of links is the same in each case. So, an improved version might be to define  $x_k$  as the sum of all page ranks  $x_j$  of the pages linking to  $k$ , so as a sum over all  $x_j$ , where  $j \in L_k$ . As a further refinement, a link to page  $k$  from a page  $j$  with a low number of links  $n_j$  might be considered worth more than a link from a page  $j$  with a high number of links. Altogether, this leads to the following proposal for the page rank

$$x_k = \sum_{j \in L_k} \frac{x_j}{n_j}. \quad (1.16)$$

Note that these are  $n$  equations (one for each page rank  $x_k$ ) and that the sum on the RHS runs over all pages  $j$  which link to page  $k$ . Eqs. (1.16) constitute a system of  $n$  linear equations for the variables  $x_1, \dots, x_n$  (while the number of links,  $n_j$ , are given constants). Perhaps this is best explained by focusing on a simple example.

Consider a very simple internet with four sites, so  $n = 4$ , and a structure of links as indicated in Fig. 3. From the figure, it is clear that the number of links on each site (equal to the number of outgoing arrows

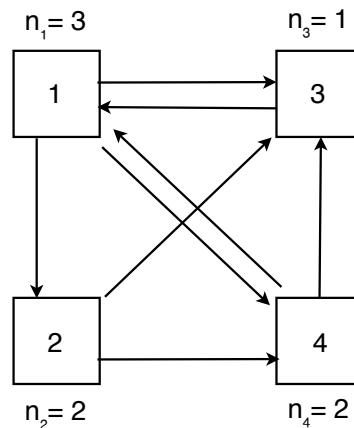


Figure 3: Example of a simple “internet” with four sites. An arrow from site  $j$  to site  $k$  indicates that site  $k$  is linked by site  $j$ .

from each site) is given by  $n_1 = 3$ ,  $n_2 = 2$ ,  $n_3 = 1$ ,  $n_4 = 2$  while the links themselves are specified by  $L_1 = \{3, 4\}$ ,  $L_2 = \{1\}$ ,  $L_3 = \{1, 2, 4\}$ ,  $L_4 = \{1, 2\}$ . To be clear,  $L_1 = \{3, 4\}$  means that site 1 is linked to by sites 3 and 4. With this data, it is straightforward to specialize the general equations (1.16) to the example and to obtain the following equations

$$\begin{aligned} x_1 &= \frac{x_3}{1} + \frac{x_4}{2} \\ x_2 &= \frac{x_1}{3} \\ x_3 &= \frac{x_1}{3} + \frac{x_2}{2} + \frac{x_4}{2} \\ x_4 &= \frac{x_1}{3} + \frac{x_2}{2} \end{aligned} \tag{1.17}$$

for the ranks of the four pages. Clearly, this is a system of linear equations. Later, we will formulate such systems using vector/matrix notation. In the present case, we can, for example, introduce

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix} \tag{1.18}$$

and re-write the Eqs. (1.17) as  $A\mathbf{x} = \mathbf{x}$ . This equation describes a so-called “eigenvalue problem”, a class of problems we will discuss in detail towards the end of the course. Of course, we will also properly introduce matrices shortly. At this stage, the main point to note is that linear systems of equations are relevant in “everyday” problems and that we need to understand their structure and develop efficient solution methods. The four equations (1.17) for our explicit example can, of course, be solved by elementary methods (adding and subtracting equations and their multiples), resulting in

$$\mathbf{x} = \alpha \begin{pmatrix} 2 \\ 2/3 \\ 3/2 \\ 1 \end{pmatrix}, \tag{1.19}$$

where  $\alpha$  is an arbitrary real number. Hence, site 1 is the highest-ranked one. In reality, the internet has an enormous number,  $n$ , of sites. Real applications of the page rank algorithm therefore involve very large systems of linear equations and vectors and matrices of corresponding size. Clearly, solving such systems will require more refined methods and a better understanding of their structure. Much of the course will be devoted to this task.

## 1.2 Vector spaces

The modern mathematical approach is to define vectors by the properties they satisfy rather than by “what they are” and we would now like to follow this route. First, however, we need to introduce the concept of a *field*.

**Field:** Prime examples of fields are the rational numbers,  $\mathbb{Q}$ , the real numbers,  $\mathbb{R}$ , and the complex numbers,  $\mathbb{C}$ . More generally, a field is an arena within which “regular calculations” involving addition and multiplication of numbers can be carried out. We will write  $F$  for a field and will usually have the real or complex numbers in mind, although much of what we will do also holds for other fields. For the more formal-minded, a proper definition of fields can be found in Appendix A.

**Example 1.3:** Examples of finite fields

There exist “unusual” fields which satisfy all the requirements listed in Appendix A. These include fields with a finite number of elements and here we introduce the simplest such examples, the finite fields  $\mathbb{F}_p = \{0, 1, \dots, p-1\}$ , where  $p$  is a prime number. Addition and multiplication of two numbers  $a, b \in \mathbb{F}_p$  are defined by

$$a + b := (a + b) \bmod p, \quad a \cdot b = (ab) \bmod p. \quad (1.20)$$

Here, the addition and multiplication on the right-hand sides of these definitions are just the usual ones for integers. The modulus operation,  $a \bmod p$ , denotes the remainder of the division of  $a$  by  $p$ . In other words, the definitions (1.20) are just the usual ones for addition and multiplication except that the modulus operation brings the result back into the required range  $\{0, 1, \dots, p-1\}$  whenever it exceeds  $p-1$ . Although these fields might seem abstract, they have important applications, for example in numerical linear algebra. They allow calculations based on a finite set of integers which avoids numerical uncertainties (as would arise for real numbers) and overflows (as may arise if all integers are used).

The smallest example of a field in this class is  $\mathbb{F}_2 = \{0, 1\}$ . Since every field must contain 0 (the neutral element of addition) and 1 (the neutral element of multiplication),  $\mathbb{F}_2$  is the smallest non-trivial field. From the definitions (1.20) its addition and multiplication tables are

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline 1 & 1 & 0 \end{array} \quad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ \hline 1 & 0 & 1 \end{array} \quad (1.21)$$

Note that, taking into account the mod 2 operation, in this field we have  $1 + 1 = 0$ . Since the elements of  $\mathbb{F}_2$  can be viewed as the two states of a bit, this field has important applications in computer science and in coding theory, to which we will return later.

We are now ready to define vector spaces.

**Definition 1.1.** A vector space  $V$  over a field  $F$  ( $= \mathbb{R}, \mathbb{C}$  or any other field) is a set with two operations:

- i) vector addition:  $(\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} + \mathbf{w} \in V$ , where  $\mathbf{v}, \mathbf{w} \in V$
- ii) scalar multiplication:  $(\alpha, \mathbf{v}) \mapsto \alpha \mathbf{v} \in V$ , where  $\alpha \in F$  and  $\mathbf{v} \in V$ .

For all  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$  and all  $\alpha, \beta \in F$ , these operations have to satisfy the following rules:

- (V1)  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$  “associativity”
- (V2) There exists a “zero vector”,  $\mathbf{0} \in V$  so that  $\mathbf{0} + \mathbf{v} = \mathbf{v}$  “neutral element”
- (V3) There exists an inverse,  $-\mathbf{v}$  with  $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$  “inverse element”
- (V4)  $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$  “commutativity”
- (V5)  $\alpha(\mathbf{v} + \mathbf{w}) = \alpha \mathbf{v} + \alpha \mathbf{w}$
- (V6)  $(\alpha + \beta)\mathbf{v} = \alpha \mathbf{v} + \beta \mathbf{v}$
- (V7)  $(\alpha\beta)\mathbf{v} = \alpha(\beta \mathbf{v})$
- (V8)  $1 \cdot \mathbf{v} = \mathbf{v}$

The elements  $\mathbf{v} \in V$  are called “vectors”, the elements  $\alpha \in F$  of the field are called “scalars”.

In short, a vector space defines an environment which allows for addition and scalar multiplication of vectors, subject to a certain number of rules. Note that the above definition does not assume anything about the nature of vectors. In particular, it is not assumed that they are made up from components.

Let us draw a few simple conclusions from these axioms to illustrate that indeed all the “usual” rules for calculations with vectors can be deduced.

i)  $-(-\mathbf{v}) = \mathbf{v}$

This follows from (V3) and (V4) which imply  $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$  and  $-(-\mathbf{v}) + (-\mathbf{v}) = \mathbf{0}$ , respectively. Combining these two equations gives  $\mathbf{v} + (-\mathbf{v}) = -(-\mathbf{v}) + (-\mathbf{v})$  and then adding  $\mathbf{v}$  to both sides, together with (V1) and (V3), leads to  $\mathbf{v} = -(-\mathbf{v})$ .

ii)  $0 \cdot \mathbf{v} = \mathbf{0}$

Since  $0 \mathbf{v} = (0 + 0)\mathbf{v} \stackrel{(V6)}{=} 0 \mathbf{v} + 0 \mathbf{v}$  and  $0 \mathbf{v} = 0 \mathbf{v} + \mathbf{0}$  it follows that  $0 \mathbf{v} = \mathbf{0}$ .

iii)  $(-1)\mathbf{v} = -\mathbf{v}$

Since  $\mathbf{0} \stackrel{(ii)}{=} 0 \mathbf{v} = (1 + (-1))\mathbf{v} \stackrel{(V6),(V8)}{=} \mathbf{v} + (-1)\mathbf{v}$  and  $\mathbf{0} = \mathbf{v} + (-\mathbf{v})$  it follows that  $(-1)\mathbf{v} = -\mathbf{v}$ .

We now follow the standard path and define the “sub-structure” associated to vector spaces, the *sub vector spaces*. These are basically vector spaces in their own right but they are contained in larger vector spaces. The formal definition is as follows.

**Definition 1.2.** *A sub vector space  $W \subset V$  is a non-empty subset of a vector space  $V$  satisfying:*

(S1)  $\mathbf{w}_1 + \mathbf{w}_2 \in W$  for all  $\mathbf{w}_1, \mathbf{w}_2 \in W$

(S2)  $\alpha \mathbf{w} \in W$  for all  $\alpha \in F$  and for all  $\mathbf{w} \in W$

*In other words, a sub vector space is a non-empty subset of a vector space which is closed under vector addition and scalar multiplication.*

This definition implies immediately that a sub vector space is also a vector space over the same field as  $V$ . Indeed, we already know that  $0\mathbf{w} = \mathbf{0}$  and  $(-1)\mathbf{w} = -\mathbf{w}$ , so from property (S2) a sub vector space  $W$  contains the zero vector and an inverse for each vector. Hence, the requirements (V2) and (V3) in Definition 1.1 are satisfied for  $W$ . All other requirements in Definition 1.1 are trivially satisfied for  $W$  simply by virtue of them being satisfied in  $V$ . Hence,  $W$  is indeed a vector space.

Every vector space  $V$  has two trivial sub vector spaces, the vector space  $\{\mathbf{0}\}$  consisting of only the zero vector and the whole space  $V$ . We will now illustrate the concepts of vector space and sub vector space by some examples.

**Example 1.4:** Some examples of vector spaces

(a) The column vectors discussed in Section 1.1 of course form a vector space. Here, we can slightly generalize these to  $V = F^n$  over  $F$ , where  $F = \mathbb{R}$  or  $\mathbb{C}$  (or indeed any other field), that is, to column vectors

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \quad (1.22)$$

where  $v_i \in F$ . Vector addition and scalar multiplication are of course defined exactly as in Eqs. (1.2), (1.3), that is, component by component. It is easy to check that all vector space axioms (V1)–(V8) are indeed satisfied for these definitions. For example consider (V6):

$$(\alpha + \beta)\mathbf{v} \stackrel{(1.3)}{=} \begin{pmatrix} (\alpha + \beta)v_1 \\ \vdots \\ (\alpha + \beta)v_n \end{pmatrix} = \begin{pmatrix} \alpha v_1 + \beta v_1 \\ \vdots \\ \alpha v_n + \beta v_n \end{pmatrix} \stackrel{(1.2)}{=} \begin{pmatrix} \alpha v_1 \\ \vdots \\ \alpha v_n \end{pmatrix} + \begin{pmatrix} \beta v_1 \\ \vdots \\ \beta v_n \end{pmatrix} \stackrel{(1.3)}{=} \alpha \mathbf{v} + \beta \mathbf{v} \quad (1.23)$$

It is useful to write the definitions of the two vector space operations in index notation as

$$(\mathbf{v} + \mathbf{w})_i := v_i + w_i, \quad (\alpha \mathbf{v})_i := \alpha v_i. \quad (1.24)$$

The subscript  $i$  on the LHS means that component number  $i$  from the vector enclosed in brackets is extracted. Using this notation, the vector space axioms can be verified in a much more concise way. For example, we can demonstrate (V7) by  $((\alpha\beta)\mathbf{v})_i = (\alpha\beta)v_i = \alpha(\beta v_i) = \alpha(\beta\mathbf{v})_i = (\alpha(\beta\mathbf{v}))_i$ .

Now, this example is not a big surprise and on its own would hardly justify the formal effort of our general definition. So, let us move on to more adventurous examples.

(b) The set of all functions  $f : S \rightarrow F$  from a set  $S$  into a field  $F$ , with vector addition and scalar multiplication defined as

$$(f + g)(x) := f(x) + g(x) \quad (1.25)$$

$$(af)(x) := af(x), \quad (1.26)$$

that is, by “pointwise” addition and multiplication, forms a vector space over  $F$ . The null “vector” is the function which is zero everywhere and all axioms (V1) – (V8) are clearly satisfied. There are many interesting specializations and sub vector spaces of this.

(c) All continuous (or differentiable) functions  $f : [a, b] \rightarrow F$  on an interval  $[a, b] \subset \mathbb{R}$  form a vector space. Indeed, since continuity (or differentiability) is preserved under addition and scalar multiplication, as defined in (b), this is a (sub) vector space. For example, consider the real-valued functions  $f(x) = 2x^2 + 3x - 1$  and  $g(x) = -2x + 4$ . Then the vector addition of these two functions and the scalar multiple of  $f$  by  $\alpha = 4$  are given by

$$(f+g)(x) = (2x^2+3x-1)+(-2x+4) = 2x^2+x+3, \quad (\alpha f)(x) = 4(2x^2+3x-1) = 8x^2+12x-4. \quad (1.27)$$

(d) In physics, many problems involve solving 2nd order, linear, homogeneous differential equations of the form

$$p(x)\frac{d^2f}{dx^2} + q(x)\frac{df}{dx} + r(x)f = 0 \quad (1.28)$$

where  $p$ ,  $q$  and  $r$  are fixed functions. The task is to find all functions  $f$  which satisfy this equation. This equation is referred to as a “linear” differential equation since every term is linear in the unknown function  $f$  (rather than, for example, quadratic). This property implies that with two solutions,  $f$ ,  $g$  of the differential equation also  $f + g$  and  $\alpha f$  (for scalars  $\alpha$ ) are solutions. Hence, the space of solutions of such an equation forms a vector space (indeed a sub vector space of the twice differentiable functions). A simple example is the differential equation

$$\frac{d^2f}{dx^2} + f = 0 \quad (1.29)$$

which is obviously solved by  $f(x) = \cos(x)$  and  $f(x) = \sin(x)$ . Since the solution space forms a vector space, we know that  $\alpha \cos(x) + \beta \sin(x)$  for arbitrary  $\alpha, \beta \in \mathbb{R}$  solves the equation. This can also be easily checked explicitly by inserting  $f(x) = \alpha \cos(x) + \beta \sin(x)$  into the differential equation.

(e) The matrices of size  $n \times m$  consist of an array of numbers in  $F$  with  $n$  rows and  $m$  columns. A matrix is usually denoted by

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}, \quad (1.30)$$

with entries  $a_{ij} \in F$ . As for vectors, we use index notation  $a_{ij}$ , where  $i = 1, \dots, n$  labels the rows and  $j = 1, \dots, m$  labels the columns, to collectively refer to all the entries. Addition of two  $n \times m$  matrices  $A$

and  $B$  with components  $a_{ij}$  and  $b_{ij}$  and scalar multiplication can then be defined as

$$A + B = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix} \quad (1.31)$$

$$:= \begin{pmatrix} a_{11} + b_{11} & \dots & a_{1m} + b_{1m} \\ \vdots & & \vdots \\ a_{n1} + b_{n1} & \dots & a_{nm} + b_{nm} \end{pmatrix} \quad (1.32)$$

$$\alpha A = \alpha \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} := \begin{pmatrix} \alpha a_{11} & \dots & \alpha a_{1m} \\ \vdots & & \vdots \\ \alpha a_{n1} & \dots & \alpha a_{nm} \end{pmatrix} \quad (1.33)$$

that is, as for vectors, component by component. Clearly, with these operations, the  $n \times m$  matrices with entries in  $F$  form a vector space over  $F$ . The zero “vector” is the matrix with all entries zero. Indeed, as long as we define vector addition and scalar multiplication component by component it does not matter whether the numbers are arranged in a column (as for vectors) or a rectangle (as for matrices). By slight abuse of notation we sometimes denote the entries of a matrix  $A$  by  $A_{ij} = a_{ij}$ . In index notation, the above operations can then be written more concisely as  $(A + B)_{ij} := A_{ij} + B_{ij}$  and  $(\alpha A)_{ij} := \alpha A_{ij}$ , in analogy with the definitions (1.24) for column vectors.

For a numerical example, consider the  $2 \times 2$  matrices

$$A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 5 \\ -1 & 8 \end{pmatrix}. \quad (1.34)$$

Their sum is given by

$$A + B = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} + \begin{pmatrix} 0 & 5 \\ -1 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad (1.35)$$

while the scalar multiplication of  $A$  with  $\alpha = 3$  gives

$$\alpha A = 3 \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 3 & -6 \\ 9 & -12 \end{pmatrix}. \quad (1.36)$$

This list of examples hopefully illustrates the strength of the general approach. Perhaps surprisingly, even many of the more “exotic” vector spaces do play a role in physics, particularly in quantum physics. Much of what follows will only be based on the general Definition 1.1 of a vector space and, hence, will apply to all of the above examples and many more.

### 1.3 Linear combinations, linear independence

Let us introduce the following useful pieces of terminology. For  $k$  vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  in a vector space  $V$  over a field  $F$  the expression

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_k \mathbf{v}_k = \sum_{i=1}^k \alpha_i \mathbf{v}_i, \quad (1.37)$$

with scalars  $\alpha_1, \dots, \alpha_n \in F$ , is called a *linear combination*. The set of all linear combinations of  $\mathbf{v}_1, \dots, \mathbf{v}_k$ ,

$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) := \left\{ \sum_{i=1}^k \alpha_i \mathbf{v}_i \mid \alpha_i \in F \right\}, \quad (1.38)$$

is called the *span* of  $\mathbf{v}_1, \dots, \mathbf{v}_k$ . Although the second definition might seem slightly abstract at first, the span has a rather straightforward geometrical interpretation. For example, the span of a single vector consists of all scalar multiples of this vector, so it can be viewed as the line through  $\mathbf{0}$  in the direction of this vector. The span of two vectors (which are not scalar multiples of each other) can be viewed as the plane through  $\mathbf{0}$  containing these two vectors and so forth.

We note that  $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$  is a sub vector space of  $V$ . This is rather easy to see. Consider two vectors  $\mathbf{u}, \mathbf{v} \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$  in the span, so that  $\mathbf{u} = \sum_i \alpha_i \mathbf{v}_i$  and  $\mathbf{v} = \sum_i \beta_i \mathbf{v}_i$ . Then, the sum  $\mathbf{u} + \mathbf{v} = \sum_i (\alpha_i + \beta_i) \mathbf{v}_i$  is clearly in the span as well, as is the scalar multiple  $\alpha \mathbf{u} = \sum_{i=1}^n (\alpha \alpha_i) \mathbf{v}_i$ . Hence, from Def. 1.2, the span is a (sub) vector space.

**Example 1.5:** The span of vectors

(a) For a simple example in  $\mathbb{R}^3$  consider the span of the first two standard unit vectors  $\text{Span}(\mathbf{e}_1, \mathbf{e}_2) = \{x\mathbf{e}_1 + y\mathbf{e}_2 \mid x, y \in \mathbb{R}\}$  which, of course, corresponds to the  $x$ - $y$  plane.

(b) For a more complicated example in  $\mathbb{R}^3$ , define the two vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}.$$

Their span is given by

$$\text{Span}(\mathbf{u}, \mathbf{v}) = \{\alpha \mathbf{u} + \beta \mathbf{v} \mid \alpha, \beta \in \mathbb{R}\} = \left\{ \begin{pmatrix} \alpha \\ 4\alpha - 3\beta \\ 2\alpha + \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}$$

which describes a plane through  $\mathbf{0}$ .

The above interpretation of the span as lines, planes, etc. through  $\mathbf{0}$  already points to a problem. Consider the span of three vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  but assume that  $\mathbf{u}$  is a linear combination of the other two. In this case,  $\mathbf{u}$  can be omitted without changing the span, so  $\text{Span}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \text{Span}(\mathbf{v}, \mathbf{w})$ . In this sense, the original set of vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  was not minimal. What we would like is a criterion for minimality of a set of vectors, so that none of them can be removed without changing the span. This leads to the concept of *linear independence* which is central to the subject. Formally, it is defined as follows.

**Definition 1.3.** Let  $V$  be a vector space over  $F$  and  $\alpha_1, \dots, \alpha_k \in F$  scalars. A set of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$  is called *linearly independent* if

$$\sum_{i=1}^k \alpha_i \mathbf{v}_i = \mathbf{0} \implies \text{all } \alpha_i = 0. \quad (1.39)$$

Otherwise, the vectors are called *linearly dependent*. That is, they are linearly dependent if  $\sum_{i=1}^k \alpha_i \mathbf{v}_i = \mathbf{0}$  has a solution with at least one  $\alpha_i \neq 0$ .

To relate this to our previous discussion the following statement should be helpful.

**Claim 1.1.** The vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  are linearly dependent  $\iff$  One vector  $\mathbf{v}_i$  can be written as a linear combination of the others.

*Proof.* The proof is rather simple but note that there are two directions to show.



“ $\Rightarrow$ ”: Assume that the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  are linearly dependent so that the equation  $\sum_{i=1}^k \alpha_i \mathbf{v}_i = \mathbf{0}$  has a solution with at least one  $\alpha_i \neq 0$ . Say,  $\alpha_1 \neq 0$ , for simplicity. Then we can solve for  $\mathbf{v}_1$  to get

$$\mathbf{v}_1 = -\frac{1}{\alpha_1} \sum_{i>1} \alpha_i \mathbf{v}_i, \quad (1.40)$$

and, hence, we have expressed  $\mathbf{v}_1$  as a linear combination of the other vectors.

“ $\Leftarrow$ ”: Now assume one vector, say  $\mathbf{v}_1$ , can be written as a linear combination of the others so that  $\mathbf{v}_1 = \sum_{i>1} \beta_i \mathbf{v}_i$ . Then it follows that  $\sum_{i=1}^n \alpha_i \mathbf{v}_i = \mathbf{0}$  with  $\alpha_1 = 1 \neq 0$  and  $\alpha_i = -\beta_i$  for  $i > 1$ . Hence, the vectors are linearly dependent.  $\square$

So for a linearly dependent set of vectors we can eliminate (at least) one vector without changing the span. A linearly independent set is one which cannot be further reduced in this way, so is “minimal” in this sense.

**Example 1.6:** : Linear independence of vectors

(a) The standard unit vectors  $\mathbf{e}_1, \dots, \mathbf{e}_n$  in  $\mathbb{R}^n$  over  $\mathbb{R}$  or  $\mathbb{C}^n$  over  $\mathbb{C}$  form a linearly independent set. This is quite easy to see using Eq. (1.39). We have

$$\sum_{i=1}^n \alpha_i \mathbf{e}_i = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \stackrel{!}{=} \mathbf{0}, \quad (1.41)$$

and this clearly implies that all  $\alpha_i = 0$ .

(b) As a less trivial example, consider the following vectors in  $\mathbb{R}^3$ :

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad (1.42)$$

Again, using Eq. (1.39), we have

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = \begin{pmatrix} \alpha_3 \\ \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_1 + 2\alpha_2 - \alpha_3 \end{pmatrix} \stackrel{!}{=} \mathbf{0}. \quad (1.43)$$

From the first component it follows that  $\alpha_3 = 0$  and combining the second and the third component leads to  $\alpha_1 = \alpha_2 = 0$ . Therefore the three vectors are linearly independent.

(c) Let us discuss linear dependence for systems of two and three vectors. First, assume that the two (non-zero) vectors  $\mathbf{u}, \mathbf{v}$  are linearly dependent. From Claim 1.1 this means that one can be written as a linear combination of the other, so,  $\mathbf{u} = \alpha \mathbf{v}$ . So for two vectors, linear dependence means that they point into the same, or opposite direction, that is they lie on the same line through  $\mathbf{0}$ .

Analogously, for three linearly dependent vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w}$ , one can be expressed as a linear combination of the other two, so, for example,  $\mathbf{u} = \alpha \mathbf{v} + \beta \mathbf{w}$ . This means that  $\mathbf{u}$  is in the plane through  $\mathbf{0}$  spanned by  $\mathbf{v}$  and  $\mathbf{w}$ .

(d) In example 1.4(d) we have explained that the solutions to homogeneous, linear second order differential equations form a vector space. A simple example of such a differential equation is

$$\frac{d^2 f}{dx^2} = -f \quad (1.44)$$

which is obviously solved by  $f(x) = \sin x$  and  $f(x) = \cos x$ . Are these two solutions linearly independent? Using Eq. (1.39) we should start with  $\alpha \sin x + \beta \cos x = 0$  and, since the zero “vector” is the function identical to zero, this equation has to be satisfied for all  $x$ . Setting  $x = 0$  we learn that  $\beta = 0$  and setting  $x = \pi/2$  it follows that  $\alpha = 0$ . Hence,  $\sin$  and  $\cos$  are linearly independent.

(e) For an example which involves linear dependence consider the three vectors

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}. \quad (1.45)$$

Forming a general linear combination gives

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = \begin{pmatrix} -2\alpha_1 + \alpha_2 \\ \alpha_2 + 2\alpha_3 \\ \alpha_1 + \alpha_2 + 3\alpha_3 \end{pmatrix} \stackrel{!}{=} \mathbf{0}. \quad (1.46)$$

This set of equations clearly has non-trivial solutions, for example  $\alpha_1 = 1$ ,  $\alpha_2 = 2$ ,  $\alpha_3 = -1$ , so that the vectors are linearly dependent. Alternatively, this could have been inferred by noting that  $\mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2$ .

## 1.4 Basis and dimension

For a given vector space  $V$ , it is useful to have a “minimal” set of vectors which still spans the whole space. Such a set of vectors is called a *basis* and its formal definition is:

**Definition 1.4.** A set  $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$  of vectors is called a *basis* of  $V$  iff:

- (B1)  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are linearly independent.
- (B2)  $V = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$

It is easy to check that the vectors in Example 1.6 (a), (b), (d) above form a basis. The concept of basis is of central importance in the theory of vector spaces. Every vector can be written as a linear combination of the basis vectors and, what is more, for a given vector this linear combination is unique:

**Claim 1.2.** If  $\mathbf{v}_1, \dots, \mathbf{v}_n$  is a basis of  $V$ , every vector  $\mathbf{v} \in V$  can be written as a unique linear combination  $\mathbf{v} = \sum_{i=1}^n \alpha_i \mathbf{v}_i$ , that is, given  $\mathbf{v}$ , the  $\alpha_i$  are uniquely determined. The coefficients  $\alpha_i$  are called the *coordinates* of  $\mathbf{v}$  with respect to  $\mathbf{v}_1, \dots, \mathbf{v}_n$ .

*Proof.* We need to show that there is indeed only one set of possible coefficients for a given vector  $\mathbf{v}$ . Let us write  $\mathbf{v}$  as two linear combinations

$$\mathbf{v} = \sum_{i=1}^n \alpha_i \mathbf{v}_i = \sum_{i=1}^n \beta_i \mathbf{v}_i \quad (1.47)$$

with coefficients  $\alpha_i$  and  $\beta_i$ . Taking the difference of these two equations implies

$$\sum_{i=1}^n (\alpha_i - \beta_i) \mathbf{v}_i = \mathbf{0}, \quad (1.48)$$

and, from linear independence of the basis, it follows that all  $\alpha_i - \beta_i = 0$ , so that indeed  $\alpha_i = \beta_i$ .  $\square$

In summary, given a basis every vector can be represented by its coordinates relative to the basis. Let us illustrate this with a few examples.

**Example 1.7:** : Coordinates relative to a basis

(a) For the standard basis  $\mathbf{e}_i$  of  $V = F^n$  over  $F$  we can write every vector  $\mathbf{w}$  as

$$\mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \sum_{i=1}^n w_i \mathbf{e}_i \quad (1.49)$$

so the coordinates are identical to the components.

(b) For a more complicated example, start with the basis

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad (1.50)$$

of  $\mathbb{R}^3$  and a general vector  $\mathbf{r}$  with components  $x, y, z$ . To write  $\mathbf{r}$  as a linear combination of the basis vectors we set

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{!}{=} \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = \begin{pmatrix} \alpha_3 \\ \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_1 + 2\alpha_2 - \alpha_3 \end{pmatrix} \quad (1.51)$$

which implies  $x = \alpha_3$ ,  $y = \alpha_1 + \alpha_2 + \alpha_3$ ,  $z = \alpha_1 + 2\alpha_2 - \alpha_3$ . Solving for the  $\alpha_i$  leads to  $\alpha_1 = -3x + 2y - z$ ,  $\alpha_2 = 2x - y + z$ ,  $\alpha_3 = x$ , so these are the coordinates of  $\mathbf{r}$  relative to the given basis.

We would like to call the number of vectors in a basis the *dimension* of the vector space. However, there are usually many different choices of basis for a given vector space. Do they necessarily all have the same number of vectors? Intuitively, it seems this has to be the case but the formal proof is more difficult than expected. It comes down to the following

**Lemma 1.1.** (*Exchange Lemma*) *Let  $\mathbf{v}_1, \dots, \mathbf{v}_n$  be a basis of  $V$  and  $\mathbf{w}_1, \dots, \mathbf{w}_m \in V$  arbitrary vectors. If  $m > n$  then  $\mathbf{w}_1, \dots, \mathbf{w}_m$  are linearly dependent.*

*Proof.* If  $\mathbf{w}_1 = \mathbf{0}$  then the vectors  $\mathbf{w}_1, \dots, \mathbf{w}_m$  are linearly dependent, so can assume that  $\mathbf{w}_1 \neq \mathbf{0}$ . Since the vectors  $\mathbf{v}_i$  form a basis we can write

$$\mathbf{w}_1 = \sum_{i=1}^n \alpha_i \mathbf{v}_i$$

with at least one  $\alpha_i$  (say  $\alpha_1$ ) non-zero (or else  $\mathbf{w}_1$  would be zero). We can, therefore, solve this equation for  $\mathbf{v}_1$  so that

$$\mathbf{v}_1 = \frac{1}{\alpha_1} \left( \mathbf{w}_1 - \sum_{i=2}^n \alpha_i \mathbf{v}_i \right)$$

This shows that we can “exchange” the vector  $\mathbf{v}_1$  for  $\mathbf{w}_1$  in the basis  $\{\mathbf{v}_i\}$  such that  $V = \text{Span}(\mathbf{w}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ . This exchange process can be repeated until all  $\mathbf{v}_i$  are replaced by  $\mathbf{w}_i$  and  $V = \text{Span}(\mathbf{w}_1, \dots, \mathbf{w}_n)$ . Since  $m > n$  there is at least one vector,  $\mathbf{w}_{n+1}$ , “left over” which can be written as a linear combination

$$\mathbf{w}_{n+1} = \sum_{i=1}^n \beta_i \mathbf{w}_i .$$

This shows that the vectors  $\mathbf{w}_1, \dots, \mathbf{w}_m$  are linearly dependent. □

Now consider two basis,  $\mathbf{v}_1, \dots, \mathbf{v}_n$  and  $\mathbf{w}_1, \dots, \mathbf{w}_m$  of  $V$ . Then the Lemma implies that both  $n > m$  and  $n < m$  are impossible and  $n = m$  follows. Hence, while a vector space usually allows many choices of basis the number of basis vectors is always the same. So we can define

**Definition 1.5.** For a basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$  of a vector space  $V$  over  $F$  we call  $\dim_F(V) := n$  the dimension of  $V$  over  $F$ .

From what we have just seen, it does not matter which basis we use to determine the dimension. Every choice leads to the same result. Let us apply this to compute the dimension for some examples.

**Example 1.8:** Dimensions of some vector spaces

(a) We have already established that the standard unit vectors  $\mathbf{e}_1, \dots, \mathbf{e}_n$  form a basis of  $\mathbb{R}^n$  and  $\mathbb{C}^n$  (seen as vector spaces of the fields  $\mathbb{R}$  and  $\mathbb{C}$ , respectively), so  $\dim_{\mathbb{R}}(\mathbb{R}^n) = \dim_{\mathbb{C}}(\mathbb{C}^n) = n$ . However, note that  $\mathbb{C}^n$  as a vector space over  $\mathbb{R}$  has a basis  $\mathbf{e}_1, \dots, \mathbf{e}_n, i\mathbf{e}_1, \dots, i\mathbf{e}_n$  and, therefore,  $\dim_{\mathbb{R}}(\mathbb{C}^n) = 2n$ .

(b) Following up from example 1.6 (d) we would like to determine the dimension of the solution space (of real-valued functions) of

$$\frac{d^2 f}{dx^2} = -f. \tag{1.52}$$

The general solution is given by  $f(x) = \alpha \sin x + \beta \cos x$  with arbitrary real coefficients  $\alpha$  and  $\beta$ . We have already seen that  $\sin$  and  $\cos$  are linearly independent. Hence, the dimension of this space is 2.

(c) Real polynomials of degree  $d$  in the variable  $x$  are of the form  $a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0$  with real coefficients  $a_i$  and they form a vector space over  $\mathbb{R}$ . What is the dimension of this space? Clearly, it is spanned by the monomials  $1, x, x^2, \dots, x^d$ . To show that the monomials are linearly independent start with

$$\sum_{i=0}^d \alpha_i x^i = 0, \tag{1.53}$$

take the  $k^{\text{th}}$  derivative with respect to  $x$  and then set  $x = 0$ . This immediately implies that  $\alpha_k = 0$  and, hence, that the monomials are linearly independent and form a basis. The dimension of the space is, therefore,  $d + 1$ .

(d) For the  $n \times m$  matrices with entries in  $F$  (as a vector space over  $F$ ) define the matrices

$$E_{(ij)} = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{1.54}$$

where  $i = 1, \dots, n$  and  $j = 1, \dots, m$  and the “1” appears in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column with all other entries zero. Clearly, these matrices form a basis, in complete analogy with the standard unit vectors. Therefore, the vector space of  $n \times m$  matrices has dimension  $nm$ .

In the following Lemma we collect a few simple conclusions about vector spaces which are spanned by a finite number of vectors.

**Lemma 1.2.** For a vector space  $V$  spanned by a finite number of vectors we have:

- (i)  $V$  has a basis
- (ii) Every linearly independent set  $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$  can be completed to a basis.
- (iii) If  $n = \dim(V)$ , any linearly independent set of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  forms a basis.
- (iv) If  $\dim_F(V) = \dim_F(W)$  and  $V \subset W$  for two vector spaces  $V, W$  then  $V = W$ .

*Proof.* : (i) By assumption,  $V$  is spanned by a finite number of vectors, say  $V = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ . If these vectors are linearly independent we have found a basis. If not, one of the vectors, say  $\mathbf{v}_k$ , can be written as a linear combination of the others and can, hence, be dropped without changing the span, so  $V = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_{k-1})$ . This process can be continued until the remaining set of vectors is linearly independent.

(ii) If the linearly independent vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  already span  $V$  we are finished. If not there exists a vector  $\mathbf{v}_{k+1} \notin \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ . Hence, the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}$  must be linearly independent. We can continue adding vectors to the list until it spans the whole space. (The process must terminate after a finite number of steps or else we would contradict the Exchange Lemma 1.1.)

(iii) If  $\dim(V) = n$  and the linearly independent set  $\mathbf{v}_1, \dots, \mathbf{v}_n$  did not span  $V$  then, from (ii), it could be completed to a basis with more than  $n$  elements. However, this is a contradiction since the number of elements in a basis is the same for any choice of basis. Hence, the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  must span the space and they form a basis.

(iv) We can choose a basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$  of  $V$ . Since  $V \subset W$ , these basis vectors are linearly independent in  $W$  and, since  $\dim_F(W) = \dim_F(V)$ , they must also form a basis of  $W$ , using (iii). Hence  $V = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n) = W$ .  $\square$

### Application: Magic Squares

An amusing application of vector spaces is to magic squares. Magic squares are  $3 \times 3$  (say) quadratic schemes of (rational) numbers – we can think of them as  $3 \times 3$  matrices – such that all rows, all columns and both diagonals sum up to the same total. A simple example is

$$M = \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix} \quad (1.55)$$

where every row, column and diagonal sums up to 15. Magic squares have long held a certain fascination and an obvious problem is to find all magic squares.

In our context, the important observation is that magic squares form a vector space. Let us agree that we add and scalar multiply magic squares in the same way as matrices (see Example 1.4 (e)), that is, component by component. Then, clearly, the sum of two magic squares is again a magic square, as is the scalar multiple of a magic square. Hence, from Def. 1.2, the  $3 \times 3$  magic squares form a sub vector space of the space of all  $3 \times 3$  matrices. The problem of finding all magic squares can now be phrased in the language of vector spaces. What is the dimension of the space of magic squares and can we write down a basis for this space?

It is relative easy to find the following three elementary examples of magic squares:

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}. \quad (1.56)$$

It is also easy to show that these three matrices are linearly independent, using Eq. (1.39). Setting a general linear combination to zero,

$$\alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 = \begin{pmatrix} \alpha_1 - \alpha_3 & \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 - \alpha_2 \\ \alpha_1 - \alpha_2 + \alpha_3 & \alpha_1 & \alpha_1 + \alpha_2 - \alpha_3 \\ \alpha_1 + \alpha_2 & \alpha_1 - \alpha_2 - \alpha_3 & \alpha_1 + \alpha_3 \end{pmatrix} \stackrel{!}{=} 0, \quad (1.57)$$

immediately leads to  $\alpha_1 = \alpha_2 = \alpha_3 = 0$ . Hence,  $M_1, M_2, M_3$  are linearly independent and span a three-dimensional vector space of magic squares. Therefore, the dimension of the magic square space is at least three. Indeed, our example (1.55) is contained in this three-dimensional space since  $M = 5M_1 + 3M_2 + M_3$ . As we will see later, this is not an accident. We will show that, in fact, the dimension of the magic square space equals three and, hence, that  $M_1, M_2, M_3$  form a basis.

## 2 Vectors in $\mathbb{R}^n$ , geometrical applications

We would now like to pause the general story (before we resume in the next chapter) and focus on a number of important topics for column vectors in  $\mathbb{R}^n$ . In particular, we will introduce the scalar and vector product for column vectors which are widely used in physics and discuss some related geometrical applications.

### 2.1 Scalar product in $\mathbb{R}^n$

The *scalar (or dot) product* for two  $n$ -dimensional column vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad (2.1)$$

is defined as

$$\mathbf{a} \cdot \mathbf{b} := \sum_{i=1}^n a_i b_i. \quad (2.2)$$

In physics it is customary to omit the sum symbol in this definition and simply write  $\mathbf{a} \cdot \mathbf{b} = a_i b_i$ , adopting the convention that an index which appears twice in a given term (such as the index  $i$  in the present case) is summed over. This is also referred to as the *Einstein summation convention*.

The scalar product satisfies a number of obvious properties, namely

$$\begin{aligned} \text{(a)} \quad & \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \\ \text{(b)} \quad & \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \\ \text{(c)} \quad & \mathbf{a} \cdot (\beta \mathbf{b}) = \beta \mathbf{a} \cdot \mathbf{b} \\ \text{(d)} \quad & \mathbf{a} \cdot \mathbf{a} > 0 \text{ for all } \mathbf{a} \neq \mathbf{0} \end{aligned} \quad (2.3)$$

Property (a) means that the dot product is symmetric. Properties (b), (c) can be expressed by saying that the scalar product is linear in the second argument (vector addition and scalar multiplication can be “pulled through”) and, by symmetry, it is therefore also linear in the first argument. It is easy to show these properties using index notation.

$$\begin{aligned} \text{(a)} \quad & \mathbf{a} \cdot \mathbf{b} = a_i b_i = b_i a_i = \mathbf{b} \cdot \mathbf{a} \\ \text{(b)} \quad & \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = a_i (b_i + c_i) = a_i b_i + a_i c_i = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \\ \text{(c)} \quad & \mathbf{a} \cdot (\beta \mathbf{b}) = a_i (\beta b_i) = \beta a_i b_i = \beta \mathbf{a} \cdot \mathbf{b} \\ \text{(d)} \quad & \mathbf{a} \cdot \mathbf{a} = \sum_{i=1}^n a_i^2 > 0 \text{ for } \mathbf{a} \neq \mathbf{0} \end{aligned}$$

Scalar products can also be defined “axiomatically” by postulating the four properties (a) – (d) and we will come back to this approach in Section 6.

The last property, (d), allows us to define the *length of a vector* as

$$|\mathbf{a}| := \sqrt{\mathbf{a} \cdot \mathbf{a}} = \left( \sum_{i=1}^n a_i^2 \right)^{1/2}. \quad (2.4)$$

**Example 2.1:** : Dot product and length of vectors

The three-dimensional vectors

$$\mathbf{a} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad (2.5)$$

have a dot product

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 2 \cdot 2 + 4 \cdot 1 + (-2) \cdot 1 = 6. \quad (2.6)$$

Their lengths are given by

$$|\mathbf{a}| = \sqrt{2^2 + 4^2 + (-2)^2} = 2\sqrt{6}, \quad |\mathbf{b}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}. \quad (2.7)$$

It follows easily that  $|\alpha \mathbf{a}| = |\alpha| |\mathbf{a}|$  for any real number  $\alpha$ . This relation means that every non-zero vector  $\mathbf{a}$  can be “normalised” to length one by defining

$$\mathbf{n} = |\mathbf{a}|^{-1} \mathbf{a}. \quad (2.8)$$

The length of  $\mathbf{n}$  is indeed one, since  $|\mathbf{n}| = ||\mathbf{a}|^{-1} \mathbf{a}| = |\mathbf{a}|^{-1} |\mathbf{a}| = 1$ .

The dot product satisfies an important inequality.

**Lemma 2.1.** (*Cauchy-Schwarz inequality*) For any two vectors  $\mathbf{a}$  and  $\mathbf{b}$  in  $\mathbb{R}^n$  we have

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}| |\mathbf{b}|.$$

*Proof.* The proof is a bit tricky. We start with the simplifying assumption that  $|\mathbf{a}| = |\mathbf{b}| = 1$ . Then

$$0 \leq |\mathbf{a} \pm \mathbf{b}|^2 = (\mathbf{a} \pm \mathbf{b}) \cdot (\mathbf{a} \pm \mathbf{b}) = |\mathbf{a}|^2 \pm 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 = 2(1 \pm \mathbf{a} \cdot \mathbf{b}),$$

which shows that  $|\mathbf{a} \cdot \mathbf{b}| \leq 1$ . Now consider arbitrary vectors  $\mathbf{a}$  and  $\mathbf{b}$ . If one of these vector is zero then the inequality is trivially satisfied so we assume that both of them are non-zero. Then the vectors

$$\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}, \quad \mathbf{v} = \frac{\mathbf{b}}{|\mathbf{b}|}$$

have both length one and, hence,  $|\mathbf{u} \cdot \mathbf{v}| \leq 1$ . Inserting the definitions of  $\mathbf{u}$  and  $\mathbf{v}$  into this inequality and multiplying by  $|\mathbf{a}|$  and  $|\mathbf{b}|$  gives the desired result.  $\square$

A closely related inequality is the famous

**Lemma 2.2.** (Triangle inequality) For any two vectors  $\mathbf{a}$  and  $\mathbf{b}$  in  $\mathbb{R}^n$  we have

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

*Proof.*

$$|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b} \leq |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}| = (|\mathbf{a}| + |\mathbf{b}|)^2,$$

where the Cauchy-Schwarz inequality has been used in the second step.  $\square$

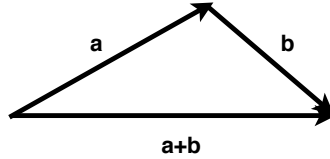


Figure 4: Geometric meaning of the triangle inequality: The length  $|\mathbf{a} + \mathbf{b}|$  is always less or equal than the sum  $|\mathbf{a}| + |\mathbf{b}|$  for the other two sides.

The triangle inequality has an obvious geometrical interpretation which is illustrated in Fig. 4. For two non-zero vectors  $\mathbf{a}$  and  $\mathbf{b}$ , the Cauchy-Schwarz inequality implies that

$$-1 \leq \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \leq 1 \quad (2.9)$$

so that there is a unique angle  $\theta \in [0, \pi]$  such that

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}. \quad (2.10)$$

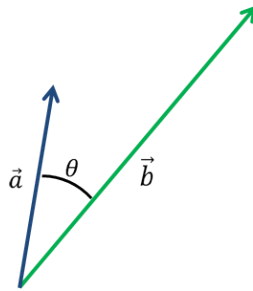


Figure 5: Angle between two vectors

This angle  $\theta$  is called the *angle between the two vectors*  $\mathbf{a}$  and  $\mathbf{b}$ , also denoted  $\sphericalangle(\mathbf{a}, \mathbf{b})$ . With this definition of the angle we can also write the scalar product as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\sphericalangle(\mathbf{a}, \mathbf{b})). \quad (2.11)$$

We call the two vectors,  $\mathbf{a}$  and  $\mathbf{b}$  *orthogonal* (or perpendicular), in symbols  $\mathbf{a} \perp \mathbf{b}$ , iff  $\mathbf{a} \cdot \mathbf{b} = 0$ . For two non-zero vectors  $\mathbf{a}$  and  $\mathbf{b}$  this means

$$\mathbf{a} \perp \mathbf{b} \iff \sphericalangle(\mathbf{a}, \mathbf{b}) = \frac{\pi}{2}. \quad (2.12)$$



**Example 2.2:** : Angle between vectors and orthogonality

(a) Recall that the two vectors  $\mathbf{a}$  and  $\mathbf{b}$  in Eq. (2.5) have a dot product  $\mathbf{a} \cdot \mathbf{b} = 6$  and lengths  $|\mathbf{a}| = 2\sqrt{6}$  and  $|\mathbf{b}| = \sqrt{6}$ . Hence, the angle between them is given by

$$\cos(\angle(\mathbf{a}, \mathbf{b})) = \frac{6}{2\sqrt{6}\sqrt{6}} = \frac{1}{2} \Rightarrow \angle(\mathbf{a}, \mathbf{b}) = \frac{\pi}{3}. \quad (2.13)$$

(b) The two vectors

$$\mathbf{a} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad (2.14)$$

have a dot product  $\mathbf{a} \cdot \mathbf{b} = 3 \cdot 1 + (-2) \cdot 2 + 1 \cdot 1 = 0$  and are, hence, orthogonal.

(c) Start with a general vector

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad (2.15)$$

in  $\mathbb{R}^2$ . It is easy to write down a vector orthogonal to  $\mathbf{a}$  by exchanging the two components and inverting the sign of one of them. Of course every scalar multiple of this new vector is also orthogonal to  $\mathbf{a}$  so the vectors

$$\mathbf{b} = \alpha \begin{pmatrix} -a_2 \\ a_1 \end{pmatrix}. \quad (2.16)$$

are orthogonal to  $\mathbf{a}$  for arbitrary  $\alpha \in \mathbb{R}$ . Indeed, we have  $\mathbf{a} \cdot \mathbf{b} = \alpha(a_1(-a_2) + a_2a_1) = 0$ . Conversely, it is easy to see that all vectors orthogonal to  $\mathbf{a}$  must be of this form.

---

## 2.2 Vector product in $\mathbb{R}^3$

For two three-dimensional vectors  $\mathbf{a}$  and  $\mathbf{b}$  the *vector product* is defined as

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} := \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}. \quad (2.17)$$

Note that this rule is relatively easy to remember: For the first entry of the cross product consider the second and third components of the two vectors and multiply them “cross-wise” with a relative minus sign between the two terms and similarly for the other two entries.

**Example 2.3:** Cross product

Using the above definition, the cross product can be carried out immediately, for example

$$\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \cdot 5 - 3 \cdot 1 \\ 3 \cdot (-2) - 2 \cdot 5 \\ 2 \cdot 1 - 4 \cdot (-2) \end{pmatrix} = \begin{pmatrix} 17 \\ -16 \\ 10 \end{pmatrix} \quad (2.18)$$

---

However, cross product calculations which involve non-numerical, symbolic expressions can become extremely tedious if done by writing out all three components explicitly. It is therefore useful to introduce

a more economical notation and adopt the Einstein summation convention. To this end, we define the following two objects:

**Kronecker delta in  $\mathbb{R}^n$**  The Kronecker delta in  $\mathbb{R}^n$  is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} , \quad (2.19)$$

where  $i, j, \dots = 1, \dots, n$ . For a vector  $a_j$  we have  $\delta_{ij}a_j = a_i$  (recall, a sum over  $j$  is implied), so the Kronecker delta acts as an “index replacer”, substituting the summed over index  $j$  by  $i$  in the previous expression. Another useful property is  $\delta_{ii} = n$  (again, note the double appearance of  $i$  means it is summed over). The dot product can also be expressed in terms of the Kronecker delta by writing  $\mathbf{a} \cdot \mathbf{b} = a_i b_i = \delta_{ij} a_i b_j$ .

**Levi-Civita tensor in  $\mathbb{R}^3$**  The Levi-Civita tensor,  $\epsilon_{ijk}$ , where  $i, j, k, \dots = 1, 2, 3$  is defined as

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2) \quad \text{“cyclic permutations”} \\ -1 & \text{if } (i, j, k) = (2, 1, 3), (3, 2, 1), (1, 3, 2) \quad \text{“anti-cyclic permutations”} \\ 0 & \text{otherwise} \end{cases} . \quad (2.20)$$

It has a number of useful properties, namely

$$(a) \quad \text{it remains unchanged under cyclic index permutations, for example } \epsilon_{ijk} = \epsilon_{jki} \quad (2.21)$$

$$(b) \quad \text{it changes sign under anti-cyclic index permutations, for example } \epsilon_{ijk} = -\epsilon_{ikj} \quad (2.22)$$

$$(c) \quad \text{it vanishes if two indices are identical, for example } \epsilon_{ijj} = 0 \quad (2.23)$$

$$(d) \quad \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \quad (2.24)$$

$$(e) \quad \epsilon_{ijk}\epsilon_{ijm} = 2\delta_{km} \quad (2.25)$$

$$(f) \quad \epsilon_{ijk}\epsilon_{ijk} = 6 \quad (2.26)$$

$$(g) \quad \epsilon_{ijk}a_j a_k = 0 . \quad (2.27)$$

The first three of these properties are obvious from the definition of the Levi-Civita tensor. Property (2.24) can be reasoned out as follows. If the index pair  $(j, k)$  is different from  $(l, m)$  (in any order) then clearly both sides of (2.24) are zero. On the other hand, if the two index pairs equal each other they can do so in the same or the opposite ordering and these two possibilities correspond precisely to the two terms on the RHS of (2.24). If we multiply (2.24) by  $\delta_{jl}$ , using the index replacing property of the Kronecker delta, we obtain

$$\epsilon_{ijk}\epsilon_{ijm} = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\delta_{jl} = 3\delta_{km} - \delta_{km} = 2\delta_{km}$$

and this is property (2.25). Further, multiplying (2.25) with  $\delta_{km}$  we have

$$\epsilon_{ijk}\epsilon_{ijk} = 2\delta_{km}\delta_{km} = 2\delta_{kk} = 6$$

and, hence, (2.26) follows. Finally, to show (2.27) we write  $\epsilon_{ijk}a_j a_k = -\epsilon_{ikj}a_k a_j = -\epsilon_{ijk}a_j a_k$ , where the summation indices  $j$  and  $k$  have been swapped in the last step, and, hence,  $2\epsilon_{ijk}a_j a_k = 0$ .

We can think of  $\delta_{ij}$  and  $\epsilon_{ijk}$  as a convenient notation for the 0’s, 1’s and  $-1$ ’s which appear in the definition of the dot and cross product. Indeed, the dot product can be written as

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i = \delta_{ij} a_i a_j , \quad (2.28)$$

while the index version of the cross product takes the form

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k . \quad (2.29)$$

To verify this last equation focus, for example, on the first component:

$$\epsilon_{1jk}a_jb_k = \epsilon_{123}a_2b_3 + \epsilon_{132}a_3b_2 = a_2b_3 - a_3b_2, \quad (2.30)$$

which indeed equals the first component of the vector product (2.17). Analogously, it can be verified that the other two components match. Note that the index expression (2.29) for the vector product is much more concise than the component version (2.17).

#### Example 2.4: Vector products in physics

Vector products make a frequent appearance in physics. Here are a few basic examples:

- (a) In mechanics the angular momentum of a mass  $m$  at position  $\mathbf{r}$  and with velocity  $\dot{\mathbf{r}}$  is given by  $\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}}$ .
- (b) The force a magnetic field  $\mathbf{B}$  exerts on a particle with charge  $q$  and velocity  $\dot{\mathbf{r}}$ , the so-called *Lorentz force*, is given by  $\mathbf{F} = q\dot{\mathbf{r}} \times \mathbf{B}$ .
- (c) The velocity of a point with coordinate  $\mathbf{r}$  in a rotating coordinate system with angular velocity  $\boldsymbol{\omega}$  is given by  $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ .

---

The vector product satisfies the following useful properties.

$$(a) \quad \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \quad (2.31)$$

$$(b) \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \quad (2.32)$$

$$(c) \quad \mathbf{a} \times (\beta\mathbf{b}) = \beta\mathbf{a} \times \mathbf{b} \quad (2.33)$$

$$(d) \quad \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2 \quad (2.34)$$

$$(e) \quad \mathbf{a} \times \mathbf{a} = \mathbf{0} \quad (2.35)$$

$$(f) \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \quad (2.36)$$

$$(g) \quad (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \quad (2.37)$$

$$(h) \quad |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \quad (2.38)$$

Property (a) means that the vector product is anti-symmetric. Properties (b) and (c) imply linearity in the second argument (vector addition and scalar multiplication can be “pulled through”) and, from anti-symmetry, linearity also holds in the first argument. Property (g) is sometimes referred to as *Lagrange identity*. The above relations can be verified by writing out all the vectors explicitly and using the definitions (2.2) and (2.17) of the dot and cross products. However, for some of the identities this leads to rather tedious calculations. It is much more economical to use index notation and express dot and cross product via Eqs. (2.28) and (2.29). The proofs are then as follows:

$$(a) \quad (\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk}a_jb_k = -\epsilon_{ikj}b_ka_j = -(\mathbf{b} \times \mathbf{a})_i$$

$$(b) \quad (\mathbf{a} \times (\mathbf{b} + \mathbf{c}))_i = \epsilon_{ijk}a_j(b_k + c_k) = \epsilon_{ijk}a_jb_k + \epsilon_{ijk}a_jc_k = (\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c})_i$$

$$(c) \quad (\mathbf{a} \times (\beta\mathbf{b}))_i = \epsilon_{ijk}a_j\beta b_k = \beta\epsilon_{ijk}a_jb_k = \beta(\mathbf{a} \times \mathbf{b})_i$$

(d) By explicit computation using the definition (2.17) with the standard unit vectors.

$$(e) \quad (\mathbf{a} \times \mathbf{a})_i = \epsilon_{ijk}a_ja_k = 0 \quad (\text{from property (2.27) of } \epsilon_{ijk})$$

- (f)  $(\mathbf{a} \times (\mathbf{b} \times \mathbf{c}))_i = \epsilon_{ijk} a_j (\mathbf{b} \times \mathbf{c})_k = \epsilon_{ijk} \epsilon_{kmn} a_j b_m c_n = \epsilon_{kij} \epsilon_{kmn} a_j b_m c_n \stackrel{(2.24)}{=} (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) a_j b_m c_n$   
 $= a_j c_j b_i - a_j b_j c_i = \mathbf{a} \cdot \mathbf{c} b_i - \mathbf{a} \cdot \mathbf{b} c_i = ((\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c})_i$
- (g)  $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \epsilon_{ijk} \epsilon_{imn} a_j b_k c_m d_n \stackrel{(2.24)}{=} (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}) a_j b_k c_m d_n = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$
- (h) Set  $\mathbf{c} = \mathbf{a}$  and  $\mathbf{d} = \mathbf{b}$  in property (g).

Note how expressions in vector notation are converted into index notation in these proofs by working from the “outside in”. For example, for property (f) we have first written the outer cross product between  $\mathbf{a}$  and  $\mathbf{b} \times \mathbf{c}$  in index notation and then, in the second step, we have converted  $\mathbf{b} \times \mathbf{c}$ . Once fully in index notation, the order of all objects can be exchanged at will - after all they are just numbers. In the proofs of (f) and (g) the Kronecker delta acts as an index replacer, as explained below Eq. (2.19).

It is worth pointing out that the cross product can also be defined axiomatically by postulating a product  $\times : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  with the properties (a) – (d) in (2.31)–(2.34). In other words, the cross product can be defined as an anti-symmetric, bi-linear operation, mapping two three-dimensional vectors into a three-dimensional vector, which acts in a simple, cyclic way (that is, as in property (d)) on the standard unit vectors. It is easy to see that the vector product is indeed completely determined by these properties. Write two vector  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$  as linear combinations of the standard unit vectors, that is,  $\mathbf{a} = \sum_i a_i \mathbf{e}_i$  and  $\mathbf{b} = \sum_j b_j \mathbf{e}_j$ , and work out their cross product using only the rules (a) – (d) (as well as the rule (e) which follows directly from (a)). This leads to

$$\mathbf{a} \times \mathbf{b} = \left( \sum_i a_i \mathbf{e}_i \right) \times \left( \sum_j b_j \mathbf{e}_j \right) \stackrel{(2.32),(2.33)}{=} \sum_{i,j} a_i b_j \mathbf{e}_i \times \mathbf{e}_j \stackrel{(2.35)}{=} \sum_{i \neq j} a_i b_j \mathbf{e}_i \times \mathbf{e}_j \quad (2.39)$$

$$\stackrel{(2.31),(2.34)}{=} (a_2 b_3 - a_3 b_2) \mathbf{e}_1 + (a_3 b_1 - a_1 b_3) \mathbf{e}_2 + (a_1 b_2 - a_2 b_1) \mathbf{e}_3, \quad (2.40)$$

which coincides with our original definition (2.17).

---

### Application: Kinetic energy of a rotating rigid body

An example which illustrates some of the above identities and techniques in the context of classical mechanics is the kinetic energy of a rigid rotating body. Consider a rigid body, as depicted in Fig. 6, which we would like to think of as consisting of (a possibly large number of) mass points, labeled by an index  $\alpha$ , each with mass  $m_\alpha$ , position vector  $\mathbf{r}_\alpha$  and velocity  $\mathbf{v}_\alpha$ . The total kinetic energy of this body is of course obtained by summing over the kinetic energy of all mass points, that is,  $E_{\text{kin}} = \frac{1}{2} \sum_\alpha m_\alpha \mathbf{v}_\alpha^2$ . From Example (2.4)(c) we know that the velocity of each mass point is related to its position by  $\mathbf{v}_\alpha = \boldsymbol{\omega} \times \mathbf{r}_\alpha$ , where  $\boldsymbol{\omega}$  is the angular velocity of the body. The kinetic energy of the rotating body can then be written as

$$\begin{aligned} E_{\text{kin}} &= \frac{1}{2} \sum_\alpha m_\alpha \mathbf{v}_\alpha^2 = \frac{1}{2} \sum_\alpha m_\alpha |\boldsymbol{\omega} \times \mathbf{r}_\alpha|^2 \stackrel{(2.38)}{=} \frac{1}{2} \sum_\alpha m_\alpha (|\boldsymbol{\omega}|^2 |\mathbf{r}_\alpha|^2 - (\boldsymbol{\omega} \cdot \mathbf{r}_\alpha)^2) \\ &= \frac{1}{2} \sum_\alpha m_\alpha (\omega_i \omega_j \delta_{ij} |\mathbf{r}_\alpha|^2 - \omega_i r_{\alpha i} \omega_j r_{\alpha j}) = \frac{1}{2} \omega_i \underbrace{\left[ \sum_\alpha m_\alpha (|\mathbf{r}_\alpha|^2 \delta_{ij} - r_{\alpha i} r_{\alpha j}) \right]}_{=: I_{ij}} \omega_j. \end{aligned}$$

The object in the square bracket, denoted by  $I_{ij}$ , is called the *moment of inertia tensor* of the rigid body. It is obviously symmetric, so  $I_{ij} = I_{ji}$ , so we can think of it as forming a symmetric matrix, and it is a characteristic quantity of the rigid body. We can think of it as playing a role in rotational motion

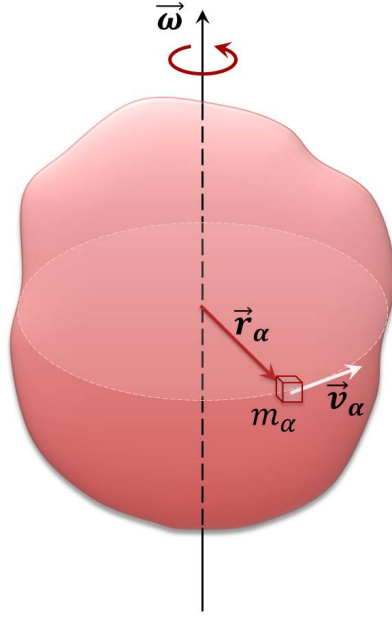


Figure 6: A rotating rigid body

analogous to that of regular mass in linear motion. Correspondingly, the total kinetic energy of the rigid body can be written as

$$E_{\text{kin}} = \frac{1}{2} \sum_{i,j} I_{ij} \omega_i \omega_j \quad (2.41)$$

This relation is of fundamental importance for the mechanics of rigid bodies, in particular the motion of tops, and we will return to it later.

Dot and cross product can be combined to a third product with three vector arguments, the *triple product*, which is defined as

$$\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle := \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) . \quad (2.42)$$

It has the following properties

$$(a) \quad \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle = \epsilon_{ijk} a_i b_j c_k = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1 \quad (2.43)$$

$$(b) \quad \text{It is linear in each argument, e.g. } \langle \alpha \mathbf{a} + \beta \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle = \alpha \langle \mathbf{a}, \mathbf{c}, \mathbf{d} \rangle + \beta \langle \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle \quad (2.44)$$

$$(c) \quad \text{It is unchanged under cyclic permutations, e.g. } \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle = \langle \mathbf{b}, \mathbf{c}, \mathbf{a} \rangle \quad (2.45)$$

$$(d) \quad \text{It changes sign for anti-cyclic permutations, e.g. } \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle = -\langle \mathbf{a}, \mathbf{c}, \mathbf{b} \rangle \quad (2.46)$$

$$(e) \quad \text{It vanishes if any two arguments are the same, e.g. } \langle \mathbf{a}, \mathbf{b}, \mathbf{b} \rangle = 0 \quad (2.47)$$

$$(f) \quad \text{The triple product of the three standard unit vectors is one, that is } \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle = 1 \quad (2.48)$$

Property (a) follows easily from the definitions of dot and cross products, Eqs. (2.28) and (2.29), in index notation and the definition (2.20) of the Levi-Civita tensor. Properties (b)–(e) are a direct consequence of (a) and the properties of the Levi-Civita tensor. Specifically, (c) and (d) follow from (2.21), (2.22) and (e) follows from (2.27). Property (f) follows from direct calculation, using the cross product relations (2.34) for the standard unit vectors.

Another notation for the triple product is

$$\det(\mathbf{a}, \mathbf{b}, \mathbf{c}) := \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle , \quad (2.49)$$

where “det” is short for *determinant*. Later we will introduce the determinant in general and for arbitrary dimensions and we will see that, in three dimensions, this general definition indeed coincides with the triple product.

Note that the six terms which appear in the explicit expression (2.43) for the triple product correspond to the six permutations of  $\{1, 2, 3\}$ , where the three terms for the cyclic permutations come with a positive sign and the other, anti-cyclic ones with a negative sign. There is a simple way to memorise these six terms. If we arrange the three vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  into the columns of a  $3 \times 3$  matrix for convenience, the six terms in the triple product correspond to the products of terms along the diagonals.

$$\det \begin{pmatrix} \times & a_1 & \times & b_1 & \times & c_1 & \times \\ \times & a_2 & \times & b_2 & \times & c_2 & \times \\ \times & a_3 & \times & b_3 & \times & c_3 & \times \end{pmatrix} = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1 \quad (2.50)$$

Here north-west to south-east lines connect the entries forming the three cyclic terms which appear with a positive sign while north-east to south-west lines connect the entries forming the anti-cyclic terms which appear with a negative sign. Corresponding lines on the left and right edge should be identified in order to collect all the factors.

**Example 2.5:** Calculating the triple product

Let us calculate the triple product  $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$  for the three vectors

$$\mathbf{a} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 4 \\ -6 \\ 3 \end{pmatrix}. \quad (2.51)$$

One way to proceed is to first work out the cross product between  $\mathbf{b}$  and  $\mathbf{c}$ , that is

$$\mathbf{b} \times \mathbf{c} = \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ -6 \\ 3 \end{pmatrix} = \begin{pmatrix} 21 \\ 10 \\ -8 \end{pmatrix}. \quad (2.52)$$

and then dot the result with  $\mathbf{a}$ , so

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 21 \\ 10 \\ -8 \end{pmatrix} = 23. \quad (2.53)$$

Alternatively and equivalently, we can use the rule (2.50) which gives

$$\begin{aligned} \det(\mathbf{a}, \mathbf{b}, \mathbf{c}) &= \det \begin{pmatrix} \times & -1 & \times & -2 & \times & 4 & \times \\ \times & 2 & \times & 5 & \times & -6 & \times \\ \times & -3 & \times & 1 & \times & 3 & \times \end{pmatrix} \\ &= (-1) \cdot 5 \cdot 3 + (-2) \cdot (-6) \cdot (-3) + 4 \cdot 2 \cdot 1 - (-1) \cdot (-6) \cdot 1 - (-2) \cdot 2 \cdot 3 - 4 \cdot 5 \cdot (-3) \\ &= -15 - 36 + 8 - 6 + 12 + 60 = 23. \end{aligned} \quad (2.54)$$

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Having introduced all the general definitions and properties we should now discuss the geometrical interpretations of the cross and triple products.

**Geometrical interpretation of cross product:**

Property (2.47) of the triple product implies that the cross product  $\mathbf{a} \times \mathbf{b}$  is perpendicular to both vectors  $\mathbf{a}$  and  $\mathbf{b}$ . For the length of a cross product it follows

$$|\mathbf{a} \times \mathbf{b}| \stackrel{(2.38)}{=} (|\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2)^{\frac{1}{2}} = |\mathbf{a}| |\mathbf{b}| \underbrace{\left(1 - \frac{(\mathbf{a} \cdot \mathbf{b})^2}{|\mathbf{a}|^2 |\mathbf{b}|^2}\right)^{\frac{1}{2}}}_{=1 - \cos^2(\angle(\mathbf{a}, \mathbf{b}))} = |\mathbf{a}| \cdot |\mathbf{b}| \sin \angle(\mathbf{a}, \mathbf{b}) \quad (2.55)$$

From this result and Fig. 7 the length,  $|\mathbf{a} \times \mathbf{b}|$ , of the cross product is equal to the area of the rectangle

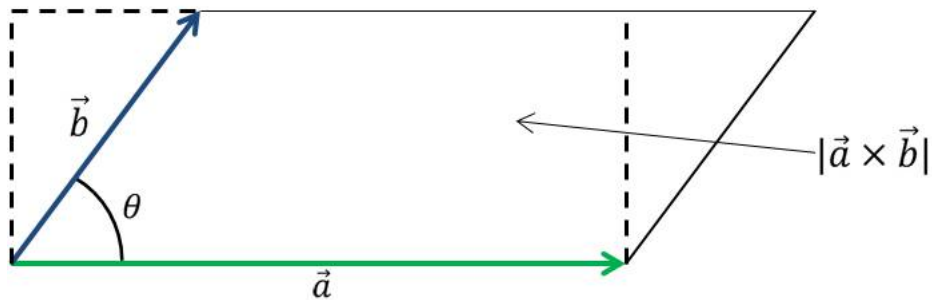


Figure 7: *Shear, leaves the area unchanged*

indicated and, as this area is left invariant by a shear, it equals the area of the parallelogram defined by the vectors  $\mathbf{a}$  and  $\mathbf{b}$ . In summary, we therefore see that the vector product  $\mathbf{a} \times \mathbf{b}$  defines a vector perpendicular to  $\mathbf{a}$  and  $\mathbf{b}$  whose length equals the area of the parallelogram defined by  $\mathbf{a}$  and  $\mathbf{b}$ .

This geometrical interpretation suggests a number of applications for the cross product. In particular, it can be used to find a vector which is orthogonal to two given vectors and to calculate the area of the parallelogram (and the triangle) defined by two vectors.

**Example 2.6:** Applications of the cross product

Consider the two vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \quad (2.56)$$

with cross product

$$\mathbf{c} := \mathbf{a} \times \mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} \quad (2.57)$$

It is easy to verify that  $\mathbf{c}$  is indeed orthogonal to  $\mathbf{a}$  and  $\mathbf{b}$ , that is,  $\mathbf{c} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{b} = 0$ . The area of the parallelogram defined by  $\mathbf{a}$  and  $\mathbf{b}$  is given by

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{41}, \quad (2.58)$$

while the area of the triangle defined by  $\mathbf{a}$  and  $\mathbf{b}$  is half the area of the parallelogram, that is  $\sqrt{41}/2$  for the example.

**Geometrical interpretation of triple product**

We first note that the triple product of three standard unit vectors is  $\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) =$

$\mathbf{e}_1 \cdot \mathbf{e}_1 = 1$  which equals the volume of the unit cube. For three arbitrary vectors  $\alpha\mathbf{e}_1$ ,  $\beta\mathbf{e}_2$ ,  $\gamma\mathbf{e}_3$  in the directions of the coordinate axis we find, from linearity (2.44) of the triple product, that  $\langle \alpha\mathbf{e}_1, \beta\mathbf{e}_2, \gamma\mathbf{e}_3 \rangle = \alpha\beta\gamma\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle = \alpha\beta\gamma$  which equals the volume of the cuboid with lengths  $\alpha$ ,  $\beta$ ,  $\gamma$ . Suppose this cuboid is sheared to a parallelepiped. As an example let us consider a shear in the direction of  $\mathbf{e}_3$ , leading to a parallelepiped defined by the vectors  $\mathbf{a} = \alpha\mathbf{e}_1 + \delta\mathbf{e}_3$ ,  $\mathbf{b} = \beta\mathbf{e}_2$  and  $\mathbf{c} = \gamma\mathbf{e}_3$ . Then, by linearity of the triple product and (2.25) we have  $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle = \alpha\beta\gamma\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle + \delta\beta\gamma\langle \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_3 \rangle = \alpha\beta\gamma$ , so the triple product is the same for the cuboid and the parallelepiped related by a shear. It is clear that this remains true for general shears. Since shears are known to leave the volume unchanged we conclude that the (absolute) value of the triple product,  $|\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle|$ , for three arbitrary vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  equals the volume of the parallelepiped defined by these vectors.

This geometrical interpretation suggests that the triple product of three linearly dependent vectors should be zero. Indeed, three linearly dependent vectors all lie in one plane and form a degenerate parallelepiped with volume zero. To be sure let us properly formulate and prove this assertion.

**Claim 2.1.**  $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle \neq 0 \implies \mathbf{a}, \mathbf{b}, \mathbf{c}$  are linearly independent.

*Proof.* If  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  are linearly dependent then one of the vectors can be written as a linear combination of the others, for example,  $\mathbf{a} = \beta\mathbf{b} + \gamma\mathbf{c}$ . From linearity of the triple product and (2.47) it then follows that

$$\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle = \langle \beta\mathbf{b} + \gamma\mathbf{c}, \mathbf{b}, \mathbf{c} \rangle = \beta\langle \mathbf{b}, \mathbf{b}, \mathbf{c} \rangle + \gamma\langle \mathbf{c}, \mathbf{b}, \mathbf{c} \rangle = 0$$

□

We will later generalize this statement to arbitrary dimensions, using the determinant, and also show that its converse holds. For the time being we note that we have obtained a useful practical way of checking if three vectors in three dimensions are linearly independent and, hence, form a basis. In short, if  $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle \neq 0$  then  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  form a basis of  $\mathbb{R}^3$ .

**Example 2.7:** Applications of the triple product

In Example 1.7 we have seen that the three vectors

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad (2.59)$$

form a basis of  $\mathbb{R}^3$ . We can now check this independently by computing the triple product of these vectors and using Claim 2.1. For the triple product we find

$$\det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} = 1. \quad (2.60)$$

Since this result is non-zero we conclude from Claim 2.1 that the three vectors are indeed linearly independent and, hence, form a basis of  $\mathbb{R}^3$ . Moreover, we have learned that the volume of the parallelepiped defined by  $\mathbf{v}_1$ ,  $\mathbf{v}_2$  and  $\mathbf{v}_3$  equals 1.

### 2.3 Some geometry, lines and planes

The methods developed in this section can be applied to a wide range of geometrical problems in three dimensions, involving objects such as lines and planes. These applications are in no way central to linear



algebra and are indeed part of another, related mathematical discipline called *affine geometry*. However, given their importance in physics we will briefly discuss some of these applications.

### 2.3.1 Affine space

The “arena” of affine geometry is *affine space*  $A_n$  defined by

$$A_n = \mathbb{R}^n = \left\{ P = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \mid p_i \in \mathbb{R} \right\}.$$

As a set this is the same as  $\mathbb{R}^n$ , however,  $A_n$  is simply considered as a space of points without a vector space structure. Vectors  $\mathbf{v} \in \mathbb{R}^n$  can act on points  $P$  in the affine space  $A_n$  by a *translation* defined as

$$P \mapsto P + \mathbf{v} = \begin{pmatrix} p_1 + v_1 \\ \vdots \\ p_n + v_n \end{pmatrix}$$

The unique vector<sup>1</sup> translating  $P = (p_1, \dots, p_n)^T \in A_n$  to  $Q = (q_1, \dots, q_n)^T \in A_n$  is denoted by  $\overrightarrow{PQ} := (q_1 - p_1, \dots, q_n - p_n)^T \in \mathbb{R}^n$ . It is easy to verify that

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR},$$

a property which is also intuitively apparent from Fig. 8. The distance  $d(P, Q)$  between points P and Q

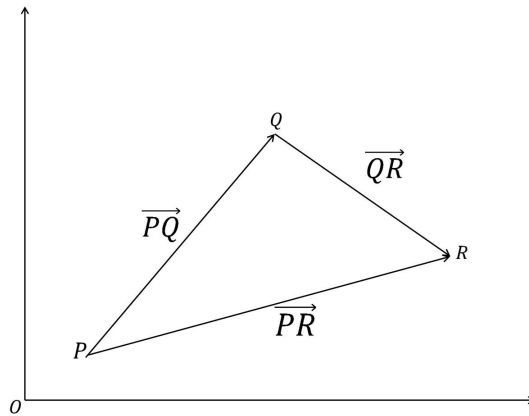


Figure 8: *Addition of translation vectors*

is defined as the length of the corresponding translation vector  $\overrightarrow{PQ}$ , so

$$d(P, Q) := |\overrightarrow{PQ}|.$$

We will frequently blur the distinction between the affine space  $A_n$  and the vector space  $\mathbb{R}^n$  and identify a point  $P \in A_n$  with the vector  $\overrightarrow{OP} \in \mathbb{R}^n$  translating the origin  $O$  to  $P$ .

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<sup>1</sup>For ease of notation we will sometimes write a column  $\mathbf{v}$  with components  $v_1, \dots, v_n$  as  $\mathbf{v} = (v_1, \dots, v_n)^T$ . The  $T$  superscript, short for “transpose”, indicates conversion of a row into a column. We will discuss transposition more systematically shortly.

### 2.3.2 Lines in $\mathbb{R}^2$

We begin by discussing *lines* in  $\mathbb{R}^2$ . In *Cartesian form* they can be described as all points  $(x, y) \in \mathbb{R}^2$  which satisfy the linear equation

$$y = ax + b \quad (2.61)$$

where  $a, b$  are fixed real numbers. Alternatively, a line can be described in *parametric vector form* as all vectors  $\mathbf{r}(t)$  given by

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \mathbf{p} + t\mathbf{q}, \quad t \in \mathbb{R}. \quad (2.62)$$

Here,  $\mathbf{p} = (p_x, p_y)^T$  and  $\mathbf{q} = (q_x, q_y)^T$  are fixed vectors and  $t$  is a parameter. The geometric interpretation of these various objects is apparent from Fig. 9.

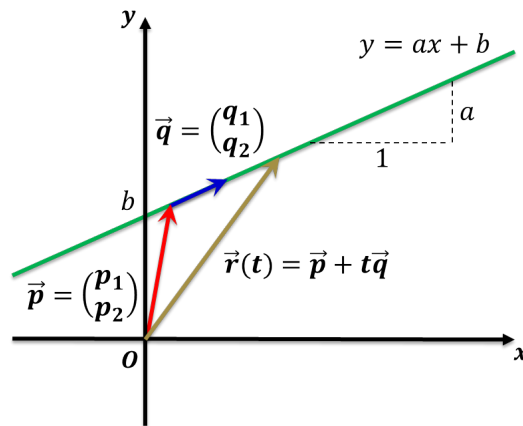


Figure 9: *Lines in two dimensions*

It is sometimes required to convert between those two descriptions of a line. To get from the Cartesian to the vector form simply use  $x$  as the parameter so that  $x(t) = t$  and  $y(t) = at + b$ . Combining these two equations into a vector equation gives

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ b \end{pmatrix}}_{\mathbf{p}} + t \underbrace{\begin{pmatrix} 1 \\ a \end{pmatrix}}_{\mathbf{q}} \quad (2.63)$$

where the vectors  $\mathbf{p}$  and  $\mathbf{q}$  are identified as indicated. For the opposite direction, to get from the vector to the Cartesian form, simply solve the two components of (2.62) for  $t$  so that

$$t = \frac{x - p_x}{q_x} = \frac{y - p_y}{q_y} \implies y = \underbrace{\frac{q_y}{q_x}}_a x + \underbrace{p_y - \frac{q_y}{q_x} p_x}_b \quad (2.64)$$

and  $a$  and  $b$  are identified as indicated.

**Example 2.8:** Conversion between Cartesian and vector form for two-dimensional lines

(a) Start with a line  $y = 2x - 3$  in Cartesian form. Setting  $x(t) = t$  and  $y(t) = 2t - 3$  the vector form of this line is given by

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t \\ 2t - 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \quad (2.65)$$

(b) Conversely, the line in vector form given by

$$\mathbf{r}(t) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (2.66)$$

can be split up into the two components  $x = 2 + t$  and  $y = -1 + 2t$ . Hence,  $t = x - 2$  and inserting this into the equation for  $y$  gives the Cartesian form  $y = 2x - 5$  of the line.

Finally, a common problem is to find the intersection of two lines given by  $\mathbf{r}_1(t_1) = \mathbf{p}_1 + t_1\mathbf{q}_1$  and  $\mathbf{r}_2(t_2) = \mathbf{p}_2 + t_2\mathbf{q}_2$ . Setting  $\mathbf{r}_1(t_1) = \mathbf{r}_2(t_2)$  leads to

$$t_1\mathbf{q}_1 - t_2\mathbf{q}_2 = \mathbf{p}_2 - \mathbf{p}_1. \quad (2.67)$$

If  $\mathbf{q}_1, \mathbf{q}_2$  are linearly independent then they form a basis of  $\mathbb{R}^2$  and, in this case, we know from Claim 1.2 that there is a unique solution  $t_1, t_2$  for this equation. The intersection point is obtained by computing  $\mathbf{r}_1(t_1)$  or  $\mathbf{r}_2(t_2)$  for these values. If  $\mathbf{q}_1, \mathbf{q}_2$  are linearly dependent then the lines are parallel and either there is no intersection or the two lines are identical.

**Example 2.9:** Intersection of two lines in two dimensions

We would like to determine the intersection point of the two lines

$$\mathbf{r}_1(t_1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \mathbf{r}_2(t_2) = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \quad (2.68)$$

Setting  $\mathbf{r}_1(t_1) = \mathbf{r}_2(t_2)$  leads to

$$t_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} - t_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \iff \begin{array}{l} -t_1 - t_2 = 2 \\ t_1 - 2t_2 = -2 \end{array}. \quad (2.69)$$

The unique solution is  $t_1 = -2$  and  $t_2 = 0$ , which are the parameter values of the intersection point. To obtain the intersection point  $\mathbf{r}_{\text{isec}}$  itself insert these parameter values into the equations for the lines which gives

$$\mathbf{r}_{\text{isec}} = \mathbf{r}_1(-2) = \mathbf{r}_2(0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}. \quad (2.70)$$

**2.3.3 Lines in  $\mathbb{R}^3$**

The vector form for 2-dimensional lines (2.62) can be easily generalized to three dimensions:

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \mathbf{p} + t\mathbf{q}. \quad (2.71)$$

Here  $\mathbf{p} = (p_x, p_y, p_z)^T$  and  $\mathbf{q} = (q_x, q_y, q_z)^T$  are fixed vectors. As before, we can get to the Cartesian form by solving the components of Eq. (2.71) for  $t$  resulting in

$$t = \frac{x - p_x}{q_x} = \frac{y - p_y}{q_y} = \frac{z - p_z}{q_z}. \quad (2.72)$$

Note that this amounts to two equations between the three coordinates  $x$ ,  $y$ ,  $z$  as should be expected for the definition of a one-dimensional object (the line) in three dimensions. The geometrical interpretation of the various vectors is indicated in Fig. 10.

**Example 2.10:** Conversion between Cartesian and vector form for lines in three dimensions

(a) We would like to convert the line in vector form given by

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (2.73)$$

into Cartesian form. Solving the three components for  $t$  immediately leads to the Cartesian form

$$t = \frac{x-2}{3} = -\frac{y+1}{5} = \frac{z-4}{2}. \quad (2.74)$$

(b) Conversely, given the Cartesian form

$$t = \frac{x-7}{2} = y+1 = \frac{z+3}{8} \quad (2.75)$$

we can solve for  $x$ ,  $y$  and  $z$  in terms of  $t$  which leads to  $x = 7 + 2t$ ,  $y = -1 + t$ ,  $z = -3 + 8t$ . Combining these three equations into one vector equation gives the vector form of the line

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ -3 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix}. \quad (2.76)$$

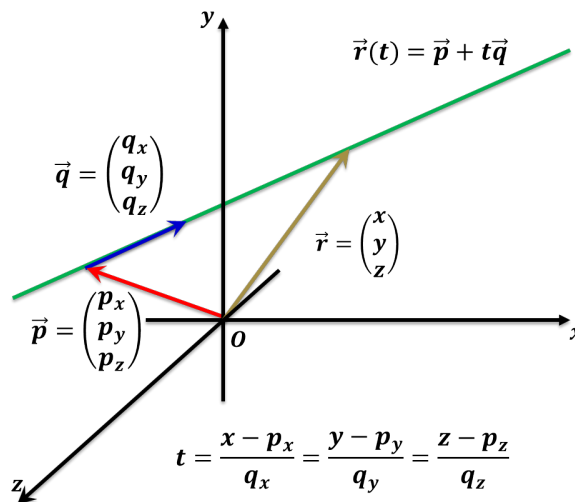


Figure 10: Lines in three dimensions

For the minimum distance of a line from a given point we have the following statement.

**Claim 2.2.** The minimum distance of a line  $\mathbf{r}(t) = \mathbf{p} + t\mathbf{q}$  from a point  $\mathbf{p}_0$  arises at  $t_{\min} = -\frac{\mathbf{d} \cdot \mathbf{q}}{|\mathbf{q}|^2}$  where  $\mathbf{d} = \mathbf{p} - \mathbf{p}_0$ . The minimal distance is given by  $d_{\min} = |\mathbf{d} \times \mathbf{q}|/|\mathbf{q}|$ .

*Proof.* We simply work out the distance square  $d^2(t) := |\mathbf{r}(t) - \mathbf{p}_0|^2$  of an arbitrary point  $\mathbf{r}(t)$  on the line from the point  $\mathbf{p}_0$  which leads to

$$d^2(t) = |\mathbf{d} + t\mathbf{q}|^2 = |\mathbf{q}|^2 t^2 + 2(\mathbf{d} \cdot \mathbf{q})t + |\mathbf{d}|^2 = \left[ |\mathbf{q}| t + \frac{\mathbf{d} \cdot \mathbf{q}}{|\mathbf{q}|} \right]^2 + |\mathbf{d}|^2 - \frac{(\mathbf{d} \cdot \mathbf{q})^2}{|\mathbf{q}|^2}. \quad (2.77)$$

This is minimal when the expression inside the square bracket vanishes which happens for  $t = t_{\min} = -\frac{\mathbf{d} \cdot \mathbf{q}}{|\mathbf{q}|^2}$ . This proves the first part of the claim. For the second part we simply compute

$$d_{\min}^2 := d^2(t_{\min}) = \frac{1}{|\mathbf{q}|^2} (|\mathbf{d}|^2 |\mathbf{q}|^2 - (\mathbf{d} \cdot \mathbf{q})^2) \stackrel{(2.38)}{=} \frac{|\mathbf{d} \times \mathbf{q}|^2}{|\mathbf{q}|^2}. \quad (2.78)$$

□

**Example 2.11:** Minimal distance of a line from a point

We would like to find the minimal distance of the line  $\mathbf{r}(t) = \mathbf{p} + t\mathbf{q}$  from the point  $\mathbf{p}_0$ , where

$$\mathbf{p} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}, \quad \mathbf{p}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad (2.79)$$

Using the notation from Claim 2.2 we have

$$\mathbf{d} = \mathbf{p} - \mathbf{p}_0 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad \mathbf{d} \times \mathbf{q} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 11 \\ 7 \\ 1 \end{pmatrix}. \quad (2.80)$$

Hence,  $|\mathbf{d} \times \mathbf{q}| = 3\sqrt{19}$  and  $|\mathbf{q}| = \sqrt{38}$  so that

$$d_{\min} = \frac{|\mathbf{d} \times \mathbf{q}|}{|\mathbf{q}|} = \frac{3}{\sqrt{2}}. \quad (2.81)$$

### 2.3.4 Planes in $\mathbb{R}^3$

To obtain the vector form of a plane in three dimensions we can generalize Eq. (2.71), the vector form of a 3-dimensional line, by introducing two parameters,  $t_1$  and  $t_2$  and define the plane as all points  $\mathbf{r}(t_1, t_2)$  given by

$$\mathbf{r}(t_1, t_2) = \begin{pmatrix} x(t_1, t_2) \\ y(t_1, t_2) \\ z(t_1, t_2) \end{pmatrix} = \mathbf{p} + t_1\mathbf{q} + t_2\mathbf{s}, \quad t_1, t_2 \in \mathbb{R}, \quad (2.82)$$

where  $\mathbf{p}$ ,  $\mathbf{q}$  and  $\mathbf{s}$  are fixed vectors in  $\mathbb{R}^3$ . Of course, for this to really define a plane (rather than a line) the vectors  $\mathbf{q}$  and  $\mathbf{s}$  must be linearly independent. A unit normal vector to this plane is given by

$$\mathbf{n} = \frac{\mathbf{q} \times \mathbf{s}}{|\mathbf{q} \times \mathbf{s}|}. \quad (2.83)$$

Multiplying the vector form (2.82) by  $\mathbf{n} = (n_x, n_y, n_z)^T$  (and remembering that  $\mathbf{n} \cdot \mathbf{q} = \mathbf{n} \cdot \mathbf{s} = 0$ ) we get to the Cartesian form of a three-dimensional plane

$$\mathbf{n} \cdot \mathbf{r} = d \quad \text{or} \quad n_x x + n_y y + n_z z = d, \quad (2.84)$$

where  $d = \mathbf{n} \cdot \mathbf{p}$  is a constant. From Eq. (2.11) we can re-write the Cartesian form as  $\cos(\theta)|\mathbf{r}| = d$ , where  $\theta = \angle(\mathbf{n}, \mathbf{r})$  is the angle between  $\mathbf{n}$  and  $\mathbf{r}$ . The distance  $|\mathbf{r}|$  of the plane from the origin is minimal if  $\cos(\theta) = \pm 1$  which shows that the constant  $d$  (or rather its absolute value  $|d|$ ) should be interpreted as the minimal distance of the plane from the origin. The geometrical meaning of the various objects is indicated in Fig. 11. Finally, to convert a plane in Cartesian form (2.84) into vector form we must first

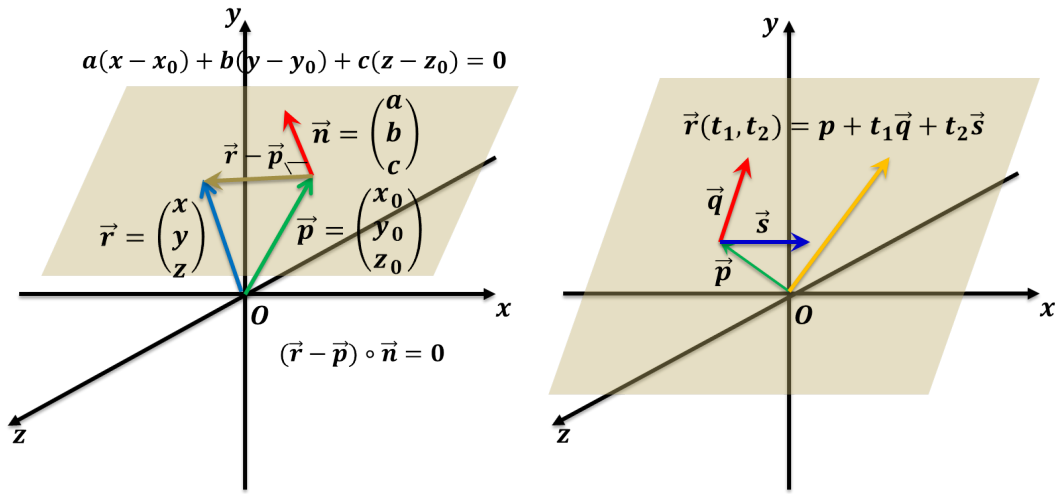


Figure 11: *Cartesian and vector descriptions of a plane*

find a vector  $\mathbf{p}$  with  $\mathbf{p} \cdot \mathbf{n} = d$  (a vector “to the plane”) and then two linearly independent vectors  $\mathbf{q}, \mathbf{r}$  satisfying  $\mathbf{q} \cdot \mathbf{n} = \mathbf{r} \cdot \mathbf{n} = 0$  (two vectors “in the plane”). These are then three suitable vectors to write down the vector form (2.82).

**Example 2.12:** Conversion between vector form and Cartesian form for a plane in three dimensions

(a) Start with a plane  $\mathbf{r}(t_1, t_2) = \mathbf{p} + t_1 \mathbf{q} + t_2 \mathbf{s}$  in vector form, where

$$\mathbf{p} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}. \quad (2.85)$$

To convert to Cartesian form, we first work out a normal vector,  $\mathbf{N}$ , to the plane, given by

$$\mathbf{N} = \mathbf{q} \times \mathbf{s} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (2.86)$$

Then, the Cartesian form reads  $\mathbf{N} \cdot \mathbf{r} = \mathbf{N} \cdot \mathbf{p}$  and with  $\mathbf{N} \cdot \mathbf{p} = -5$  this leads to  $x + y + z = 5$ .

(b) Conversely, start with the plane  $2x - 3y + z = 4$  in Cartesian form with normal vector  $\mathbf{N} = (2, -3, 1)^T$ . We need to find two linearly independent vectors “in the plane”, that is, vectors  $\mathbf{q}$  and  $\mathbf{r}$  satisfying  $\mathbf{N} \cdot \mathbf{q} = \mathbf{N} \cdot \mathbf{s} = 0$ . Obvious simple choices are  $\mathbf{q} = (1, 1, 1)^T$  and  $\mathbf{s} = (1, 0, -2)^T$ . Further, we need a vector

“in the plane”, that is, a vector  $\mathbf{p}$  satisfying  $\mathbf{N} \cdot \mathbf{p} = 4$  and we can choose, for example,  $\mathbf{p} = (2, 0, 0)^T$ . Combining these results the vector form of the plane reads

$$\mathbf{r}(t_1, t_2) = \mathbf{p} + t_1\mathbf{q} + t_2\mathbf{s} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}. \quad (2.87)$$

### 2.3.5 Intersection of line and plane

To study the intersection of a line and a plane first write down vector equations for each, so

$$\mathbf{r}_L(t) = \mathbf{a} + t\mathbf{b}, \quad \mathbf{r}_P(t_1, t_2) = \mathbf{p} + t_1\mathbf{q} + t_2\mathbf{s}. \quad (2.88)$$

Equating these two, that is  $\mathbf{r}_L(t) = \mathbf{r}_P(t_1, t_2)$ , leads to

$$t_1\mathbf{q} + t_2\mathbf{s} - t\mathbf{b} = \mathbf{a} - \mathbf{p}. \quad (2.89)$$

Let us assume that the triple product  $\langle \mathbf{b}, \mathbf{q}, \mathbf{s} \rangle$  is non-zero so that, from Claim (2.1), the vectors  $\mathbf{b}, \mathbf{q}, \mathbf{s}$  form a basis. In this case, Eq. (2.89) has a unique solution for  $t_1, t_2, t$  which corresponds to the parameter values of the intersection point. This solution can be found by splitting (2.89) up into its three component equations and explicitly solving for  $t_1, t_2, t$ . Perhaps a more elegant way to proceed is to multiply (2.89) by  $(\mathbf{q} \times \mathbf{s})$ , so that the terms proportional to  $t_1$  and  $t_2$  drop out. The resulting equation can easily be solved for  $t$  which leads to

$$t_{\text{isec}} = \frac{\langle \mathbf{p} - \mathbf{a}, \mathbf{q}, \mathbf{s} \rangle}{\langle \mathbf{b}, \mathbf{q}, \mathbf{s} \rangle}. \quad (2.90)$$

This is the value of the line parameter  $t$  at the intersection point and we obtain the intersection point itself by evaluating  $\mathbf{r}_L(t_{\text{isec}})$ .

#### Example 2.13: Intersection of line and plane in three dimensions

Consider a line  $\mathbf{r}_L(t) = \mathbf{a} + t\mathbf{b}$  and a plane  $\mathbf{r}_P(t_1, t_2) = \mathbf{p} + t_1\mathbf{q} + t_2\mathbf{s}$  in vector form with

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{p} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}. \quad (2.91)$$

By straightforward calculation we have  $\langle \mathbf{p} - \mathbf{a}, \mathbf{q}, \mathbf{s} \rangle = -4$  and  $\langle \mathbf{b}, \mathbf{q}, \mathbf{s} \rangle = -1$  so that, from Eq. (2.90), the value of the line parameter  $t$  at the intersection point is  $t_{\text{isec}} = 4$ . The intersection point,  $\mathbf{r}_{\text{isec}}$ , is then obtained by evaluating the equation for the line at  $t = t_{\text{isec}} = 4$ , so

$$\mathbf{r}_{\text{isec}} = \mathbf{r}_L(4) = \begin{pmatrix} 5 \\ -4 \\ -3 \end{pmatrix}. \quad (2.92)$$

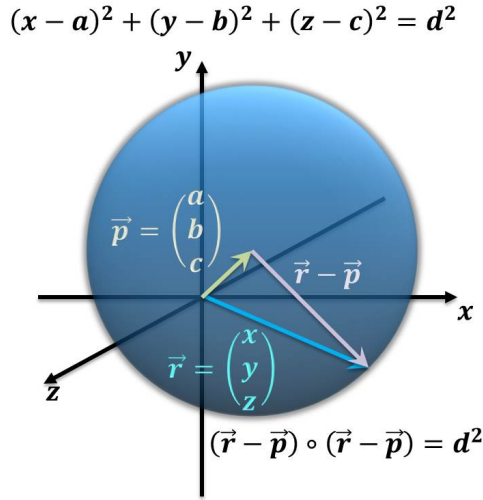


Figure 12: *Sphere in three dimensions*

### 2.3.6 Minimal distance of two lines

Two lines in three dimensions do not generically intersect but we can still ask about their minimal distance. We begin with the two lines

$$\mathbf{r}_i(t_i) = \mathbf{p}_i + t_i \mathbf{q}_i, \quad \text{where } i = 1, 2 \quad (2.93)$$

in vector form. One way to proceed would be in analogy with the proof of Claim 2.2, that is, by finding the values of  $t_1, t_2$  for which the distance  $|\mathbf{r}_1(t_1) - \mathbf{r}_2(t_2)|$  is minimal. However, this requires minimization of a function of two variables, a technique you may not yet be familiar with. Alternatively we can introduce the unit vector

$$\mathbf{n} = \frac{\mathbf{q}_1 \times \mathbf{q}_2}{|\mathbf{q}_1 \times \mathbf{q}_2|} \quad (2.94)$$

which is evidently perpendicular to both lines and start with the intuitive assertion that the vector of minimal length between the two lines is in the direction of  $\mathbf{n}$ . This leads to the relation

$$\mathbf{p}_1 + t_1 \mathbf{q}_1 - \mathbf{p}_2 - t_2 \mathbf{q}_2 = \pm d_{\min} \mathbf{n}, \quad (2.95)$$

where  $t_1, t_2$  are the parameter values of the points of minimal distance on the two lines and the sign on the RHS should be chosen so that  $d_{\min} \geq 0$  (since we would like a distance to be positive). By multiplying the last equation with  $\mathbf{n}$  it then follows easily that

$$d_{\min} = |(\mathbf{p}_1 - \mathbf{p}_2) \cdot \mathbf{n}| \quad (2.96)$$

**Example 2.14:** Minimal distance of two lines in three dimensions

We begin with two lines  $\mathbf{r}_1(t_1) = \mathbf{p}_1 + t_1 \mathbf{q}_1$  and  $\mathbf{r}_2(t_2) = \mathbf{p}_2 + t_2 \mathbf{q}_2$  in vector form, where

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{q}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{p}_2 = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \quad \mathbf{q}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}. \quad (2.97)$$

The vector (2.94) normal to both lines is then given by

$$\mathbf{q}_1 \times \mathbf{q}_2 = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}, \quad \mathbf{n} = \frac{\mathbf{q}_1 \times \mathbf{q}_2}{|\mathbf{q}_1 \times \mathbf{q}_2|} = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}. \quad (2.98)$$



From Eq. (2.96) this gives the minimal distance  $d_{\min} = |(\mathbf{p}_1 - \mathbf{p}_2) \cdot \mathbf{n}| = 11/3$  between the two lines.

### 2.3.7 Spheres in $\mathbb{R}^3$

A sphere in  $\mathbb{R}^3$  with radius  $\rho$  and center  $\mathbf{p} = (a, b, c)^T$  consists of all points  $\mathbf{r} = (x, y, z)^T$  with  $|\mathbf{r} - \mathbf{p}| = \rho$ . Written out in coordinates this reads explicitly

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = \rho^2 \quad (2.99)$$

In particular, setting  $a = b = c = 0$ , a sphere around the origin is described by  $x^2 + y^2 + z^2 = \rho^2$ . The RHS of this equation is a particular example of a *quadratic form*, polynomials which consist of terms quadratic in the variables. We will study quadratic forms in more detail later.

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#### Application: The perceptron - a basic building block of artificial neural networks

Artificial neural networks constitute an important set of methods in modern computing which are motivated by the structure of the human brain. Many of the operating principles of artificial neural networks can be formulated and understood in terms of linear algebra. Here, we would like to introduce one of the basic building blocks of artificial neural networks - the perceptron.

The structure of the perceptron is schematically illustrated in Fig. 13. It receives  $n$  real input values

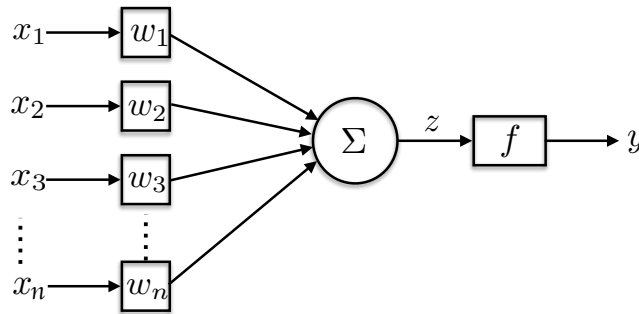


Figure 13: *Schematic representation of the perceptron*

$x_1, \dots, x_n$  which can be combined into an  $n$ -dimensional input vector  $\mathbf{x} = (x_1, \dots, x_n)^T$ . The internal state of the perceptron is determined by three pieces of data: the real values  $w_1, \dots, w_n$ , called the “weights”, which can be arranged into the  $n$ -dimensional weight vector  $\mathbf{w} = (w_1, \dots, w_n)^T$ , a real number  $\theta$ , called the “threshold” of the perceptron and a real function  $f$ , referred to as the “activation function”. In terms of this data, the perceptron computes the output values  $y$  from the input values  $\mathbf{x}$  as

$$z = \mathbf{w} \cdot \mathbf{x} - \theta, \quad y = f(z). \quad (2.100)$$

The activation function can, for example, be chosen as the sign function

$$f(z) = \begin{cases} +1 & \text{for } z \geq 0 \\ -1 & \text{for } z < 0 \end{cases}. \quad (2.101)$$

Given this set-up, the functioning of the perceptron can be phrased in geometrical terms. Consider the equation in  $\mathbb{R}^n$  with coordinates  $\mathbf{x}$  given by

$$\mathbf{w} \cdot \mathbf{x} = \theta. \quad (2.102)$$

Note that this is simply the equation of a plane (or a hyper-plane in dimensions  $n > 3$ ) in Cartesian form. If a point  $\mathbf{x} \in \mathbb{R}^n$  is “above” (or on) this plane, so that  $\mathbf{w} \cdot \mathbf{x} - \theta \geq 0$ , then, from Eqs. (2.100) and (2.101), the output of the perceptron is  $+1$ . On the other hand, for a point  $\mathbf{x} \in \mathbb{R}^n$  below this plane, so that  $\mathbf{w} \cdot \mathbf{x} - \theta < 0$ , the perceptron’s output is  $-1$ . In other words, the purpose of the perceptron is to “decide” whether a certain point  $\mathbf{x}$  is above or below a given plane.

So far this does not seem to hold much interest - all we have done is to re-formulate a sequence of simple mathematical operations related to the Cartesian form a plane, in a different language. The point is that the internal state of the perceptron, that is the choice of a plane specified by the weight vector  $\mathbf{w}$  and the threshold  $\theta$ , is not inserted “by hand” but rather determined by a learning process. This works as follows. Imagine a certain quantity,  $y$ , rapidly changes from  $-1$  to  $+1$  across a certain (hyper-) plane in  $\mathbb{R}^n$  whose location is not a priori known. Let us perform  $m$  measurements of  $y$ , giving measured values  $y^{(1)}, \dots, y^{(m)} \in \{-1, +1\}$  at locations  $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)} \in \mathbb{R}^n$ . These measurements can then be used to train the perceptron. Starting from random values  $\mathbf{w}^{(1)}$  and  $\theta^{(1)}$  of the weight vector and the threshold we can iteratively improve those values by carrying out the operations

$$\mathbf{w}^{(a+1)} = \mathbf{w}^{(a)} + \lambda(y^{(a)} - y) \mathbf{x}^{(a)} , \quad \theta^{(a+1)} = \theta^{(a)} - \lambda(y^{(a)} - y) . \quad (2.103)$$

Here,  $y$  is the output value produced by the perceptron given the input vector  $\mathbf{x}^{(a)}$  and  $\lambda$  is a real value, typically chosen in the interval  $[0, 1]$ , called the learning rate of the perceptron. Evidently, if the value  $y$  produced by the perceptron differs from the true, measured value  $y^{(a)}$ , the weight vector and the threshold of the perceptron are adjusted according to Eqs. (2.103). This training process continues until all measurements are used up and the final values  $\mathbf{w} = \mathbf{w}^{(m+1)}$  and  $\theta = \theta^{(m+1)}$  have been obtained. In this state the perceptron can then be used to “predict” the value of  $y$  for new input vectors  $\mathbf{x}$ . Essentially, the perceptron has “learned” about the location of the plane via the training process and is now able to decide whether a given point is located above or below.

In the context of artificial neural networks the perceptron corresponds to a single neuron. Proper artificial neural networks can be constructed by combining a number of perceptrons into a network, using the output of certain perceptrons within the network as input for others. Such networks of perceptrons correspond to collections of (hyper-) planes and are, for example, applied in the context of pattern recognition. The details of this are beyond the present scope but are not too difficult to understand by generalising the above discussion for a single perceptron.

### 3 Linear maps and matrices

We now return to the general story and consider arbitrary vector spaces. The next logical step is to study maps between vector spaces which are “consistent” with the vector space structure - they are called *linear maps*. As we will see, linear maps are closely related to matrices.

#### 3.1 Linear maps

Before we consider linear maps we need to collect a few basic facts for general maps between arbitrary sets. All these are elementary and considered part of basic mathematical literacy. They will help us to deal with linear maps - and distinguish general properties of maps from more specific ones of linear maps - but they are also foundational for other parts of mathematics.

##### 3.1.1 General maps between sets and their properties

You probably have an intuitive understanding of a map between two sets but to be clear let us start with the following definition.

**Definition 3.1.** A map between two sets  $X$  and  $Y$  assigns to each  $x \in X$  a  $y \in Y$  which is written as  $y = f(x)$  and referred to as the image of  $x$  under  $f$ . In symbols,

$$f : X \rightarrow Y, \quad x \mapsto f(x).$$

The set  $X$  is called domain of the map  $f$ ,  $Y$  is called the co-domain of  $f$ . The set

$$\text{Im}(f) = \{f(x) | x \in X\} \subseteq Y$$

is called the image of  $f$  and consists of all elements of the co-domain which can be obtained as images under  $f$ .

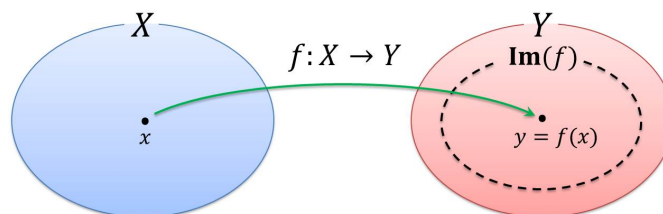


Figure 14: Visual representation of a map, with domain, co-domain, and image.

So a map assigns to each element of the domain an element of the co-domain. However, note that not all elements of the co-domain necessarily need to be obtained in this way. This is precisely what is encoded by the image,  $\text{Im}(f)$ , the set of elements in the co-domain which can be “reached” by  $f$ . If  $\text{Im}(f) = Y$  then all elements of the co-domain are obtained as images, otherwise some are not. Also note that, while each element of the domain is assigned to a unique element of the co-domain, two different elements of the domain may well have the same image. It is useful to formalize these observations by introducing the following definitions.

**Definition 3.2.** Let  $f : X \rightarrow Y$  be a map between two sets  $X$  and  $Y$ . The map  $f$  is called

(i) *one-to-one (or injective)* if every element of the co-domain is the image of at most one element of the domain. Equivalently,  $f$  is one-to-one iff  $f(x) = f(\tilde{x}) \implies x = \tilde{x}$  for all  $x, \tilde{x} \in X$ .

(ii) *onto (or surjective)* if every element of the co-domain is the image of at least one element of the domain. Equivalently,  $f$  is onto iff  $\text{Im}(f) = Y$ .

(iii) *bijective*, if it is injective and surjective, that is, if every element of the co-domain is the image of precisely one element of the domain.

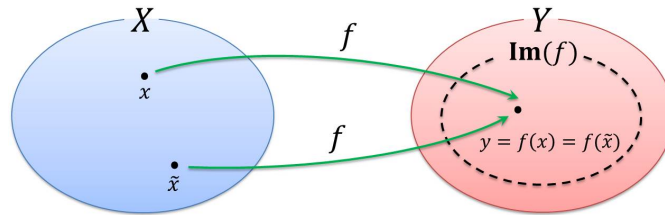


Figure 15: The above map is not one-to-one (injective), since  $f(x) = f(\tilde{x})$  but  $x \neq \tilde{x}$ .

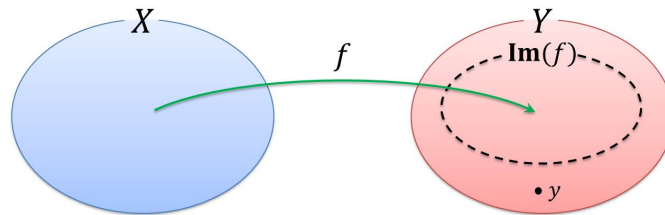


Figure 16: The above map is not onto (surjective), since  $\text{Im}(f) \neq Y$ .

**Example 3.1:** Some simple examples of maps

(a) The map  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = ax$ , for a non-zero real constant  $a$ , is bijective.

(b) The map  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = x^2$  is neither injective (since  $f(x) = f(-x)$ ) nor surjective (since  $f(x) \geq 0$  always). However, it can be made surjective if we choose the co-domain to be  $\mathbb{R}^{\geq 0}$  instead of  $\mathbb{R}$ . If we restrict both domain and co-domain to the positive numbers, so consider it as a map  $f : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ , it is bijective.

(c) The sin function  $g(x) = \sin x$  seen as a map  $g : \mathbb{R} \rightarrow \mathbb{R}$  is neither injective (since  $g(x) = g(x + 2\pi)$ ) nor surjective (since  $|g(x)| \leq 1$ ). However, restricted to  $g : [-\pi/2, \pi/2] \rightarrow [-1, 1]$  it is bijective.

Two maps can be applied one after the other provided the co-domain of the first map is the same as the domain of the second. This process is called *composition of maps* and is formally defined as follows.

**Definition 3.3.** For two maps  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  the compositive map  $g \circ f : X \rightarrow Z$  is defined by

$$(g \circ f)(x) := g(f(x)).$$

From this definition it is easy to show that map composition is associative, that is,  $h \circ (g \circ f) = (h \circ g) \circ f$ , since

$$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x))) = (h \circ g)(f(x)) = ((h \circ g) \circ f)(x). \quad (3.1)$$

**Example 3.2:** Map composition

For a simple example of map composition consider the maps in Example (3.1) (a), (c), that is  $f(x) = ax$  and  $g(x) = \sin(x)$ . Their composition is  $g \circ f(x) = \sin(ax)$  or, in the opposite order,  $f \circ g(x) = a \sin(x)$ . In other words, if the maps are given as explicit functions then the composition is obtained by “inserting one function into the other”. The example also shows that map composition is not commutative.

A trivial but useful map is the *identity map*  $\text{id}_X : X \rightarrow X$  which maps every element in  $X$  onto itself, that is,  $\text{id}_X(x) = x$  for all  $x \in X$ . It is required to define the important concept of *inverse map*.

**Definition 3.4.** Given a map  $f : X \rightarrow Y$ , a map  $g : Y \rightarrow X$  is called an *inverse of  $f$*  if

$$(g \circ f) = \text{id}_X \quad \text{and} \quad (f \circ g) = \text{id}_Y.$$

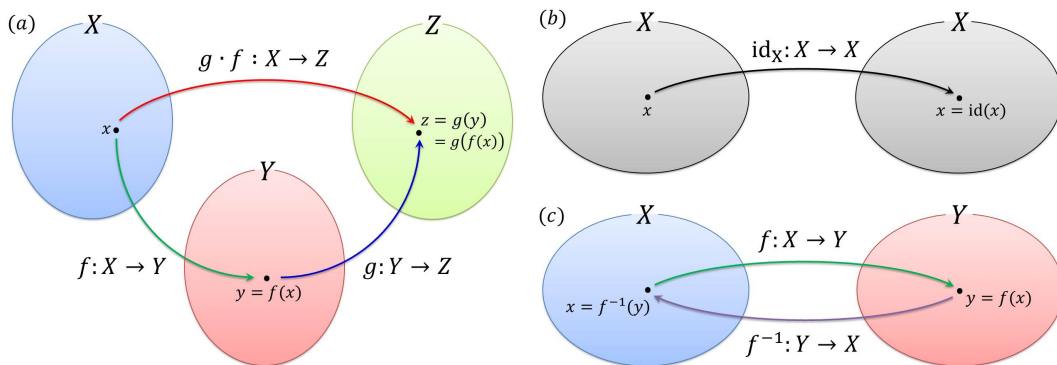


Figure 17: (a) The map  $g \circ f : X \rightarrow Z$  is the composition of maps  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ , (b) The identity map  $\text{id}_X : X \rightarrow X$ , (c) The map  $f^{-1} : Y \rightarrow X$  is the inverse map of  $f : X \rightarrow Y$ .

The inverse map “undoes” the effect of the original map and in order to construct such a map we need to assign to each  $y \in Y$  in the co-domain an  $x \in X$  in the domain such that  $y = f(x)$ . If the map is not surjective this is impossible for some  $y$  since they are not in the image of  $f$ . On the other hand, if the map is not injective some  $y \in Y$  are images of more than one element in the domain so that the required assignment is not unique. Finally, if the map is bijective every  $y \in Y$  is the image of precisely one  $x \in X$ , so we can attempt to define the inverse by setting  $g(y) = x$  for this unique  $x$ . Altogether this suggests the following

**Theorem 3.1.** The map  $f : X \rightarrow Y$  has an inverse if and only if  $f$  is bijective. If the inverse exists it is unique and denoted by  $f^{-1} : Y \rightarrow X$ .

*Proof.* “ $\Rightarrow$ ” We assume that  $f : X \rightarrow Y$  has an inverse  $g : Y \rightarrow X$  with  $f \circ g = \text{id}_Y$  and  $g \circ f = \text{id}_X$ . To show that  $f$  is injective start with  $f(x) = f(\tilde{x})$  and apply  $g$  from the left. It follows immediately that  $x = \tilde{x}$ . To show surjectivity of we need to find, for a given  $y \in Y$ , an  $x \in X$  such that  $f(x) = y$ . We can choose  $x = g(y)$  since  $f(x) = f \circ g(y) = \text{id}_Y(y) = y$ . In conclusion  $f$  is bijective.

“ $\Leftarrow$ ” We assume that  $f$  is bijective. Hence, for every  $y \in Y$  there is precisely one  $x \in X$  with  $f(x) = y$ .

We define the prospective inverse map by  $g(y) = x$ . Then  $f \circ g(y) = f(x) = y$  and  $g \circ f(x) = g(y) = x$ .

To show uniqueness we consider two maps  $g : Y \rightarrow X$  and  $\tilde{g} : Y \rightarrow X$  with  $g \circ f(x) = x = \tilde{g} \circ f(x)$ . Setting  $y = f(x)$  it follows that  $g(y) = \tilde{g}(y)$  and, since  $f$  is surjective this holds for all  $y \in Y$ . Hence,  $g = \tilde{g}$ .  $\square$

If the maps  $f$  and  $g$  are both bijective, then it is easy to see that the composite map  $f \circ g$  is also bijective and, hence, has an inverse. This inverse of the composite map can be computed from the formula

$$(f \cdot g)^{-1} = g^{-1} \cdot f^{-1} . \quad (3.2)$$

Note the change of order on the RHS of this equation. This relation follows from  $(f \circ g)^{-1} \circ (f \circ g) = \text{id}$  and  $(g^{-1} \circ f^{-1}) \circ (f \circ g) = \text{id}$  which implies that both  $(f \circ g)^{-1}$  and  $(g^{-1} \circ f^{-1})$  provide an inverse for  $f \circ g$ . Uniqueness of the inverse function then leads to Eq. (3.2). Further we have

$$(f^{-1})^{-1} = f \quad (3.3)$$

from the uniqueness of the inverse and the fact that both  $f$  and  $(f^{-1})^{-1}$  provide an inverse for  $f^{-1}$ .

### Example 3.3: Inverse function

(a) In Example 3.1 (a), we have seen that the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = ax$ , where  $a \in \mathbb{R}$  is bijective for  $a \neq 0$ . Hence, there is an inverse function, which is explicitly given by  $f^{-1}(x) = x/a$ .

(b) In Example 3.1 (a), we have considered the function  $g(x) = \sin(x)$  which was neither surjective nor injective as a function  $g : \mathbb{R} \rightarrow \mathbb{R}$ . However, with domain and co-domain restricted as  $g : [-\pi/2, \pi/2] \rightarrow [-1, 1]$  it is bijective and the inverse function is  $g^{-1}(x) = \arcsin(x)$ .

### 3.1.2 Linear maps: Definition and basic properties

We are now ready to discuss *linear maps* which are maps between two vector spaces - rather than general sets as above - which, in addition, are consistent with the two vector space operations, that is, vector addition and scalar multiplication. More precisely what we mean is:

**Definition 3.5.** A map  $f : V \rightarrow W$  between two vector spaces  $V$  and  $W$  over a field  $F$  is called linear if

$$(L1) f(\mathbf{v}_1 + \mathbf{v}_2) = f(\mathbf{v}_1) + f(\mathbf{v}_2)$$

$$(L2) f(\alpha \mathbf{v}) = \alpha f(\mathbf{v})$$

for all  $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in V$  and for all  $\alpha \in F$ . Further, the set  $\text{Ker}(f) = \{\mathbf{v} \in V \mid f(\mathbf{v}) = \mathbf{0}\} \subset V$  is called the kernel of  $f$ .

A few remarks are in order. First, note that the two conditions for linearity are the obvious ones for consistency with the vector space structure. Condition (L1) says that vector addition can be “pulled through” linear maps and condition (L2) is a similar statement for scalar multiplication, that is, scalars can be “pulled out” of linear maps. In short, linear maps are the maps between vector spaces which are consistent with vector addition and scalar multiplication. As for any map, we can define the image  $\text{Im}(f) = \{f(\mathbf{v}) \mid \mathbf{v} \in V\} \subset W$  of the linear map  $f : V \rightarrow W$ , which is a subset of the co-domain vector space  $W$ . Since vector spaces have a special element - the zero vector  $\mathbf{0}$  - we can define a further set associated to a linear map, namely the kernel,  $\text{Ker}(f)$ . It is the subset of the domain vector space  $V$  which consists of all vectors  $\mathbf{v} \in V$  mapped to the zero vector, so  $f(\mathbf{v}) = \mathbf{0}$ .

Following the standard mathematical approach we have defined linear maps by their properties rather than by “what they are”. As we proceed we will gain some insight into the structure of linear maps and also discuss many examples. We begin by summarizing a number of simple but important properties of linear maps which follow from the above definition.

**Lemma 3.1.** (Properties of linear maps) A linear map  $f : V \rightarrow W$  between two vector spaces  $V$  and  $W$  over  $F$  has the following properties:

- (i) The zero vectors are mapped into each other, so  $f(\mathbf{0}) = \mathbf{0}$ . Hence  $\mathbf{0} \in \text{Ker } f$ .
- (ii) The kernel of  $f$  is a sub vector space of  $V$ .
- (iii) The image of  $f$  is a sub vector space of  $W$ .
- (iv)  $f$  surjective  $\Leftrightarrow \text{Im}(f) = W \Leftrightarrow \dim \text{Im}(f) = \dim W$
- (v)  $f$  injective  $\Leftrightarrow \text{Ker}(f) = \{\mathbf{0}\} \Leftrightarrow \dim \text{Ker}(f) = 0$
- (vi) The scalar multiple  $\alpha f$ , where  $\alpha \in F$ , is linear.
- (vii) For another linear map  $g : V \rightarrow W$ , the sum  $f + g$  is linear.
- (viii) For another linear map  $g : W \rightarrow U$ , the composition  $g \circ f$  is linear.

*Proof.* (i)  $f(\mathbf{0}) = f(0\mathbf{0}) \stackrel{(L2)}{=} 0f(\mathbf{0}) = \mathbf{0}$ .

(ii) We need to check the two conditions in Def. 1.2. If  $\mathbf{v}_1, \mathbf{v}_2 \in \text{Ker}(f)$  then, by definition of the kernel,  $f(\mathbf{v}_1) = f(\mathbf{v}_2) = \mathbf{0}$ . It follows that  $f(\mathbf{v}_1 + \mathbf{v}_2) = f(\mathbf{v}_1) + f(\mathbf{v}_2) = \mathbf{0}$  so that  $\mathbf{v}_1 + \mathbf{v}_2 \in \text{Ker}(f)$ . Similarly, if  $\mathbf{v} \in \text{Ker}(f)$ , so that  $f(\mathbf{v}) = \mathbf{0}$  it follows that  $f(\alpha\mathbf{v}) = \alpha f(\mathbf{v}) = \mathbf{0}$ , hence,  $\alpha\mathbf{v} \in \text{Ker}(f)$ .

(iii) This is very similar to the proof in (ii) and we leave it as an exercise.

(iv) The first part,  $f$  surjective  $\Leftrightarrow \text{Im}(f) = W$ , is clear by the definition of surjective and the image. Clearly, if  $\text{Im}(f) = W$ , then both spaces have the same dimension. Conversely, from Lemma 1.2, two vector spaces with the same dimension and one contained in the other (here  $\text{Im}(f) \subset W$ ) must be identical.

(v) Suppose  $f$  is injective and consider a vector  $\mathbf{v} \in \text{Ker } f$ . Then  $f(\mathbf{v}) = f(\mathbf{0}) = \mathbf{0}$ , which implies that  $\mathbf{v} = \mathbf{0}$  and, hence,  $\text{Ker}(f) = \{\mathbf{0}\}$ . Conversely, assume that  $\text{Ker}(f) = \{\mathbf{0}\}$ . Then, from linearity,  $f(\mathbf{v}_1) = f(\mathbf{v}_2)$  implies that  $f(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}$  so that  $\mathbf{v}_1 - \mathbf{v}_2 \in \text{Ker}(f) = \{\mathbf{0}\}$ . Hence,  $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$  and  $f$  is injective.

(vi) Simply check the linearity conditions for  $\alpha f$ , for example  $(\alpha f)(\mathbf{v}_1 + \mathbf{v}_2) = \alpha f(\mathbf{v}_1 + \mathbf{v}_2) = \alpha(f(\mathbf{v}_1) + f(\mathbf{v}_2)) = \alpha f(\mathbf{v}_1) + \alpha f(\mathbf{v}_2) = (\alpha f)(\mathbf{v}_1) + (\alpha f)(\mathbf{v}_2)$ .

(vii) Check the linearity conditions for  $f + g$ , similar to what has been done in (vi).

(viii) Simply check the linearity conditions for  $g \circ f$ , given that they are satisfied for  $f$  and  $g$ .  $g \circ f(\mathbf{v} + \mathbf{w}) = g(f(\mathbf{v} + \mathbf{w})) = g(f(\mathbf{v}) + f(\mathbf{w})) = g(f(\mathbf{v})) + g(f(\mathbf{w})) = g \circ f(\mathbf{v}) + g \circ f(\mathbf{w})$  and  $g \circ f(\alpha\mathbf{v}) = g(f(\alpha\mathbf{v})) = g(\alpha f(\mathbf{v})) = \alpha g(f(\mathbf{v})) = \alpha g \circ f(\mathbf{v})$ .  $\square$

The above Lemma contains a number of extremely useful statements. First of all, both the image and the kernel of a linear map are sub vector spaces - as one would hope for maps designed to be consistent with the vector space structure. This means we can assign dimensions to both of them. In fact, the dimension of the image is of particular importance and is given a special name.

**Definition 3.6.** The dimension of the image of a linear map  $f$  is called the rank of  $f$ , in symbols  $\text{rk}(f) := \dim \text{Im}(f)$ .

It might be difficult to check if a map is surjective or injective, using the original definitions of these properties. For linear maps, Lemma (3.1) gives simple criteria for both properties in terms of the dimensions of image and kernel.

We have seen earlier that spaces of functions, if appropriately restricted, form vector spaces. The same is in fact true for linear maps, a fact which will become important later when we discuss dual vector spaces and which we formulate in the following

**Lemma 3.2.** The set of all linear maps  $f : V \rightarrow W$  forms a vector space, also denoted  $\text{Hom}(V, W)$ , (“homomorphisms from  $V$  to  $W$ ”). Vector addition and scalar multiplication are defined by  $(f + g)(\mathbf{v}) = f(\mathbf{v}) + g(\mathbf{v})$  and  $(\alpha f)(\mathbf{v}) = \alpha f(\mathbf{v})$ .

*Proof.* From Lemma 3.1, (vi), (vii) the scalar multiple of a linear map and the sum of two linear maps is again a linear map. From Def. 1.2 this shows that the set of linear maps does indeed form a (sub) vector space.  $\square$

To get a better intuitive feel for the action of linear maps we recall our interpretation of sub vector spaces as lines, planes and their higher-dimensional analogues through  $\mathbf{0}$ . We should think of both the kernel and the image of a linear map in this way, the former residing in the domain vector space, the latter in the co-domain.

To be concrete, let us consider a linear map  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  and let us assume that  $\dim \text{Ker}(f) = 2$ , that is, the kernel of  $f$  is a plane through the origin in  $\mathbb{R}^3$ . This situation is schematically shown in Fig. 18. Recall that all vectors in the kernel of  $f$ , that is all vectors in the corresponding plane (the blue plane in Fig. 18) are mapped to the zero vector. What is more, consider two vectors  $\mathbf{v}_1, \mathbf{v}_2 \in \text{Ker}(f) + \mathbf{k}$  which both lie in a plane parallel to  $\text{Ker}(f)$ , shifted by a vector  $\mathbf{k}$  (the pink plane in Fig. 18). Then we have  $\mathbf{v}_1 - \mathbf{v}_2 \in \text{Ker}(f)$  so that  $f(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}$  and, hence, by linearity  $f(\mathbf{v}_1) = f(\mathbf{v}_2)$ . Therefore, not only do all vectors in the kernel get mapped to the zero vector, but all vectors in a plane parallel to the kernel get mapped to the same (although non-zero) vector. Effectively, the action of the linear map “removes” the two dimensions parallel to the kernel plane and only keeps the remaining dimension perpendicular to it. Hence, the image of this linear map is one-dimensional, that is a line through the origin, as indicated in Fig. 18. This structure is indeed general as expressed by the following theorem.

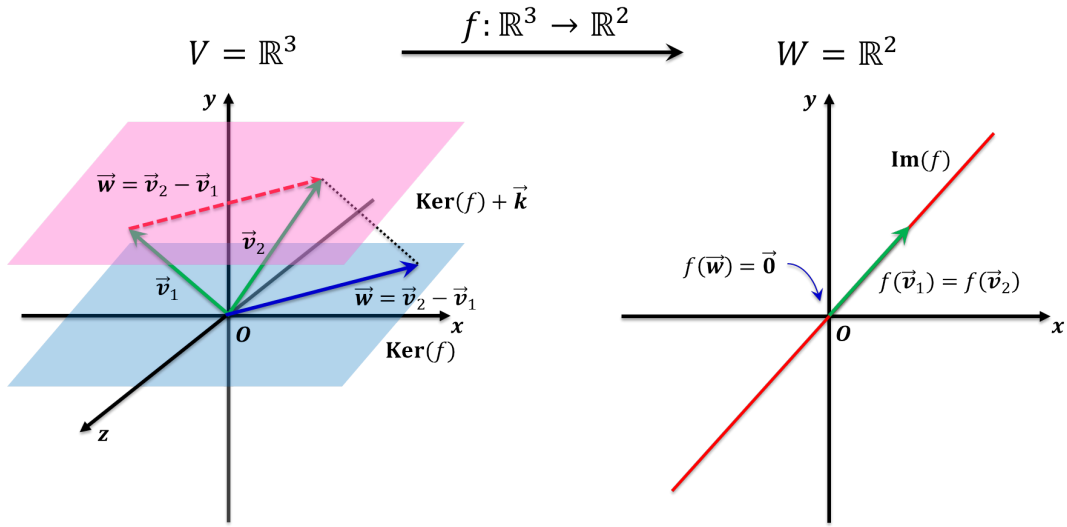


Figure 18: *Geometric representation of a linear map  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ . If  $\dim(V) = 3$  and  $\dim \text{Ker}(f) = 2$  it follows that  $\dim \text{Im}(f) = 1$ .*

**Theorem 3.2.** *For a linear map  $f : V \rightarrow W$  we have*

$$\dim \text{Ker}(f) + \text{rk}(f) = \dim(V) \tag{3.4}$$

*Proof.* For simplicity of notation, set  $k = \dim \text{Ker}(f)$  and  $n = \dim(V)$ . Let  $\mathbf{v}_1, \dots, \mathbf{v}_k$  be a basis of  $\text{Ker}(f)$  which we complete to a basis  $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n$  of  $V$ . We will show that  $f(\mathbf{v}_{k+1}), \dots, f(\mathbf{v}_n)$  forms a basis of  $\text{Im}(f)$ . To do this we need to check the two conditions in Definition 1.4.

(B1) First we need to show that  $\text{Im}(f)$  is spanned by  $f(\mathbf{v}_{k+1}), \dots, f(\mathbf{v}_n)$ . We begin with an arbitrary



vector  $\mathbf{w} \in \text{Im}(f)$ . This vector must be the image of a  $\mathbf{v} \in V$ , so that  $\mathbf{w} = f(\mathbf{v})$ . We can expand  $\mathbf{v}$  as a linear combination

$$\mathbf{v} = \sum_{i=1}^n \alpha_i \mathbf{v}_i$$

of the basis in  $V$ . Acting on this equation with  $f$  and using linearity we find

$$\mathbf{w} = f(\mathbf{v}) = f\left(\sum_{i=1}^n \alpha_i \mathbf{v}_i\right) = \sum_{i=1}^n \alpha_i f(\mathbf{v}_i) = \sum_{i=k+1}^n \alpha_i f(\mathbf{v}_i),$$

Hence, we have written  $w$  as a linear combination of the vectors  $f(\mathbf{v}_{k+1}), \dots, f(\mathbf{v}_n)$  which, therefore, span the image of  $f$ .

(B2) For the second step, we have to show that the vectors  $f(\mathbf{v}_{k+1}), \dots, f(\mathbf{v}_n)$  are linearly independent. As usual, we start with the equation

$$\sum_{i=k+1}^n \alpha_i f(\mathbf{v}_i) = \mathbf{0} \quad \Rightarrow \quad f\left(\sum_{i=k+1}^n \alpha_i \mathbf{v}_i\right) = \mathbf{0}.$$

The second of these equations means that the vector  $\sum_{i=k+1}^n \alpha_i \mathbf{v}_i$  is in the kernel of  $f$  and, given that  $\mathbf{v}_1, \dots, \mathbf{v}_k$  form a basis of the kernel, there are coefficients  $\alpha_1, \dots, \alpha_k$  such that

$$\sum_{i=k+1}^n \alpha_i \mathbf{v}_i = -\sum_{i=1}^k \alpha_i \mathbf{v}_i \quad \Rightarrow \quad \sum_{i=1}^n \alpha_i \mathbf{v}_i = \mathbf{0}.$$

Since  $\mathbf{v}_1, \dots, \mathbf{v}_n$  form a basis of  $V$  it follows that all  $\alpha_i = 0$  and, hence,  $f(\mathbf{v}_{k+1}), \dots, f(\mathbf{v}_n)$  are linearly independent.

Altogether, it follows that  $f(\mathbf{v}_{k+1}), \dots, f(\mathbf{v}_n)$  form a basis of  $\text{Im}(f)$ . Hence, by counting the number of basis elements  $\dim \text{Im}(f) = n - k = \dim(V) - \dim \text{Ker}(f)$ .  $\square$

The dimensional formula (3.4) is a profound statement about linear maps and it will be immensely helpful to understand the solution structure of systems of linear equations. For now we draw a few easy conclusions:

**Claim 3.1.** For a linear map  $f : V \rightarrow W$  we have:

- (i)  $f$  bijective (has an inverse) implies that  $\dim(V) = \dim(W)$ .
- (ii) For  $\dim(V) = \dim(W) = n$  the following conditions are equivalent:  
 $f$  is bijective (has an inverse)  $\iff \dim \text{Ker}(f) = 0 \iff \text{rk}(f) = n$
- (iii) If  $f$  is invertible then the inverse  $f^{-1} : W \rightarrow V$  is also a linear map.

*Proof.* (i) If  $f$  is bijective, it is injective and surjective, so from Lemma 3.1 (v),(vi)  $\dim \text{Ker}(f) = 0$  and  $\dim \text{Im}(f) = \dim(W)$ . Then, from Eq. (3.4),  $\dim(V) = \dim \text{Ker}(f) + \dim \text{Im}(f) = 0 + \dim(W) = \dim(W)$ .

(ii) This is an easy consequence of Theorem 3.2 and Lemma 3.1 (v), (vi) and we leave the proof as an exercise.

(iii) We set  $\mathbf{w}_1 = f(\mathbf{v}_1)$ ,  $\mathbf{w}_2 = f(\mathbf{v}_2)$  and  $\mathbf{w} = f(\mathbf{v})$  and check the linearity conditions in Def. 3.5 for  $f^{-1}$ .

$$\begin{aligned} f^{-1}(\mathbf{w}_1 + \mathbf{w}_2) &= f^{-1}(f(\mathbf{v}_1) + f(\mathbf{v}_2)) = f^{-1}(f(\mathbf{v}_1 + \mathbf{v}_2)) = \mathbf{v}_1 + \mathbf{v}_2 = f^{-1}(\mathbf{w}_1) + f^{-1}(\mathbf{w}_2) \\ f^{-1}(\alpha \mathbf{w}) &= f^{-1}(\alpha f(\mathbf{v})) = f^{-1}(f(\alpha \mathbf{v})) = \alpha \mathbf{v} = \alpha f^{-1}(\mathbf{w}) \end{aligned}$$

$\square$

Part (i) of the above claim says we can have invertible linear maps only between vector spaces of the same dimension. If the dimensions are indeed the same, part (ii) tells us the map is invertible precisely if its rank is maximal, that is, if the rank is equal to the dimension of the space.

**Example 3.4:** Examples of linear maps

(a) Linear maps  $f : F^n \rightarrow F^m$  from matrices

Consider an  $n$ -dimensional column vector  $\mathbf{v} \in F^n$  with components  $v_i$  and an  $m \times n$  matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad (3.5)$$

with entries  $a_{ij} \in F$ . We denote the column vector which consists of the entries in the  $i^{\text{th}}$  row of  $A$  by  $\mathbf{a}_i$ . To map  $n$ -dimensional into  $m$ -dimensional column vectors, we need to provide  $m$  functions each of which can, in general, depend on all  $n$  coordinates  $v_i$ . To obtain a linear map it seems intuitive that we should choose these functions linear in the coordinates  $v_i$  and the most general such possibility can be written down using the coefficients of the above matrix  $A$ . It is given by

$$f(\mathbf{v}) = \begin{pmatrix} a_{11}v_1 + \cdots + a_{1n}v_n \\ \vdots \\ a_{m1}v_1 + \cdots + a_{mn}v_n \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{v} \\ \vdots \\ \mathbf{a}_m \cdot \mathbf{v} \end{pmatrix} =: A\mathbf{v}. \quad (3.6)$$

By the last equality, we have defined the multiplication  $A\mathbf{v}$  of an  $m \times n$  matrix  $A$  with an  $n$ -dimensional vector  $\mathbf{v}$ . By definition, this multiplication is carried out by forming the dot product between the vector and the rows of the matrix, as indicated above<sup>2</sup>. Evidently, this only makes sense if “sizes fit”, that is, if the number of components of the vector equals the number of columns of the matrix. The outcome is a column vector whose dimension equals the number of rows of the matrix. Using index notation, multiplication of vectors by matrices and the above linear map can more concisely be written as

$$(f(\mathbf{v}))_i = \sum_{j=1}^n a_{ij}v_j = a_{ij}v_j, \quad (3.7)$$

where a sum over  $j$  is implied by the Einstein summation convention in the last expression. Using this notation it is quite straightforward to check that  $f$  satisfies the conditions for a linear map in Definition 3.5.

$$f(\mathbf{v} + \mathbf{w})_i = a_{ij}(v_j + w_j) = a_{ij}v_j + a_{ij}w_j = f(\mathbf{v})_i + f(\mathbf{w})_i \quad (3.8)$$

$$f(\alpha\mathbf{v})_i = a_{ij}(\alpha v_j) = \alpha(a_{ij}v_j) = \alpha f(\mathbf{v})_i \quad (3.9)$$

We conclude that Eq. (3.6) indeed defines a linear map and that, via the multiplication of matrices with vectors, we can define such a map for each matrix  $A$ . In short, multiplication of  $n$ -dimensional column vectors by a  $m \times n$  matrix  $A$  corresponds to a linear map  $f : F^n \rightarrow F^m$ . Conversely, we can ask if every linear map between column vectors can be obtained from a matrix in this way. We will return to this question shortly and see that the answer is “yes”.

Since this is the first time we encounter the multiplication of matrices and vectors an explicit numerical

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<sup>2</sup>Strictly, we have called this expression dot product only for the real case,  $F = \mathbb{R}$ . For the present purpose we mean by dot product any expression  $\mathbf{a} \cdot \mathbf{v} = \sum_i a_i v_i$ , where  $a_i, v_i \in F$  and  $F$  is an arbitrary field.

example might be helpful. Consider, the  $4 \times 3$  matrix and the three-dimensional vector

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ -2 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}. \quad (3.10)$$

Their product is obtained by dotting  $\mathbf{v}$  into the rows of  $A$ , so

$$A\mathbf{v} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ -2 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 9 \\ -1 \\ 12 \end{pmatrix} =: \mathbf{w}, \quad (3.11)$$

resulting in the four-dimensional vector  $\mathbf{w}$ . Stated another way, we can view this matrix as a linear map  $\mathbb{R}^3 \rightarrow \mathbb{R}^4$  and we have just explicitly computed the image  $\mathbf{w}$  of the vector  $\mathbf{v}$  under this linear map.

### (b) Coordinate maps

We have seen earlier that a vector can be uniquely described by its coordinates relative to a basis, see Claim 1.2. We can now use the notion of linear maps to make this more precise. Consider an  $n$ -dimensional vector space  $V$  over  $F$  with basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$ . By  $\boldsymbol{\alpha}, \boldsymbol{\beta} \in F^n$  we denote  $n$ -dimensional column vectors with components  $\alpha_i$  and  $\beta_i$ , respectively, and we define the coordinate map  $\varphi : F^n \rightarrow V$  by

$$\varphi(\boldsymbol{\alpha}) = \sum_{i=1}^n \alpha_i \mathbf{v}_i. \quad (3.12)$$

This map assigns to a coordinate vector the corresponding vector, relative to the given basis. It is easy to verify that it is linear.

$$\varphi(\boldsymbol{\alpha} + \boldsymbol{\beta}) = \sum_{i=1}^n (\alpha_i + \beta_i) \mathbf{v}_i = \sum_{i=1}^n \alpha_i \mathbf{v}_i + \sum_{i=1}^n \beta_i \mathbf{v}_i = \varphi(\boldsymbol{\alpha}) + \varphi(\boldsymbol{\beta}) \quad (3.13)$$

$$\varphi(a\boldsymbol{\alpha}) = \sum_{i=1}^n (a\alpha_i) \mathbf{v}_i = a \sum_{i=1}^n \alpha_i \mathbf{v}_i = a\varphi(\boldsymbol{\alpha}). \quad (3.14)$$

Since  $\mathbf{v}_1, \dots, \mathbf{v}_n$  forms a basis it is clear that  $\text{Im}(\varphi) = V$  and, hence, from Claim 3.1 (ii),  $\varphi$  is bijective and has an inverse  $\varphi^{-1} : V \rightarrow F^n$ . Clearly, the inverse map assigns to a vector  $\mathbf{v} = \sum_{i=1}^n \alpha_i \mathbf{v}_i \in V$  its coordinate vector  $\boldsymbol{\alpha}$ , so explicitly

$$\varphi^{-1}(\mathbf{v}) = \varphi^{-1} \left( \sum_{i=1}^n \alpha_i \mathbf{v}_i \right) = \boldsymbol{\alpha} \quad (3.15)$$

A linear and bijective map between two vector spaces is also referred to as a (vector space) *isomorphism* and two vector spaces related by such a map are called *isomorphic*. What the above discussion shows is that every  $n$ -dimensional vector space  $V$  over  $F$  is isomorphic to  $F^n$  by means of a coordinate map  $\varphi$ . However, it should be noted that this isomorphism is not unique as it depends on the choice of basis.

For an explicit example of a coordinate map consider Example 1.7, where we have considered  $\mathbb{R}^3$  with basis

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}. \quad (3.16)$$

The coordinate map for this basis is given by

$$\varphi(\boldsymbol{\alpha}) = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = \begin{pmatrix} \alpha_3 \\ \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_1 + 2\alpha_2 - \alpha_3 \end{pmatrix}. \quad (3.17)$$

(c) Linear differential operators

A linear differential operator of order  $n$  has the form

$$L = \sum_{k=0}^n p_k(x) \frac{d^k}{dx^k}$$

where  $p_k(x)$  are fixed real-valued functions of  $x \in \mathbb{R}$ . If we denote by  $V$  the vector space of infinitely many times differentiable functions  $g : \mathbb{R} \rightarrow \mathbb{R}$  then we can view this differential operator as a map  $L : V \rightarrow V$ . Since single differentiation and multiplication with fixed functions are each linear operations and  $L$  is a composition of such operations it is clear that  $L$  is a linear map. We can also verify this explicitly:

$$L(g_1 + g_2) = \sum_{k=0}^n p_k(x) \frac{d^k}{dx^k} (g_1 + g_2) = \sum_{k=0}^n p_k(x) \frac{d^k g_1}{dx^k} + \sum_{k=0}^n p_k(x) \frac{d^k g_2}{dx^k} = L(g_1) + L(g_2) \quad (3.18)$$

$$L(\alpha g) = \sum_{k=0}^n p_k(x) \frac{d^k}{dx^k} (\alpha g) = \alpha \sum_{k=0}^n p_k(x) \frac{d^k g}{dx^k} = \alpha L(g). \quad (3.19)$$

The solutions to the homogeneous differential equation

$$L(g) = 0 \quad (3.20)$$

are given by the kernel,  $\text{Ker}(L)$ . For any linear map the kernel is a (sub) vector space and, for the present example, this means that any linear combination of solutions of the differential equation is also a solution.

As an explicit example consider the second order linear differential operator

$$L = \frac{d^2}{dx^2} + 4 \frac{d}{dx} - 5. \quad (3.21)$$

The associated homogeneous differential equation  $L(g) = 0$  has the two solutions,  $g_1(x) = \exp(x)$  and  $g_2(x) = \exp(-5x)$ , but, from linearity, any linear combination  $g(x) = \alpha g_1(x) + \beta g_2(x) = \alpha \exp(x) + \beta \exp(-5x)$  is also a solution.

### 3.2 Matrices and their properties

In the previous section we have introduced linear maps and we have seen that a prominent class of examples for such maps can be obtained from matrices. It is now time to be more serious about matrices and develop their theory more systematically, both to obtain practical computational tools for matrices and to gain more insight into the nature of linear maps. We begin low-key by reviewing some of the matrix properties encountered so far and by adding a few further basic definitions.

### 3.2.1 Basic matrix properties

We consider matrices of arbitrary size  $n \times m$  (that is, with  $n$  rows and  $m$  columns) given by

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}. \quad (3.22)$$

Here  $a_{ij} \in F$ , with  $i = 1, \dots, n$  and  $j = 1, \dots, m$ , are the (real or complex) entries. The matrix  $A$  is called *quadratic* if  $n = m$ , that is if it has as many rows as columns. It is often useful to be able to refer to the entries of a matrix by the same symbol and, by slight abuse of notation, we will therefore also denote the entries by  $A_{ij} = a_{ij}$ . We already know that the matrices of a given size form a vector space with vector addition and scalar multiplication defined component by component, that is,  $(A + B)_{ij} = A_{ij} + B_{ij}$  and  $(\alpha A)_{ij} = \alpha A_{ij}$ . We will frequently need to refer to the row and column vectors of a matrix for which we introduce the following notation:

$$\mathbf{A}_i = \begin{pmatrix} A_{i1} \\ \vdots \\ A_{im} \end{pmatrix}, \quad \mathbf{A}^j = \begin{pmatrix} A_{1j} \\ \vdots \\ A_{nj} \end{pmatrix}. \quad (3.23)$$

Hence,  $\mathbf{A}_i$  is an  $m$ -dimensional column vector which contains the entries in the  $i^{\text{th}}$  row of  $A$  and  $\mathbf{A}^j$  is an  $n$ -dimensional column vector which contains the entries in the  $j^{\text{th}}$  column of  $A$ . We also recall that  $A$  defines a linear map  $A : F^m \rightarrow F^n$  via multiplication of matrices and vectors which can be written as

$$A : \mathbf{v} \mapsto A\mathbf{v} = \begin{pmatrix} \mathbf{A}_1 \cdot \mathbf{v} \\ \vdots \\ \mathbf{A}_n \cdot \mathbf{v} \end{pmatrix} \quad \text{or} \quad (A\mathbf{v})_i = \sum_{j=1}^m A_{ij}v_j. \quad (3.24)$$

A very specific matrix is the *unit matrix*  $\mathbb{1}_n : F^n \rightarrow F^n$  given by

$$\mathbb{1}_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \quad (3.25)$$

Its row and columns vectors are given by the standard unit vectors, so  $\mathbf{A}_i = \mathbf{e}_i$  and  $\mathbf{A}^j = \mathbf{e}_j$ , and its components  $(\mathbb{1}_n)_{ij} = \delta_{ij}$  equal the Kronecker delta symbol introduced earlier. For its action on a vector we have

$$(\mathbb{1}\mathbf{v})_i = \delta_{ij}v_j = v_i \quad (3.26)$$

so, seen as a linear map, the unit matrix corresponds to the identity map.

More generally, a *diagonal matrix* is a matrix  $D$  with non-zero entries only along the diagonal, so  $D_{ij} = 0$  for all  $i \neq j$ . It can be written as

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} =: \text{diag}(d_1, \dots, d_n) \quad (3.27)$$

The *complex conjugate*  $A^* : F^m \rightarrow F^n$  of a matrix  $A : F^m \rightarrow F^n$  is simply the matrix whose entries are the complex conjugates of the entries in  $A$ , so in component form,  $(A^*)_{ij} = (A_{ij})^*$ . Of course, for matrices with only real entries complex conjugation is a trivial operation which leaves the matrix unchanged.

The *transpose* of an  $n \times m$  matrix  $A : F^m \rightarrow F^n$  is a  $m \times n$  matrix  $A^T : F^n \rightarrow F^m$  obtained by exchanging the rows and columns of  $A$ . In component form, this means  $(A^T)_{ij} := A_{ji}$ . A quadratic matrix  $A$  is said to be *symmetric* if  $A = A^T$  or, in index notation, if all entries satisfy  $A_{ij} = A_{ji}$ . It is called *anti-symmetric* if  $A = -A^T$  or,  $A_{ij} = -A_{ji}$  for all entries. Note that all diagonal entries  $A_{ii}$  of an anti-symmetric matrix vanish. We have

$$(A + B)^T = A^T + B^T, \quad (\alpha A)^T = \alpha A^T \quad (3.28)$$

for  $n \times m$  matrices  $A, B$  and scalars  $\alpha$ . In particular, the sum of two symmetric matrices is again symmetric as is the scalar multiple of a symmetric matrix (and similarly for anti-symmetric matrices). This means that symmetric and anti-symmetric  $n \times n$  matrices each form a sub vector space within the vector space of all  $n \times n$  matrices.

**Example 3.5:** Transpose of a matrix, symmetry and anti-symmetry

(a) An explicit example for a matrix and its transpose is

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix} \quad (3.29)$$

Note that, for non-quadratic matrices, the transpose changes the “shape” of the matrix. While  $A$  above is a  $3 \times 2$  matrix defining a linear map  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ , its transpose is a  $2 \times 3$  matrix which defines a linear map  $A^T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ .

(b) The general form of  $2 \times 2$  symmetric and anti-symmetric matrices is

$$A_{\text{symm}} = \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \quad A_{\text{anti-symm}} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \quad (3.30)$$

with arbitrary numbers  $a, b, d \in F$ . The dimension of the vector space of symmetric  $2 \times 2$  matrices over  $F$  is three (as they depend on three independent numbers) while the dimension of the vector space of anti-symmetric  $2 \times 2$  matrices over  $F$  is one (as they depend on one parameter). In particular, note that the diagonal of an anti-symmetric matrix vanishes. It is easy to write down a basis for these vector spaces and also to generalize these statements to matrices of arbitrary size.

Finally, a combination of the previous two operations is the *hermitian conjugate* of an  $n \times m$  matrix  $A : F^m \rightarrow F^n$  which is defined as a  $m \times n$  matrix  $A^\dagger : F^n \rightarrow F^m$  obtained by taking the complex conjugate of the transpose of  $A$ , that is,  $A^\dagger := (A^T)^*$ . For matrices with only real entries, hermitian conjugation is of course the same as transposition. A quadratic matrix  $A$  is said to be *hermitian* if the matrix is invariant under hermitian conjugation, that is, if  $A = A^\dagger$ , and *anti-hermitian* if  $A = -A^\dagger$ . In analogy with the properties of transposition we have

$$(A + B)^\dagger = A^\dagger + B^\dagger, \quad (\alpha A)^\dagger = \alpha^* A^\dagger. \quad (3.31)$$

The first property means that the sum of two hermitian (anti-hermitian) matrices is again hermitian (anti-hermitian). More care is required for scalar multiples. The scalar multiple of a hermitian (anti-hermitian) matrix with a real scalar remains hermitian (anti-hermitian). However, for a complex scalar this is no longer generally the case due to the complex conjugation of the scalar in the second equation (3.31). This means the  $n \times n$  hermitian (anti-hermitian) matrices form a sub vector space of the vector space of all  $n \times n$  matrices with complex entries, but only if the underlying field is taken to be the real numbers.

**Example 3.6:** Hermitian conjugate

(a) An explicit example for a  $3 \times 3$  matrix with complex entries and its hermitian conjugate is

$$A = \begin{pmatrix} i & 1 & 2-i \\ 2 & 3 & -3i \\ 1-i & 4 & 2+i \end{pmatrix}, \quad A^\dagger = \begin{pmatrix} -i & 2 & 1+i \\ 1 & 3 & 4 \\ 2+i & 3i & 2-i \end{pmatrix}. \quad (3.32)$$

Note that, in addition to the transposition carried out by exchanging rows and columns, all entries are complex conjugated.

(b) The hermitian conjugate of an arbitrary  $2 \times 2$  matrix with complex entries is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \quad (3.33)$$

For  $A$  to be hermitian we need that  $a = a^*$ ,  $d = d^*$ , so that the diagonal is real, and  $c = b^*$ . Hence, the most general hermitian  $2 \times 2$  matrix has the form

$$A_{\text{herm}} = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix}, \quad a, d \in \mathbb{R}, \quad b \in \mathbb{C}. \quad (3.34)$$

These matrices form a four-dimensional vectors space (over  $\mathbb{R}$ ) since they depend on four real parameters. For an anti-hermitian matrix, the corresponding conditions are  $a = -a^*$ ,  $d = -d^*$ , so that the diagonal must be purely imaginary, and  $c = -b^*$ . The most general such matrices are

$$A_{\text{anti-herm}} = \begin{pmatrix} a & b \\ -b^* & d \end{pmatrix}, \quad a, d \in i\mathbb{R}, \quad b \in \mathbb{C}. \quad (3.35)$$

and they form a four-dimensional vector space over  $\mathbb{R}$ .

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### 3.2.2 Rank of a matrix

Previously, we have defined the rank of a linear map as the dimension of its image. Since every matrix defines a linear map we can, therefore, talk about the rank of a matrix. Can we be more specific about what the rank of a matrix is and how it can be determined?

Consider an  $n \times m$  matrix  $A : F^m \rightarrow F^n$  with columns  $\mathbf{A}^1, \dots, \mathbf{A}^m$ , and a vector  $\mathbf{v} \in F^m$  with components  $v_i$ . The image of  $\mathbf{v}$  under the action of  $A$  can then be written as

$$A\mathbf{v} = \sum_{i=1}^m v_i \mathbf{A}^i, \quad (3.36)$$

and is hence given by a linear combination of the column vectors  $\mathbf{A}^j$  with the coefficients equal to the components of  $\mathbf{v}$ . This observation tells us that

$$\text{Im}(A) = \text{Span}(\mathbf{A}^1, \dots, \mathbf{A}^m), \quad (3.37)$$

so the image of the matrix is spanned by its column vectors. For the rank of the matrix this implies that

$$\text{rk}(A) = \dim \text{Span}(\mathbf{A}^1, \dots, \mathbf{A}^m) = \text{maximal number of lin. indep. column vectors of } A. \quad (3.38)$$

For obvious reasons this is also sometimes called the *column rank* of the matrix  $A$ . This terminology suggests we can also define the *row rank* of the matrix  $A$  as the maximal number of linearly independent row vectors of  $A$ . Having two types of ranks available for a matrix seems awkward but fortunately we have the following

**Theorem 3.3.** *Row and column rank are equal for any matrix.*

*Proof.* Suppose one row, say  $\mathbf{A}_1$ , of a matrix  $A$  can be written as a linear combination of the others. Then, by dropping  $\mathbf{A}_1$  from  $A$  we arrive at a matrix with one less row, but its row rank unchanged from that of  $A$ . The key observation is that the column rank also remains unchanged under this operation. This can be seen as follows. Write

$$\mathbf{A}_1 = \sum_{j=2}^n \alpha_j \mathbf{A}_j, \quad \boldsymbol{\alpha} = \begin{pmatrix} \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

with some coefficients  $\alpha_2, \dots, \alpha_n$  which we have arranged into the vector  $\boldsymbol{\alpha}$ . Further, let us write the column vectors of  $A$  as

$$\mathbf{A}^i = \begin{pmatrix} a_i \\ \mathbf{b}_i \end{pmatrix},$$

that is, we split off the entries in the first row, denoted by  $a_i$ , from the entries in the remaining  $n-1$  rows which are contained in the vectors  $\mathbf{b}_i$ . It follows that  $a_i = A_{1i} = (\mathbf{A}_1)_i = \sum_{j=2}^n \alpha_j A_{ji} = \sum_{j=2}^n \alpha_j (\mathbf{A}^i)_j = \boldsymbol{\alpha} \cdot \mathbf{b}_i$ , so that the column vectors can also be written as

$$\mathbf{A}^i = \begin{pmatrix} \boldsymbol{\alpha} \cdot \mathbf{b}_i \\ \mathbf{b}_i \end{pmatrix},$$

Hence, the entries in the first row are not relevant for the linear independence of the column vectors  $\mathbf{A}^i$  - merely using the vectors  $\mathbf{b}_i$  will lead to the same conclusions for linear independence. As a result we can drop a linearly dependent row without changing the row and the column rank of the matrix. Clearly, an argument similar to the above can be made if we drop a linearly dependent column vectors - again both the row and column rank remain unchanged.

In this way, we can continue dropping linearly dependent row and column vectors from  $A$  until we arrive at a (generally smaller) matrix  $A'$  which has linearly independent row and column vectors and the same row and column ranks as  $A$ . On purely dimensional grounds, a matrix with all row vectors and all column vectors linearly independent must be quadratic (For example, consider a  $3 \times 2$  matrix. Its three 2-dimensional row vectors cannot be linearly independent.). Therefore, row and column rank are the same for  $A'$  and, hence, for  $A$ .  $\square$

The above interpretation of the rank of a matrix as the maximal number of linearly independent row or column vectors gives us a practical way to determine the rank of the matrix, using the methods to check linear independence we have introduced in Section 1. Efficient, algorithmic methods for this will be introduced in the next sub-section but for smaller matrices the rank can often be found “by inspection”, as in the following example.

**Example 3.7:** Rank of a matrix by inspection, kernel and image of a matrix

(a) The  $2 \times 2$  matrix

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.39}$$

defines a map  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . Clearly,  $A$  has rank two since its two columns are linearly independent. This means that the image of  $A$  is two-dimensional and, hence,  $\text{Im}(A) = \mathbb{R}^2$ . From the dimensional formula (3.4) it also follows that  $\dim \text{Ker}(A) = 0$ , so that the kernel is trivial,  $\text{Ker}(A) = \{\mathbf{0}\}$ .



(b) Consider the  $3 \times 3$  matrix

$$A = \begin{pmatrix} -1 & 4 & 3 \\ 2 & -3 & -1 \\ 3 & 2 & 5 \end{pmatrix}. \quad (3.40)$$

which defines a map  $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ . It is clear that the first two columns of this matrix are linearly independent (they are not multiples of each other) and that the third column is the sum of the first two. Hence, the rank of this matrix is two. This means that the dimension of its image is two while, from Eq. (3.4), the dimension of its kernel is one. To find the kernel of  $A$  explicitly we have to solve  $A\mathbf{v} = \mathbf{0}$ . With  $\mathbf{v} = (x, y, z)^T$  this leads to

$$-x + 4y + 3z = 0, \quad 2x - 3y - z = 0, \quad 3x + 2y + 5z = 0, \quad (3.41)$$

and these equations are solved precisely if  $x = y = -z$ . The image of  $A$  is, in general, spanned by the column vectors, but since  $\mathbf{A}^3 = \mathbf{A}^1 + \mathbf{A}^2$  it is already spanned by the first two columns. In conclusion, we have

$$\text{Ker}(A) = \text{Span} \left( \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right), \quad \text{Im}(A) = \text{Span} \left( \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} \right). \quad (3.42)$$

### 3.2.3 Linear maps between column vectors

Now we return to the question posed in the previous section. We saw how an  $n \times m$  matrix  $A$  corresponds to a linear map  $f : F^m \rightarrow F^n$ . Is it the case that *every* linear map  $f : F^m \rightarrow F^n$  can be obtained from a matrix in this way?

To answer this question, we begin with an arbitrary linear map  $f : F^m \rightarrow F^n$ , the standard unit vectors  $\mathbf{e}_i$  of  $F^m$  and the standard unit vectors  $\tilde{\mathbf{e}}_i$  of  $F^n$ . Let us consider the images,  $f(\mathbf{e}_i)$ , of these standard unit vectors under our linear map. While we do not know what these images are explicitly it is clear that they can be expanded in terms of the basis  $\tilde{\mathbf{e}}_i$ , that is, we can write

$$f(\mathbf{e}_j) = \sum_{i=1}^n a_{ij} \tilde{\mathbf{e}}_i, \quad (3.43)$$

for some suitable set of coefficients  $a_{ij}$ . Now consider an arbitrary vector  $\mathbf{v} \in F^m$  written as a linear combination  $\mathbf{v} = \sum_{j=1}^m v_j \mathbf{e}_j$ . For the image under  $f$  of this vector we find

$$f(\mathbf{v}) = f \left( \sum_{j=1}^m v_j \mathbf{e}_j \right) = \sum_{j=1}^m v_j f(\mathbf{e}_j) = \sum_{j=1}^m v_j \sum_{i=1}^n a_{ij} \tilde{\mathbf{e}}_i = \sum_{i=1}^n \left( \sum_{j=1}^m a_{ij} v_j \right) \tilde{\mathbf{e}}_i. \quad (3.44)$$

Hence, for the  $i^{\text{th}}$  component of this image we have

$$[f(\mathbf{v})]_i = \sum_{j=1}^m a_{ij} v_j = (A\mathbf{v})_i, \quad (3.45)$$

where  $A$  is the matrix with entries  $a_{ij}$ , these being the coefficients which appear in the parametrization of the images (3.43). We have therefore succeeded in expressing the action of our arbitrary linear map  $f$  in terms of a matrix and we conclude that all linear maps between column vectors are given by matrices. We summarize this in the following

**Lemma 3.3.** *Every linear map  $f : F^m \rightarrow F^n$  can be written in terms of an  $n \times m$  matrix  $A$ , such that  $f(\mathbf{v}) = A\mathbf{v}$  for all  $\mathbf{v} \in F^m$ . If  $f(\mathbf{e}_j) = \sum_{i=1}^n a_{ij}\tilde{\mathbf{e}}_i$  for the standard unit vectors  $\mathbf{e}_i$  of  $F^m$  and  $\tilde{\mathbf{e}}_i$  of  $F^n$ , then  $a_{ij}$  are the entries of  $A$ .*

**Example 3.8:** Matrix describing a linear map  $\mathbb{R}^3 \rightarrow \mathbb{R}^3$

Consider a fixed vector  $\mathbf{n} \in \mathbb{R}^3$  and a map  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by  $f(\mathbf{v}) = \mathbf{n} \times \mathbf{v}$ . From the properties (2.31)–(2.33) of the vector product, it is easy to show that this map is linear:

$$f(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{n} \times (\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{n} \times \mathbf{v}_1 + \mathbf{n} \times \mathbf{v}_2 = f(\mathbf{v}_1) + f(\mathbf{v}_2) \quad (3.46)$$

$$f(\alpha\mathbf{v}) = \mathbf{n} \times (\alpha\mathbf{v}) = \alpha\mathbf{n} \times \mathbf{v} = \alpha f(\mathbf{v}) \quad (3.47)$$

Hence, we know from Lemma 3.3 that  $f$  can be described by a  $3 \times 3$  matrix  $A$  which can be worked out by studying the action of  $f$  on the standard unit vectors  $\mathbf{e}_i$ . Writing  $\mathbf{n} = (n_1, n_2, n_3)^T$  we find by explicit computation

$$\begin{aligned} f(\mathbf{e}_1) &= \mathbf{n} \times \mathbf{e}_1 = n_3\mathbf{e}_2 - n_2\mathbf{e}_3 \\ f(\mathbf{e}_2) &= \mathbf{n} \times \mathbf{e}_2 = -n_3\mathbf{e}_1 + n_1\mathbf{e}_3 \\ f(\mathbf{e}_3) &= \mathbf{n} \times \mathbf{e}_3 = n_2\mathbf{e}_1 - n_1\mathbf{e}_2 . \end{aligned} \quad (3.48)$$

Lemma 3.3 states that the coefficients in front of the standard unit vectors on the right-hand sides of these equations are the entries of the desired matrix  $A$ . More precisely, the coefficients which appear in the expression for  $f(\mathbf{e}_j)$  form the  $j^{\text{th}}$  column of the matrix  $A$ . Hence, the desired matrix is

$$A = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_2 \\ -n_2 & n_1 & 0 \end{pmatrix} , \quad (3.49)$$

and we have  $f(\mathbf{v}) = \mathbf{n} \times \mathbf{v} = A\mathbf{v}$  for all vectors  $\mathbf{v} \in \mathbb{R}^3$ . The interesting conclusion is that vector products with a fixed vector  $\mathbf{n}$  can also be represented by multiplication with the anti-symmetric matrix (3.49). Everything is much more elegant in index notation where

$$A_{ij} = [f(\mathbf{e}_j)]_i = [\mathbf{n} \times \mathbf{e}_j]_i = \epsilon_{ikl}n_k[\mathbf{e}_j]_l = \epsilon_{ikl}n_k\delta_{jl} = \epsilon_{ikj}n_k , \quad (3.50)$$

so that  $A_{ij} = \epsilon_{ikj}n_k$ , in agreement with Eq. (3.49).

### 3.2.4 Matrix multiplication

We have seen earlier that the composition of linear maps is again linear and we have just shown that all linear maps between column vectors are matrices. Hence, the composition of two matrices must again be a matrix. To work this out more explicitly, we start with an  $n \times m$  matrix  $A$  and an  $r \times n$  matrix  $B$  which generate linear maps according to the chain  $F^m \xrightarrow{A} F^n \xrightarrow{B} F^r$ . We would like to determine the matrix  $C$  which describes the composite map  $B \circ A : F^m \rightarrow F^r$ . By a straightforward computation we find

$$(B(A\mathbf{v}))_i = \sum_{j=1}^n B_{ij}(A\mathbf{v})_j = \sum_{k=1}^m \underbrace{\left( \sum_{j=1}^n B_{ij}A_{jk} \right)}_{C_{ik}} v_k \stackrel{!}{=} \sum_{k=1}^m C_{ik}v_k = (C\mathbf{v})_i \quad (3.51)$$

so that the entries  $C_{ik}$  of  $C$  are given by

$$C_{ik} = \sum_{j=1}^n B_{ij} A_{jk} = \mathbf{B}_i \cdot \mathbf{A}^k \quad (3.52)$$

This equation represents the component version of what we refer to as *matrix multiplication*. We obtain the entries of the new matrix  $C$  - which corresponds to the composition of  $B$  and  $A$  - by performing the summing over the entries of  $B$  and  $A$  as indicated in the middle of (3.52) or, equivalently, by dotting the columns of  $A$  into the rows of  $B$ , as indicated on the RHS of (3.52). In matrix notation this can also be written as

$$C = BA := \begin{pmatrix} \mathbf{B}_1 \cdot \mathbf{A}^1 & \cdots & \mathbf{B}_1 \cdot \mathbf{A}^m \\ \vdots & & \vdots \\ \mathbf{B}_r \cdot \mathbf{A}^1 & \cdots & \mathbf{B}_r \cdot \mathbf{A}^m \end{pmatrix}. \quad (3.53)$$

Note that the product of the  $r \times n$  matrix  $B$  with the  $n \times m$  matrix  $A$  results in the  $r \times m$  matrix  $C = BA$ . The important conclusion is that composition of matrices - in their role as linear maps - is accomplished by matrix multiplication.

We should discuss some properties of matrix multiplication. First note that the matrix product  $BA$  only makes sense if “sizes fit”, that is, if  $A$  has as many rows as  $B$  columns - otherwise the dot products in (3.53) do not make sense. This consistency condition is of course a direct consequence of the role of matrices as linear maps. The maps  $B$  and  $A$  can only be composed to  $BA$  if the co-domain of  $A$  has the same dimension as the domain of  $B$ . Let us illustrate this with the following

**Example 3.9:** Matrix multiplication

Consider the two matrices

$$B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & -2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} \quad (3.54)$$

of sizes  $2 \times 3$  and  $3 \times 3$ , respectively. Dotting the column vectors of  $A$  into the rows of  $B$  we can compute their product

$$BA = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ 4 & 4 & 3 \end{pmatrix}, \quad (3.55)$$

a  $2 \times 3$  matrix. Note, however, that the product  $AB$  is ill-defined since  $B$  has two rows but  $A$  has 3 columns.

Matrix multiplication is associative, so

$$A(BC) = (AB)C. \quad (3.56)$$

This is a direct consequence of the associativity of map composition (see the discussion around Eq. (3.1)) but can also be verified directly from the definition of matrix multiplication. This is most easily done in index notation (using Eq. (3.52)) which gives

$$(A(BC))_{ij} = A_{ik}(BC)_{kj} = A_{ik}B_{kl}C_{lj} = (AB)_{il}C_{lj} = ((AB)C)_{ij}. \quad (3.57)$$

However, matrix multiplication is not commutative, that is, typically,  $AB \neq BA$ . The “degree of non-commutativity” of two matrices is often measured by the commutator defined as

$$[A, B] := AB - BA \quad (3.58)$$

Evidently, the matrices  $A, B$  commute if and only if  $[A, B] = 0$ .

**Example 3.10:** Non-commutativity of matrix multiplication

(a) Consider the two matrices

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix} \quad (3.59)$$

By straightforward computation we have

$$AB = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ -3 & 1 \end{pmatrix}, \quad BA = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 2 & 0 \end{pmatrix}, \quad (3.60)$$

so that indeed  $AB \neq BA$ .

(b) Note that matrices with a specific structure may still commute. For example, it is easy to check that the matrices

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad B = \begin{pmatrix} c & d \\ d & c \end{pmatrix} \quad (3.61)$$

for arbitrary real numbers  $a, b, c, d$  satisfy  $[A, B] = 0$ .

What is the relation between multiplication and transposition of matrices? The answer is

$$(AB)^T = B^T A^T. \quad (3.62)$$

Note the change of order on the RHS! A proof of this relation is most easily accomplished in index notation:

$$((AB)^T)_{ij} = (AB)_{ji} = A_{jk} B_{ki} = B_{ki} A_{jk} = (B^T)_{ik} (A^T)_{kj} = (B^T A^T)_{ij}. \quad (3.63)$$

For the complex conjugation of a matrix product we have of course  $(AB)^* = A^* B^*$ , so together with Eq. (3.62) this means for the hermitian conjugate that

$$(AB)^\dagger = B^\dagger A^\dagger. \quad (3.64)$$

Finally, using matrix terminology, we can think of vectors in a slightly different way. A column vector  $\mathbf{v}$  with components  $v_1, \dots, v_m$  can also be seen as an  $m \times 1$  matrix and the action  $A\mathbf{v}$  of an  $n \times m$  matrix  $A$  on  $\mathbf{v}$  as a matrix multiplication. The transpose,  $\mathbf{v}^T = (v_1, \dots, v_m)$  is an  $m$  dimensional row vector and, hence, the dot product of two  $m$ -dimensional (column) vectors  $\mathbf{v}$  and  $\mathbf{w}$  can also be written as

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}, \quad (3.65)$$

that is, as a matrix product between the  $1 \times m$  matrix  $\mathbf{v}^T$  and the  $m \times 1$  matrix  $\mathbf{w}$ .

### 3.2.5 The inverse of a matrix

Recall from Claim 3.1, that a linear map  $f : V \rightarrow W$  can only have an inverse if  $\dim(V) = \dim(W)$ . Hence, for a matrix  $A : F^m \rightarrow F^n$  to have an inverse it must be quadratic, so that  $n = m$ . Focusing on quadratic  $n \times n$  matrices  $A$  we further know from Claim 3.1 that we have an inverse precisely when  $\text{rk}(A) = n$ , that is, when the rank of  $A$  is maximal. In this case, the inverse of  $A$ , denoted  $A^{-1}$ , is the unique linear map (and, therefore, also a matrix) satisfying

$$AA^{-1} = A^{-1}A = \mathbb{1}_n. \quad (3.66)$$

Note that this is just the general Definition (3.4) of an inverse map applied to a matrix, using the fact that matrices correspond to linear maps and map composition corresponds to matrix multiplication. We summarize the properties of the matrix inverse in the following

**Lemma 3.4.** (*Properties of matrix inverse*) *A quadratic  $n \times n$  matrix  $A : F^n \rightarrow F^n$  is invertible if and only if its rank is maximal, that is, iff  $\text{rk}(A) = n$ . If  $A, B$  are two invertible  $n \times n$  matrices we have*

- (a) *The inverse matrix, denoted  $A^{-1}$ , is the unique matrix satisfying  $AA^{-1} = A^{-1}A = \mathbb{1}_n$ .*
- (b)  *$(AB)^{-1} = B^{-1}A^{-1}$*
- (c)  *$A^{-1}$  is invertible and  $((A^{-1})^{-1})^{-1} = A$*
- (d)  *$A^T$  is invertible and  $(A^T)^{-1} = (A^{-1})^T$*

*Proof.* (a) This has already been shown above.

(b) (c) These are direct consequences of the corresponding properties (3.2), (3.3) for general maps.

(d) Recall from Claim 3.1 that a matrix is invertible iff its rank is maximal. Since, from Theorem 3.3,  $\text{rk}(A) = \text{rk}(A^T)$ , we conclude that  $A^T$  is indeed invertible which proves the first part of the claim. For the second part, we transpose  $A^{-1}A = AA^{-1} = \mathbb{1}$ , using Eq. (3.62), to arrive at  $A^T(A^{-1})^T = (A^{-1})^T A^T = \mathbb{1}$ . On the other hand, from the definition of the inverse for  $A^T$ , we have  $A^T(A^T)^{-1} = (A^T)^{-1}A^T = \mathbb{1}$ . Comparing the two equations shows that both  $(A^{-1})^T$  and  $(A^T)^{-1}$  provide an inverse for  $A^T$  and, hence, from the uniqueness of the inverse, they must be equal.  $\square$

#### Application: Matrices in graph theory

Graphs are objects consisting of a certain number of vertices,  $V_1, \dots, V_n$  and links connecting these vertices. A simple example with five vertices is shown in Fig. 19. Here we focus on undirected graphs as in Fig. 19

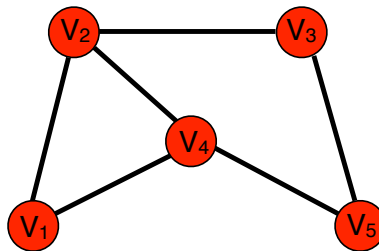


Figure 19: A simple (undirected) graph with five vertices.

for which the links have no direction, but our considerations can easily be generalized to directed graphs. Graphs can be related to linear algebra via the *adjacency matrix* which is defined by

$$M_{ij} = \begin{cases} 1 & \text{if } V_i \text{ and } V_j \text{ are linked} \\ 0 & \text{otherwise} \end{cases} \quad (3.67)$$

For example, for the graph in Fig. 19 the adjacency matrix is given by

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}. \quad (3.68)$$

This matrix is symmetric due to the underlying graph being undirected. The following fact (which we will not try to prove here) makes the adjacency matrix a useful object.

**Fact:** The number of possible walks from vertex  $V_i$  to vertex  $V_j$  over precisely  $n$  links in a graph is given by  $(M^n)_{ij}$ , where  $M$  is the adjacency matrix of the graph.

To illustrate this compute the low powers of the adjacency matrix  $M$  in Eq. (3.68).

$$M^2 = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 2 \\ 1 & 0 & 2 & 2 & 0 \\ 1 & 1 & 2 & 3 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{pmatrix}, \quad M^3 = \begin{pmatrix} 2 & 4 & 2 & 4 & 2 \\ 4 & 2 & 5 & 6 & 1 \\ 2 & 5 & 0 & 1 & 4 \\ 4 & 6 & 1 & 2 & 5 \\ 2 & 1 & 4 & 5 & 0 \end{pmatrix}. \quad (3.69)$$

For example, the number of possible walks from  $V_1$  to  $V_3$  over three links is given by  $(M^3)_{13} = 2$ . By inspecting Fig. 19 it can be seen that these two walks correspond to  $V_1 \rightarrow V_4 \rightarrow V_5 \rightarrow V_3$  and  $V_1 \rightarrow V_4 \rightarrow V_2 \rightarrow V_3$ .

### Application: Matrices in cryptography

Matrices can be used for encryption. Here is a basic example for how this works. Suppose we would like to encrypt the text: "linear algebra". First, we translate this text into numerical form using the simple code  $\square \rightarrow 0, a \rightarrow 1, b \rightarrow 2, \dots$  and then we split the resulting sequence of numbers into blocks of the same size. Here we use blocks of size three for definiteness. Next, we arrange these numbers into a matrix, with each block forming a column of the matrix. For our sample text this results in

$$T = \begin{pmatrix} 12 & 5 & 0 & 7 & 18 \\ 9 & 1 & 1 & 5 & 1 \\ 14 & 18 & 12 & 2 & 0 \end{pmatrix} \quad \text{for} \quad \begin{matrix} \text{l} & \text{e} & \square & \text{g} & \text{r} \\ \text{i} & \text{a} & \text{a} & \text{e} & \text{a} \\ \text{n} & \text{r} & \text{l} & \text{b} & \square \end{matrix}.$$

So far, this is relatively easy to decode, even if we had decided to permute the assignment of letters to numbers. As long as same letters are represented by same numbers, the code can be deciphered by a frequency analysis, at least for a sufficiently long text. To do this, the relative frequency of each number is determined and compared with the typical frequency with which letters appear in the English language. Matching similar frequencies leads to the key.

For a more sophisticated encryption, define a quadratic "encoding" matrix whose size equals the length of the blocks, so a  $3 \times 3$  matrix for our case. Basically, the only other restriction on this matrix is that it should be invertible. For our example, let us choose

$$A = \begin{pmatrix} -1 & -1 & 1 \\ 2 & 0 & -1 \\ -2 & 1 & 1 \end{pmatrix}.$$

To encode the text, carry out the matrix multiplication

$$T_{\text{enc}} = AT = \begin{pmatrix} -1 & -1 & 1 \\ 2 & 0 & -1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 12 & 5 & 0 & 7 & 18 \\ 9 & 1 & 1 & 5 & 1 \\ 14 & 18 & 12 & 2 & 0 \end{pmatrix} = \begin{pmatrix} -7 & 12 & 11 & -10 & -19 \\ 10 & -8 & -12 & 12 & 36 \\ -1 & 9 & 13 & -7 & -35 \end{pmatrix}$$

Note that in  $T_{\text{enc}}$  same letters are now represented by different numbers. For example, the letter “a” which appears three times, and corresponds to the three 1’s in  $T$ , is represented by three different numbers in  $T_{\text{enc}}$ . Without knowledge of the encoding matrix  $A$  it is quite difficult to decypher  $T_{\text{enc}}$ , particularly for large block sizes. The legitimate receiver of the text should be provided with the inverse  $A^{-1}$  of the encoding matrix, for our example given by

$$A^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 2 \end{pmatrix},$$

and can then recover the message by the simply matrix multiplication

$$T = A^{-1}T_{\text{enc}}.$$

### 3.3 Row/column operations, Gaussian elimination

We should now develop more systematic, algorithmic methods to compute properties of matrices. At the heart of these methods are *elementary row operations* which are defined as follows.

**Definition 3.7.** *The following manipulations of a matrix are called elementary row operations.*

- (R1) *Exchange two rows.*
- (R2) *Add a multiple of one row to another.*
- (R3) *Multiply a row with a non-zero scalar.*

*Analogous definitions hold for elementary column operations.*

For definiteness, we will focus on elementary row operations but most of our statements have analogues for elementary column operations. As we will see, elementary row operations will allow us to devise methods to compute the rank and the inverse of matrices and, later on, to formulate a general algorithm to solve linear systems of equations.

A basic but important observation about elementary row operations (which is indeed the main motivation for defining them) is that they do not change the span of the row vectors. Recall that the rank of a matrix is given by the maximal number of linearly independent row (or column) vectors. Hence, the rank of a matrix is also unchanged under elementary row operations. This suggests a possible strategy to compute the rank of a matrix: By a succession of elementary row operations, we should bring the matrix into a (simpler) form where the rank can easily be read off. Suppose a matrix has the form

$$A = \begin{pmatrix} \cdots & \boxed{a_{1j_1}} & & & * \\ & & \boxed{a_{2j_2}} & & \ddots \\ & & & \ddots & \\ & & & & \boxed{a_{rj_r}} & \cdots \\ 0 & & & & & \vdots \end{pmatrix}$$

where the entries  $a_{ij_i}$  are non-zero for all  $i = 1, \dots, r$ , all other entries above the solid line are arbitrary (indicated by the  $*$ ) and all entries below the solid line are zero. This form of a matrix is called (*upper*)

*echelon form*. Clearly, the first  $r$  row vectors in this matrix are linearly independent and, hence, the rank of a matrix in upper echelon form can be easily read off and is given by

$$\text{rk}(A) = r = (\text{number of steps in upper echelon form}) . \quad (3.70)$$

The important fact is that every matrix can be brought into upper echelon form by a sequence of elementary row operations. This works as follows.

**Algorithm to bring matrix into upper echelon form**

We consider an  $n \times m$  matrix. The algorithm proceeds row by row. Let us assume that we have already dealt with the first  $i - 1$  rows of the matrix. Then, for the  $i^{\text{th}}$  row we should carry out three steps.

- (1) Find the leftmost column  $j$  which has at least one non-zero entry in rows  $i, \dots, n$ .
- (2) If the  $(i, j)$  entry is zero exchange row  $i$  with one of the rows  $i + 1, \dots, n$  (the one which contains the non-zero entry identified in step 1) so that the new  $(i, j)$  entry is non-zero.
- (3) Subtract suitable multiples of row  $i$  from all rows  $i + 1, \dots, n$  such that all entries  $(i + 1, j), \dots, (n, j)$  in column  $j$  and below row  $i$  vanish.

Continue with the next row until no more non-zero entries can be found in step 1.

This procedure of bringing a matrix into its upper echelon form using elementary row operations is called *Gaussian elimination* (sometimes also referred to as *row reduction*). In summary, our procedure to compute the rank of a matrix involves, first, to bring the matrix into upper echelon form using Gaussian elimination and then to read off the rank from the number of steps in the upper echelon form. This is probably best explained with an example.

**Example 3.11:** Gaussian elimination and rank of a matrix

Consider the  $3 \times 3$  matrix

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 3 & -2 \\ 2 & 1 & 0 \end{pmatrix} .$$

Then, Gaussian elimination amounts to

$$\begin{pmatrix} 0 & 1 & -1 \\ 2 & 3 & -2 \\ 2 & 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 2 & 1 & 0 \\ 2 & 3 & -2 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_2/2} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

We have indicated the row operation from one step to the next above the arrow, referring to the  $i^{\text{th}}$  row by  $R_i$ . The final matrix is in upper echelon form. There are two steps so that  $\text{rk}(A) = 2$ .

A neat and very useful fact about elementary row operations is that they can be generated by multiplying with certain, specific matrices from the left. In other words, to perform a row operation on a matrix  $A$ , we can find a suitable matrix  $P$  such that the row operation is generated by  $A \rightarrow PA$ . For example, consider a simple  $2 \times 2$  case where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \quad P = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} . \quad (3.71)$$





total. These conditions can be cast into the seven linear equations

$$\left\{ \begin{array}{l} d + e + f = a + b + c \\ g + h + i = a + b + c \\ a + d + g = a + b + c \\ b + e + h = a + b + c \\ c + f + i = a + b + c \\ a + e + i = a + b + c \\ c + e + g = a + b + c \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} -a - b - c + d + e + g = 0 \\ -a - b - c + g + h + i = 0 \\ -b - c + d + g = 0 \\ -a - c + e + h = 0 \\ -a - b + f + i = 0 \\ -b - c + e + i = 0 \\ -a - b + e + g = 0 \end{array} \right.$$

In matrix form, this system of equations can be written as follows.

$$\underbrace{\begin{pmatrix} -1 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}}_A \underbrace{\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{pmatrix}}_x = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_0,$$

or, in short,  $A\mathbf{x} = \mathbf{0}$ . The magic squares are precisely the solutions to this equation which shows that the magic square vector space is the kernel,  $\text{Ker}(A)$ , of the matrix  $A$ . By Gaussian elimination and with a bit of calculation, the matrix  $A$  can be brought into upper echelon form and the rank can be read off as  $\text{rk}(A) = 6$ . Then, the dimension formula (3.4) leads to  $\dim \text{Ker}(A) = 9 - \text{rk}(A) = 3$  and, hence, the dimension of the magic square vector space is indeed three. In summary, the three matrices  $M_1, M_2, M_3$  in Eq. (1.56) form a basis of the magic square vector space and every magic square is given as a (unique) linear combination of these three matrices.

### Application: Coding theory

Coding theory deals with the problem of errors in information such as they may arise when information is transmitted in the presence of noise. Whenever information may be faulty, methods are required for both error detection and error correction. A simple but potentially inefficient method is to transmit the information repeatedly. Here, we would like to discuss a more sophisticated method, referred to as *Hamming code*, which is based on some of the linear algebra methods we have explored.

Information is conveniently described in binary form, that is, as a sequence of bits,  $\beta_1, \dots, \beta_n \in \{0, 1\}$ . Mathematically, a bit can be seen as an element of the finite field  $\mathbb{F}_2 = \{0, 1\}$  which we have introduced in Example 1.3 and information encoded by  $n$  bits can be seen as an element of the  $n$ -dimensional vectors space  $V = \mathbb{F}_2^n$  over the field  $\mathbb{F}_2$ . In other words, we can think of the above bit sequence as a column vector  $(\beta_1, \dots, \beta_n)^T \in \mathbb{F}_2^n$ . Through this simple re-interpretation all the tools of linear algebra are now available to deal with information.

To be specific we focus on the case of four bits,  $\beta = (\beta_1, \dots, \beta_4)^T$ , but the method can be generalized to arbitrary dimensions. We begin by writing down the matrix

$$H = (\mathbf{H}^1, \dots, \mathbf{H}^7) = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad (3.74)$$

whose columns consist of all non-zero vectors of  $\mathbb{Z}_2^3$ . Clearly,  $\text{rk}(H) = 3$  (since  $\mathbf{H}^1, \mathbf{H}^2, \mathbf{H}^4$  are linearly independent) and, therefore, its kernel has dimension  $\dim \text{Ker}(H) = 7 - 3 = 4$ . It is easy to see that this four-dimensional kernel has a basis  $\mathbf{K}_1 = (1, 0, 0, 0, 0, 1, 1)^T$ ,  $\mathbf{K}_2 = (0, 1, 0, 0, 0, 1, 0)^T$ ,  $\mathbf{K}_3 = (0, 0, 1, 0, 1, 1, 0)^T$ ,  $\mathbf{K}_4 = (0, 0, 0, 1, 1, 1, 1)^T$  which we can arrange into the rows of a  $4 \times 7$  matrix

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \quad (3.75)$$

The key idea is now to encode the information stored in  $\beta_1, \dots, \beta_4$  by forming the linear combination of these numbers with the basis vectors of  $\text{Ker}(H)$  which we have just determined. In other words, we encode the information in the seven-dimensional vector

$$\mathbf{v} = \sum_{i=1}^4 \beta_i \mathbf{K}_i = \boldsymbol{\beta}^T K. \quad (3.76)$$

Note that, given the structure of the matrix  $K$ , the first four bits in  $\mathbf{v}$  coincide with the actual information  $\beta_1, \dots, \beta_4$ . By construction, the vector  $\mathbf{v}$  is an element of  $\text{Ker}(H)$ .

Now suppose that the transmission of  $\mathbf{v}$  has resulted in a vector  $\mathbf{w}$  which can have an error in at most one bit. How do we detect whether such an error has occurred? We note that the seven-dimensional standard unit vectors  $\mathbf{e}_1, \dots, \mathbf{e}_7$  are not in the kernel of  $H$ . Further, if  $\mathbf{v}$  is in the kernel then none of the vectors  $\mathbf{w} = \mathbf{v} + \mathbf{e}_i$  is. This means the transmitted information  $\mathbf{w}$  is free of (one-bit) errors if and only if  $\mathbf{w} \in \text{Ker}(H)$ , a condition which can be easily tested.

Assuming  $\mathbf{w} \notin \text{Ker}(H)$  so that the information is faulty, how can the error be corrected? Assume that bit number  $i$  has changed in  $\mathbf{w}$  so that the correct original vector is  $\mathbf{v} = \mathbf{w} - \mathbf{e}_i$ . Since  $\mathbf{v} \in \text{Ker}(H)$  it follows that  $H\mathbf{w} = H\mathbf{e}_i = \mathbf{H}^i$ , so that the product  $H\mathbf{w}$  coincides with one of the columns  $\mathbf{H}^i$  of  $H$ . This means, if  $H\mathbf{w}$  equals column  $i$  of  $H$  then we should flip bit number  $i$  in  $\mathbf{w}$  to correct for the error.

Let us carry all this out for an explicit example. Suppose that the transmitted message is  $\mathbf{w} = (1, 1, 0, 0, 0, 1, 1)^T$  and that it contains at most one error. Then we work out

$$H\mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \mathbf{H}^2. \quad (3.77)$$

First,  $\mathbf{w}$  is not in the kernel of  $H$  so an error has indeed occurred. Secondly, the vector  $H\mathbf{w}$  corresponds to the second column vector of  $H$  so we should flip the second bit to correct for the error. This means,  $\mathbf{v} = (1, 0, 0, 0, 0, 1, 1)^T$  and the original information (which is contained in the first four entries of  $\mathbf{v}$ ) is  $\boldsymbol{\beta} = (1, 0, 0, 0)^T$ .

By paying the price of enhancing the transmitted information from four bits (in  $\boldsymbol{\beta}$ ) to seven bits (in  $\mathbf{v}$ ) both a detection and correction of one-bit errors can be carried out with this method. Compare this with the naive method of simply transmitting the information in  $\boldsymbol{\beta}$  twice which corresponds to an enhancement from four to eight bits. In this case, one-bit errors can of course be detected. However, without further information they cannot be corrected since it is impossible to decide which of the two transmissions is the correct one.

Our next task is to devise an algorithm to compute the inverse of a matrix, using elementary row operations. The basic observation is that every quadratic, invertible  $n \times n$  matrix  $A$  can be converted into the

unit matrix  $\mathbb{1}_n$  by a sequence of row operations. Schematically, this works as follows:

$$A \xrightarrow{\text{echelon form}} \begin{pmatrix} a'_{11} & & * \\ & a'_{22} & \\ & & \ddots \\ 0 & & & a'_{nn} \end{pmatrix} \xrightarrow{(R1), (R2)} \begin{pmatrix} a'_{11} & & 0 \\ & a'_{22} & \\ & & \ddots \\ 0 & & & a'_{nn} \end{pmatrix} \xrightarrow{(R3)} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \mathbb{1}_n$$

In the first step, we bring  $A$  into upper echelon form, by the algorithm already discussed. At this point we can read off the rank of the matrix. If  $\text{rk}(A) < n$  the inverse does not exist and we can stop. On the other hand, if  $\text{rk}(A) = n$  then all diagonal entries  $a'_{ii}$  in the upper echelon form must be non-zero (or else we would not have  $n$  steps). This means, in a second step, we can make all entries above the diagonal zero. We start with the last column and subtract suitable multiples of the last row from the others until all entries in the last column except  $a'_{nn}$  are zero. We proceed in a similar way, column by column from the right to the left, using row operations of type (R1) and (R2). In this way we arrive at a diagonal matrix, with diagonal entries  $a'_{ii} \neq 0$  which, in the final step, can be converted into the unit matrix by row operations of type (R3).

This means we can find a set of matrices  $P_1, \dots, P_k$  of the type (3.73), generating the appropriate elementary row operations, such that

$$\mathbb{1}_n = \underbrace{P_1 \cdots P_k}_{A^{-1}} A \quad \Rightarrow \quad A^{-1} = P_1 \cdots P_k \mathbb{1}_n. \quad (3.78)$$

These equations imply an explicit algorithm to compute the inverse of a square matrix. We convert  $A$  into the unit matrix  $\mathbb{1}_n$  using elementary row operations as described above, and then simply carry out the same operations on  $\mathbb{1}_n$  in parallel. When we are done the unit matrix will have been converted into  $A^{-1}$ . Again, we illustrate this procedure by means of an example.

**Example 3.12:** Computing the inverse of a matrix with row operations

$$\begin{array}{l} A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & -2 \\ 1 & -4 & 0 \end{pmatrix} \qquad \mathbb{1}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \\ R_3 \rightarrow R_3 - R_1 : \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & -2 \\ 0 & -4 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\ \\ R_3 \rightarrow R_3 + \frac{4}{3}R_2 : \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & -2 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \leftarrow \text{rk}(A) = 3 \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & \frac{4}{3} & 1 \end{pmatrix} \\ \\ R_2 \rightarrow R_2 - 3R_3 : \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 3 & -3 & -3 \\ -1 & \frac{4}{3} & 1 \end{pmatrix} \\ \\ R_1 \rightarrow R_1 - 3R_3 : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \qquad \begin{pmatrix} 4 & -4 & -3 \\ 3 & -3 & -3 \\ -1 & \frac{4}{3} & 1 \end{pmatrix} \\ \\ R_2 \rightarrow \frac{R_2}{3} : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \qquad \begin{pmatrix} 4 & -4 & -3 \\ 1 & -1 & -1 \\ -1 & \frac{4}{3} & 1 \end{pmatrix} \\ \\ R_3 \rightarrow -\frac{3}{2}R_3 : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}_3 \qquad \begin{pmatrix} 4 & -4 & -3 \\ 1 & -1 & -1 \\ \frac{3}{2} & -2 & -\frac{3}{2} \end{pmatrix} = A^{-1} \end{array}$$

As a final check we show that

$$AA^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & -2 \\ 1 & -4 & 0 \end{pmatrix} \begin{pmatrix} 4 & -4 & -3 \\ 1 & -1 & -1 \\ \frac{3}{2} & -2 & -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}_3 \quad \checkmark$$

and thus confirm that we have correctly computed the inverse of  $A$ .

### 3.4 Relation between linear maps and matrices

We have now fully understood linear maps between column vector spaces - they are described by matrices. The action of such linear maps is given by multiplication of a matrix with column vectors, composition of maps is via matrix multiplication and the inverse map corresponds to the inverse matrix. Further we have introduced the computational tools to work with matrices. However, we still do not have a clear picture of linear maps between arbitrary vector space and this is what we will analyze now.

Start with a linear map  $f : V \rightarrow W$  between two vector spaces  $V$  and  $W$  over  $F$  with dimensions  $n$  and  $m$ , respectively. We introduce a basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$  of  $V$  and a basis  $\mathbf{w}_1, \dots, \mathbf{w}_m$  of  $W$ . Then, all vectors in  $\mathbf{v} \in V$  and  $\mathbf{w} \in W$  can be written as linear combinations  $\mathbf{v} = \sum_{i=1}^n \alpha_i \mathbf{v}_i$  with coordinate vectors  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^T$  and  $\mathbf{w} = \sum_{j=1}^m \beta_j \mathbf{w}_j$  with coordinate vectors  $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m)^T$ , respectively. Following Example (3.4) (b), we can introduce coordinate maps  $\varphi : F^n \rightarrow V$  and  $\psi : F^m \rightarrow W$  relative to each basis which act as

$$\varphi(\boldsymbol{\alpha}) = \sum_{i=1}^n \alpha_i \mathbf{v}_i, \quad \psi(\boldsymbol{\beta}) = \sum_{j=1}^m \beta_j \mathbf{w}_j. \quad (3.79)$$

The images  $f(\mathbf{v}_i)$  of the  $V$  basis vectors can always be written as a linear combination of the basis vectors for  $W$  so we have

$$f(\mathbf{v}_j) = \sum_{i=1}^m a_{ij} \mathbf{w}_i \quad (3.80)$$

for some coefficients  $a_{ij} \in F$ . The situation so far can be summarized by the following diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \varphi \uparrow & & \uparrow \psi \\ F^n & \xrightarrow{A=?} & F^m \end{array} \quad (3.81)$$

Essentially, we are describing vectors by their coordinate vectors (relative to the chosen basis) and we would like to find a matrix  $A$  which acts on these coordinate vectors “in the same way” as the original linear map  $f$  on the associated vectors. In this way, we can describe the action of the linear map by a matrix. How do we find this matrix  $A$ ? Abstractly, it is given by

$$A = \psi^{-1} \circ f \circ \varphi, \quad (3.82)$$

as can be seen by going from  $F^n$  to  $F^m$  in the diagram (3.81) using the “upper path”, that is, via  $V$  and  $W$ . From Lemma 3.3 we know that we can work out the components of a matrix by letting it act on the standard unit vectors.

$$A\mathbf{e}_j \stackrel{(3.82)}{=} \psi^{-1} \circ f \circ \varphi(\mathbf{e}_j) \stackrel{(3.79)}{=} \psi^{-1} \circ f(\mathbf{v}_j) \stackrel{(3.80)}{=} \psi^{-1} \left( \sum_{i=1}^m a_{ij} \mathbf{w}_i \right) \stackrel{\text{linearity}}{=} \sum_{i=1}^m a_{ij} \psi^{-1}(\mathbf{w}_i) \stackrel{(3.79)}{=} \sum_{i=1}^m a_{ij} \tilde{\mathbf{e}}_i \quad (3.83)$$

Comparing with Lemma 3.3 it follows that  $a_{ij}$  are the entries of the desired matrix  $A$ . Also, Eq. (3.82) implies that  $\text{Im}(A) = \psi^{-1}(\text{Im}(f))$ . If we denote by  $\chi := \psi^{-1}|_{\text{Im}(f)}$  the restriction of  $\psi^{-1}$  to  $\text{Im}(f)$  we have  $\dim \text{Ker}(\chi) = 0$  since  $\psi^{-1}$  is an isomorphism and hence  $\dim \text{Ker}(\psi^{-1}) = 0$ . We conclude, using Theorem 3.2, that

$$\text{rk}(A) = \dim \text{Im}(A) = \text{rk}(\chi) = \dim \text{Im}(f) - \dim \text{Ker}(\chi) = \text{rk}(f), \quad (3.84)$$

which means that the linear map  $f$  and the matrix  $A$  representing  $f$  have the same rank, that is,  $\text{rk}(A) = \text{rk}(f)$ .

While this discussion might have been somewhat abstract it has a simple and practically useful conclusion. To find the matrix  $A$  which represents a linear map relative to a basis, compute the images  $f(\mathbf{v}_j)$  of the (domain) basis vectors and write them as a linear combinations of the (co-domain) basis vectors  $\mathbf{w}_i$ , as in Eq. (3.80). The coefficients in these linear combinations form the matrix  $A$ . More precisely, by careful inspection of the indices in Eq. (3.80), it follows that the coefficients which appear in the image of the  $j^{\text{th}}$  basis vector form the  $j^{\text{th}}$  column of the matrix  $A$ . We summarize these conclusions in

**Lemma 3.5.** *Let  $f : V \rightarrow W$  be a linear map,  $\mathbf{v}_1, \dots, \mathbf{v}_n$  a basis of  $V$  and  $\mathbf{w}_1, \dots, \mathbf{w}_m$  a basis of  $W$ . The entries  $a_{ij}$  of the  $m \times n$  matrix  $A$  which describes this linear map relative to this choice of basis can be read off from the images of the basis vectors as*

$$f(\mathbf{v}_j) = \sum_{i=1}^m a_{ij} \mathbf{w}_i. \quad (3.85)$$

We have  $\text{rk}(A) = \text{rk}(f)$  and, in particular,  $A$  is invertible if and only if  $f$  is.

The relation between linear maps and matrices is a key fact of linear algebra which we would like to illustrate with two examples.

**Example 3.13:** Relation between linear maps and matrices

(a) Consider the linear map  $B : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by the matrix

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}. \quad (3.86)$$

For simplicity, we choose the same basis for the domain and the co-domain, namely  $\mathbf{v}_1 = \mathbf{w}_1 = (1, 2)^T$  and  $\mathbf{v}_2 = \mathbf{w}_2 = (-1, 1)^T$ . Then, the images of the basis vectors under  $B$ , written as linear combinations of the same basis, are

$$B\mathbf{v}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix} = -1\mathbf{v}_1 - 2\mathbf{v}_2, \quad B\mathbf{v}_2 = \begin{pmatrix} -1 \\ -2 \end{pmatrix} = -1\mathbf{v}_1 + 0\mathbf{v}_2. \quad (3.87)$$

Arranging the coefficients from  $B\mathbf{v}_1$  into the first column of a matrix and the coefficients from  $B\mathbf{v}_2$  into the second column we find

$$B' = \begin{pmatrix} -1 & -1 \\ -2 & 0 \end{pmatrix}.sq \quad (3.88)$$

This is the matrix representing the linear map  $B$  relative to the basis  $\{\mathbf{v}_1, \mathbf{v}_2\}$ . It might be useful to be explicit about what exactly this means. Write an arbitrary 2-dimensional vector as

$$\begin{pmatrix} x \\ y \end{pmatrix} = x'\mathbf{v}_1 + y'\mathbf{v}_2 = \begin{pmatrix} x' - y' \\ 2x' + y' \end{pmatrix} \quad (3.89)$$

so that a vector  $(x, y)^T$  is described, relative to the basis  $\{\mathbf{v}_1, \mathbf{v}_2\}$ , by the coordinate vector  $(x', y')^T$ . Consider the example  $(x, y) = (1, 8)$  with associated coordinate vector  $(x', y') = (3, 2)$ . Then

$$\begin{array}{ccc} B & \begin{pmatrix} 1 \\ 8 \end{pmatrix} & = & \begin{pmatrix} 1 \\ -16 \end{pmatrix} \\ & \updownarrow & & \updownarrow \\ B' & \begin{pmatrix} 3 \\ 2 \end{pmatrix} & = & \begin{pmatrix} -5 \\ -6 \end{pmatrix} \end{array} \quad (3.90)$$

The vectors connected by arrows relate exactly as in Eq. (3.89), that is the vectors in the lower equation are the coordinate vectors of their counterparts in the upper equation. Eqs. (3.90) are basically a specific instance of the general diagram (3.81). This illustrates exactly how the representing matrix acts “in the same way” as the linear map: If the linear map relates two vectors then the representing matrix relates their two associated coordinate vectors.

(b) It might be useful to discuss a linear map which, originally, is not defined by a matrix. To this end, we consider the vector space  $V = \{a_2x^2 + a_1x + a_0 \mid a_i \in \mathbb{R}\}$  of all quadratic polynomials with real coefficients and the linear map  $f = \frac{d}{dx} : V \rightarrow V$ , that is, the linear map obtained by taking the first derivative. As before we choose the same basis for domain and co-domain, namely the standard basis  $1, x, x^2$  of monomials. We would like to find the matrix  $A$  representing the first derivative, relative to this basis.

As before, we work out the images of the basis vectors and write them as linear combinations of the same basis:

$$\begin{aligned} \frac{d}{dx}1 &= 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ \frac{d}{dx}x &= 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ \frac{d}{dx}x^2 &= 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 \end{aligned}$$

Arranging the coefficients in each row into the columns of a matrix we arrive at

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.91)$$

This matrix generates the first derivative of quadratic polynomials relative to the standard monomial basis. As before, let us be very explicit about what this means. Consider the polynomial  $p(x) = 5x^2 + 3x + 7$  with coordinate vector  $(7, 3, 5)^T$  and its first derivative  $p'(x) = 10x + 3$  with coordinate vector  $(3, 10, 0)^T$ . Then we have

$$A \begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ 0 \end{pmatrix}, \quad (3.92)$$

that is,  $A$  indeed maps the coordinate vector for  $p$  into the coordinate vector for  $p'$ .

The correspondence between operators acting on functions and matrices acting on vectors illustrated by this example is at the heart of quantum mechanics. Historically, Schrödinger’s formulation of quantum mechanics is in terms of (wave) functions and operators, while Heisenberg’s formulation is in terms of matrices. The relation between those two formulations is precisely as in the above example.

### 3.5 Change of basis

We have seen that a linear map can be described, relative to a basis in the domain and co-domain, by a matrix. It is clear from the previous discussion that, for a fixed linear map, this matrix depends on the specific choice of basis. In other words, if we choose another basis the matrix describing the same linear map will change. We would now like to work out how precisely the representing matrix transforms under a change of basis.

To simplify the situation, we consider a linear map  $f : V \rightarrow V$  from a vector space to itself and choose the same basis on domain and co-domain. (The general situation of a linear map between two different vector spaces is a straightforward generalization.) The two sets of basis vectors, coordinate maps and representing matrices are then denoted by

basis of $V$	coordinate map	coordinate vector	representing matrix
$\mathbf{v}_1, \dots, \mathbf{v}_n$	$\varphi(\alpha) = \sum_{i=1}^n \alpha_i \mathbf{v}_i$	$\alpha = (\alpha_1, \dots, \alpha_n)^T$	$A = \varphi^{-1} \circ f \circ \varphi$
$\mathbf{v}'_1, \dots, \mathbf{v}'_n$	$\varphi'(\alpha') = \sum_{i=1}^n \alpha'_i \mathbf{v}'_i$	$\alpha' = (\alpha'_1, \dots, \alpha'_n)^T$	$A' = \varphi'^{-1} \circ f \circ \varphi'$

(3.93)

We would like to find the relationship between  $A$  and  $A'$ , that is, between the representing matrices for  $f$  relative to the unprimed and the primed basis. From Eq. (3.82) we know that the two matrices can be written as  $A = \varphi^{-1} \circ f \circ \varphi$  and  $A' = \varphi'^{-1} \circ f \circ \varphi'$ , so that

$$\begin{aligned}
 A' &= \varphi'^{-1} \circ f \circ \varphi' = \varphi'^{-1} \circ \varphi \circ \varphi^{-1} \circ f \circ \varphi \circ \varphi^{-1} \circ \varphi' \\
 &= \underbrace{\varphi'^{-1} \circ \varphi}_=: P \circ \underbrace{\varphi^{-1} \circ f \circ \varphi}_= A \circ \underbrace{\varphi^{-1} \circ \varphi'}_=: P^{-1} = PAP^{-1}.
 \end{aligned}
 \tag{3.94}$$

Note that all we have done is to insert two identity maps,  $\varphi \circ \varphi^{-1}$ , in the second step and then combined maps differently in the third step. What is the interpretation of  $P = \varphi'^{-1} \circ \varphi$ ? For a given vector  $\mathbf{v} \in V$  and its coordinate vectors  $\alpha = \varphi^{-1}(\mathbf{v})$  and  $\alpha' = \varphi'^{-1}(\mathbf{v})$  relative to the unprimed and primed basis we have  $\alpha' = \varphi'^{-1}(\mathbf{v}) = \varphi'^{-1} \circ \varphi(\alpha) = P\alpha$ , so in summary

$$\alpha' = P\alpha. \tag{3.95}$$

Hence,  $P$  converts unprimed coordinate vectors  $\alpha$  into the corresponding primed coordinate vector  $\alpha'$  and, as a linear map between column vector, it is a matrix. In short,  $P$  describes the change of basis under consideration. The corresponding transformation of the representing matrix under this basis change is then

$$A' = PAP^{-1}. \tag{3.96}$$

This is one of the key equations of linear algebra. For example, we can ask if we can choose a basis for which the representing matrix is particularly simple. Eq. (3.96) is the starting point for answering this question to which we will return later. Note that Eq. (3.96) makes intuitive sense. Acting with the equation on a primed coordinate vector  $\alpha'$ , the first we obtain on the RHS is  $P^{-1}\alpha'$ . This is the corresponding unprimed coordinate vector on which the matrix  $A$  can sensibly act, thereby converting it into another unprimed coordinate vector. The final action of  $P$  converts this back into a primed coordinate vector. Altogether, this is the action of the matrix  $A'$  on  $\alpha'$  as required by the equation.

Another way to think about the matrix  $P$  is by relating the primed and the unprimed basis vectors. In general, from Lemma 3.3, we can write  $P\mathbf{e}_j = \sum_i P_{ij}\mathbf{e}_i$ . Multiplying this equation with  $\varphi'$  from the left and using  $\mathbf{v}_j = \varphi(\mathbf{e}_j)$ ,  $\mathbf{v}'_i = \varphi'(\mathbf{e}_i)$  we find

$$\mathbf{v}_j = \sum_i P_{ij}\mathbf{v}'_i \iff \mathbf{v}'_j = \sum_i (P^{-1})_{ij}\mathbf{v}_i. \tag{3.97}$$



Hence, the entries of the matrix  $P$  can be calculated by expanding the unprimed basis vectors in terms of the primed basis.

**Example 3.14:** Basis transformation of a matrix

Relative to the unprimed basis  $\mathbf{v}_1 = \mathbf{e}_1$ ,  $\mathbf{v}_2 = \mathbf{e}_2$  of standard unit vectors, a linear map is described by the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We would like to determine the matrix  $A'$  which describes the same linear map relative to the basis

$$\mathbf{v}'_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}'_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

One way to proceed is as before, by applying Lemma 3.5, and compute the images of the basis vectors in order to read off  $A'$ . This leads to

$$A\mathbf{v}'_1 = 0\mathbf{v}'_1 + 1\mathbf{v}'_2, \quad A\mathbf{v}'_2 = 1\mathbf{v}'_1 + 0\mathbf{v}'_2,$$

and arranging the coefficients on the right-hand sides into the column of a matrix gives

$$A' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.98)$$

Alternatively, we should be able to determine  $A'$  from Eq. (3.96). To work out the relation between the primed and un-primed coordinate vectors  $\boldsymbol{\alpha}' = (\alpha'_1, \alpha'_2)^T$  and  $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)^T$  we write

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \stackrel{!}{=} \alpha'_1\mathbf{v}'_1 + \alpha'_2\mathbf{v}'_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha'_1 + \alpha'_2 \\ -\alpha'_1 + \alpha'_2 \end{pmatrix}.$$

Comparing this with the general relation (3.95) between the coordinate vectors we can read off the coordinate transformation  $P^{-1}$  as

$$P^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Applying the basis transformation (3.96) with this matrix  $P$  we find

$$A' = PAP^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

in accordance with the earlier result (3.98).

## 4 Systems of linear equations

We will now apply our general results and methods to the problem of solving linear equations. This will lead to an understanding of the structure of the solutions and to explicit solution methods.

### 4.1 General structure of solutions

Consider a linear map  $f : V \rightarrow W$ . We are looking for all solutions  $\mathbf{x} \in V$  of the equation

$$f(\mathbf{x}) = \mathbf{b} \tag{4.1}$$

where  $\mathbf{b} \in W$  is a fixed vector. For  $\mathbf{b} \neq \mathbf{0}$  this is called an *inhomogenous linear equation* and

$$f(\mathbf{x}) = \mathbf{0} \tag{4.2}$$

is the *associated homogenous equation*. Its general solution is  $\text{Ker}(f)$ . The solutions of the inhomogeneous and associated homogeneous equations are related in an interesting way.

**Lemma 4.1.** *If  $\mathbf{x}_0 \in V$  solves the inhomogenous equation, that is  $f(\mathbf{x}_0) = \mathbf{b}$ , then the affine space*

$$\mathbf{x}_0 + \text{Ker}(f)$$

*is the general solution of the inhomogenous equation.*

*Proof.* If  $\mathbf{x}$  is a solution to  $f(\mathbf{x}) = \mathbf{b}$  then  $f(\mathbf{x} - \mathbf{x}_0) = f(\mathbf{x}) - f(\mathbf{x}_0) = \mathbf{b} - \mathbf{b} = \mathbf{0}$ , so  $\mathbf{x} - \mathbf{x}_0 \in \text{Ker}(f)$ . Conversely, if  $\mathbf{x} \in \mathbf{x}_0 + \text{Ker}(f)$ , then we can write  $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}$  for some vector  $\mathbf{v} \in \text{Ker}(f)$ . Then,  $f(\mathbf{x}) = f(\mathbf{x}_0 + \mathbf{v}) = f(\mathbf{x}_0) + f(\mathbf{v}) = \mathbf{b} + \mathbf{0} = \mathbf{b}$ .  $\square$

In short, the Lemma says that the general solution of the inhomogeneous equation is obtained by the sum of a special solution to the inhomogeneous equation and all solutions to the homogeneous equation. Recall that  $\text{Ker}(f)$  is a sub vector space, so a line, a plane, etc. through  $\mathbf{0}$  with dimension  $\dim(\text{Ker}(f)) = \dim(V) - \text{rf}(f)$  (see Eq. (3.4)). This shows that the geometry of the solution is schematically as indicated in Fig. 20. Lemma (4.1) is helpful in order to find the general solution to inhomogenous, linear differential equations as in the following

**Example 4.1:** Solution to inhomogenous linear differential equation.

The previous Lemma has a prominent application to inhomogeneous, linear (second order) differential equations, that is differential equations for  $y = y(x)$  of the form

$$p(x) \frac{d^2 y}{dx^2} + q(x) \frac{dy}{dx} + r(x)y = s(x) ,$$

where  $p, q, r$  and  $s$  are fixed functions. The relevant vector space is the space of (infinitely many times) differentiable functions, the linear map  $f$  corresponds to the linear differential operator  $p(x) \frac{d^2}{dx^2} + q(x) \frac{d}{dx} + r(x)$  and the inhomogeneity  $\mathbf{b}$  given by  $s(x)$ . From Lemma 4.1, the general solution to this equation can be obtained by finding a special solution,  $y_0$ , and then adding to it all solutions of the associated homogeneous equation

$$p(x) \frac{d^2 y}{dx^2} + q(x) \frac{dy}{dx} + r(x)y = 0 .$$

To be specific consider the differential equation

$$\frac{d^2 y}{dx^2} + y = x .$$

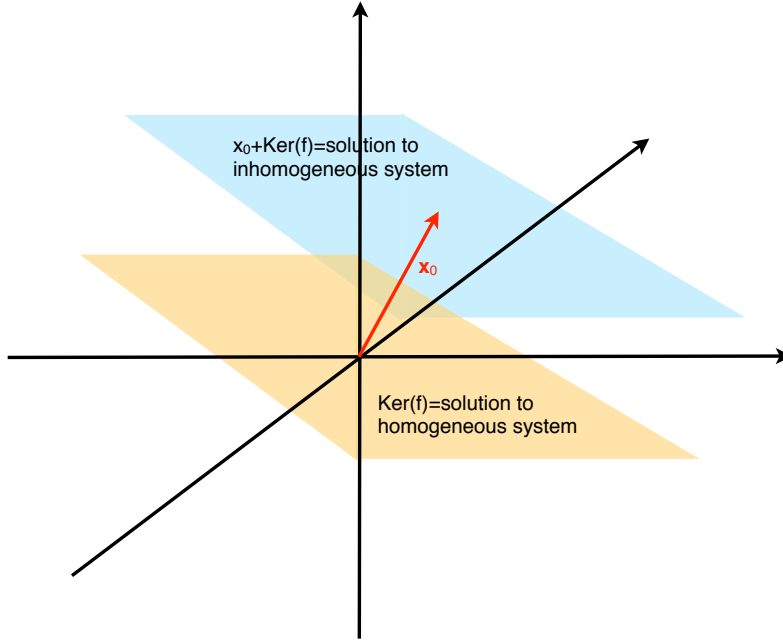


Figure 20: Solutions to homogeneous and inhomogeneous linear equations.

An obvious special solution is the function  $y_0(x) = x$ . The general solution of the associated homogeneous equation

$$\frac{d^2y}{dx^2} + y = 0$$

is  $a \sin(x) + b \cos(x)$  for arbitrary real constants  $a, b$ . Hence, the general solution to the inhomogeneous equation is

$$y(x) = x + a \sin(x) + b \cos(x).$$

Our main interest is of course in systems of linear equations, that is, the case where the linear map is an  $m \times n$  matrix  $A : F^n \rightarrow F^m$  with entries  $a_{ij}$ . For  $\mathbf{x} = (x_1, \dots, x_n)^T \in F^n$  and a fixed vector  $\mathbf{b} = (b_1, \dots, b_m)^T \in F^m$  the system of linear equations can be written as

$$\begin{array}{rcc}
 \mathbf{Ax} = \mathbf{b} & \text{or} & \begin{array}{cccc}
 a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
 \vdots & & \vdots & & \vdots & & \vdots \\
 a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & = & b_m
 \end{array} \\
 & & & & & & (4.3)
 \end{array}$$

This is a system of  $m$  equations in  $n$  variables with associated homogeneous system

$$\begin{array}{rcc}
 \mathbf{Ax} = \mathbf{0} & \text{or} & \begin{array}{cccc}
 a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & 0 \\
 \vdots & & \vdots & & \vdots & & \vdots \\
 a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & = & 0
 \end{array} \\
 & & & & & & (4.4)
 \end{array}$$

The solution space of the homogeneous system is  $\text{Ker}(A)$ , a (sub) vector space whose dimension is given by the dimension formula  $\dim \text{Ker}(A) = n - \text{rk}(A)$  (see Eq. (3.4)). If the inhomogeneous system has a solution,  $\mathbf{x}_0$ , then its general solution is  $\mathbf{x}_0 + \text{Ker}(A)$  and such a “special” solution  $\mathbf{x}_0$  exists if and only if

$\mathbf{b} \in \text{Im}(A)$ . If  $\text{rk}(A) = m$  then  $\text{Im}(A) = F^m$  and a solution exists for any choice of  $\mathbf{b}$ . On the other hand, if  $\text{rk}(A) < m$ , there is no solution for “generic” choices of  $\mathbf{b}$ . For example, if  $m = 3$  and  $\text{rk}(A) = 2$  then the image of  $A$  is a plane in a three-dimensional space and we need to choose  $\mathbf{b}$  to lie in this plane for a solution to exist. Clearly this corresponds to a very special choice of  $\mathbf{b}$  and generic vectors  $\mathbf{b}$  will not lie in this plane. To summarize the general structure of the solution to  $A\mathbf{x} = \mathbf{b}$ , where  $A$  is an  $m \times n$  matrix, we should, therefore distinguish two cases.

(1)  $\text{rk}(A) = m$

In this case there exists a solution,  $\mathbf{x}_0$ , for any choice of  $\mathbf{b}$  and the general solution is given by

$$\mathbf{x}_0 + \text{Ker}(A) \tag{4.5}$$

The number of free parameters in this solution equals  $\dim \text{Ker}(A) = n - \text{rk}(A) = n - m$ .

(2)  $\text{rk}(A) < m$

(a) If  $\mathbf{b} \in \text{Im}(A)$  we have a solution with  $\dim \text{Ker}(A) = n - \text{rk}(A)$  free parameters.

(b) If  $\mathbf{b} \notin \text{Im}(A)$  there is no solution.

For a quadratic  $n \times n$  matrix  $A$  we can be slightly more specific and the above cases are as follows.

(1)  $\text{rk}(A) = n$

A solution exists for any choice of  $\mathbf{b}$  and there are no free parameters since  $\dim \text{Ker}(A) = n - n = 0$ . Hence, the solution is unique. Indeed, in this case, the matrix  $A$  is invertible (see Lemma 3.4) and the unique solution is given by  $\mathbf{x} = A^{-1}\mathbf{b}$ .

(2)  $\text{rk}(A) < n$

(a) If  $\mathbf{b} \in \text{Im}(A)$  we have a solution with  $n - \text{rk}(A)$  free parameters.

(b) If  $\mathbf{b} \notin \text{Im}(A)$  there is no solution.

The main message of this discussion is that, given the size of the matrix  $A$  and its rank, we are able to draw a number of conclusions about the qualitative structure of the solution, without any explicit calculation. We will see below how this can be applied to explicit examples.

We can also think about the solutions to a system of linear equations in a geometrical way. With the row vectors  $\mathbf{A}_i$  of the matrix  $A$ , the linear system (4.3) can be re-written as  $m$  equations for (hyper) planes (that is  $n - 1$ -dimensional planes) in  $n$  dimensions:

$$\mathbf{A}_i \cdot \mathbf{x} = b_i, \quad i = 1, \dots, m. \tag{4.6}$$

Geometrically, we should then think of the solutions to the linear system as the common intersection of these  $m$  (hyper) planes. For example, if we consider a  $3 \times 3$  matrix we should consider the common intersection of three planes in three dimensions. Clearly, depending on the case, these planes can intersect in a point, a line, a plane or not intersect at all. In other words, we may have no solution or the solution may have 0, 1 or 2 free parameters. This corresponds precisely to the cases discussed above.

## 4.2 Solution by ”explicit calculation”

We begin our discussion of solution methods and examples with the most basic approach: Explicit calculation by which we mean the addition of suitable multiples of the various equations so solve for the

variables. To be specific, we consider the following system with three variables  $\mathbf{x} = (x, y, z)^T$  and three equations

$$\begin{aligned} E_1 : \quad & 2x + 3y - z = -1 \\ E_2 : \quad & -x - 2y + z = 3 \\ E_3 : \quad & ax + y - 2z = b. \end{aligned} \tag{4.7}$$

To make matters more interesting, we have introduced two parameters  $a, b \in \mathbb{R}$ . We would like to find the solution to this system for arbitrary real values of these parameters. We can also write the above system in matrix form,  $A\mathbf{x} = \mathbf{b}$ , with

$$A = \begin{pmatrix} 2 & 3 & -1 \\ -1 & -2 & 1 \\ a & 1 & -2 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ b \end{pmatrix}. \tag{4.8}$$

Before we embark on the explicit calculation, let us apply the results of our previous general discussion and predict the qualitative structure of the solution. The crucial piece of information required for this discussion is the rank of the matrix  $A$ . Of course, this can be determined from the general methods based on row reduction which we have introduced in Section 3.3. But, as explained before, for small matrices the rank can often be inferred “by inspection”. For the matrix  $A$  in (4.8) it is clear that the second and third column vectors,  $\mathbf{A}^2$  and  $\mathbf{A}^3$ , are linearly independent. Hence, its ranks is at least two. The first column vector,  $\mathbf{A}^1$ , depends on the parameter  $a$  so we have to be more careful. For generic  $a$  values  $\mathbf{A}^1$  does not lie in the plane spanned by  $\mathbf{A}^2, \mathbf{A}^3$ , so the generic rank of  $A$  is three. In this case, from our general results, there is a unique solution to the linear system for any value of the other parameter  $b$ . For a specific  $a$  value  $\mathbf{A}^1$  will be in the plane spanned by  $\mathbf{A}^2, \mathbf{A}^3$  and the rank is reduced to two. Then, the image of  $A$  is two-dimensional, that is a plane. For generic values of  $b$  the vector  $\mathbf{b}$  will not lie in this plane so there is no solution. However, for a specific  $b$  value, when  $\mathbf{b}$  does lie in this plane, there is a solution with  $\dim \text{Ker}(A) = 3 - \text{rk}(A) = 1$  parameter, that is, a solution line. So, in summary we expect the following qualitative structure for the solution to the system (4.7).

- 1) For generic values of  $a$  the rank of  $A$  is three and there is a unique solution for all values of  $b$ .
- 2a) For a specific value of  $a$  (when  $\text{rk}(A) = 2$ ) and for a specific value of  $b$  there is a line of solutions.
- 2b) For the above specific value of  $a$  and generic  $b$  there is no solution.

Let us now confirm this expectation by an explicit calculation. We begin by adding appropriate multiples of Eqs. (4.7), namely

$$E_1 + E_2 : \quad x + y = 2 \tag{4.9}$$

$$E_3 + 2E_2 : \quad (a - 2)x - 3y = b + 6. \tag{4.10}$$

Eliminating  $y$  from these two equations then leads to

$$(a + 1)x = b + 12. \tag{4.11}$$

This equation allows us to explicitly identify the various cases we expect.

- 1)  $a \neq -1$ : We can divide Eq. (4.11) by  $(a + 1)$  to solve for  $x$  and then insert into Eq. (4.9) and the first Eq. (4.7) to get  $y$  and  $z$ . So in this case we have a unique solution for any  $b$  given by

$$x = \frac{b + 12}{a + 1}, \quad y = \frac{2a - b - 10}{a + 1}, \quad z = \frac{7a - b - 5}{a + 1}. \tag{4.12}$$

2a)  $a = -1$  and  $b = -12$ : In this case, Eq. (4.11) becomes trivial and we are left with only two independent equations. Solving Eq. (4.9) and the first Eq. (4.7) for  $x$  and  $z$  in terms of  $y$  we find

$$x = 2 - y \quad z = 5 + y . \quad (4.13)$$

that is, a line of solutions parametrized by  $y$ .

2b)  $a = -1$  and  $b \neq -12$ : In this case, Eq. (4.11) leads to a contradiction so there is no solution.

### 4.3 Solution by row reduction

While “explicit calculation” as in the previous sub-section is probably the fastest “by hand” method for relatively small systems, larger linear systems require a more systematic method. For a specific case, a linear system  $\mathbf{Ax} = \mathbf{b}$  with a quadratic and invertible matrix  $A$ , we already know how this works. The unique solution in this case is  $\mathbf{x} = A^{-1}\mathbf{b}$  and the inverse of  $A$  can be computed by the row reduction method introduced in Section (3.3). We will now generalize this method so it can be applied to all linear systems.

So let us start with an arbitrary linear system with  $m$  equations for  $n$  variables, so a system of the form  $\mathbf{Ax} = \mathbf{b}$  with an  $m \times n$  matrix  $A$ , inhomogeneity  $\mathbf{b} \in F^m$  and variables  $\mathbf{x} \in F^n$ . We can multiply the linear system with one of the  $m \times m$  matrices  $P$  from Eq. (3.73), generating the elementary row operations, to get the linear system  $P\mathbf{Ax} = P\mathbf{b}$ . This new system has the same solutions as the original one since  $P$  is invertible. This means we do not change the solutions to the linear system if we carry out elementary row operations simultaneously on the matrix  $A$  and the inhomogeneity  $\mathbf{b}$ . This suggests we should encode the linear system by the *augmented matrix* defined by

$$A' = (A|\mathbf{b}) , \quad (4.14)$$

an  $m \times (n+1)$  matrix which consists of  $A$  plus one additional column formed by the vector  $\mathbf{b}$ . We can now reformulate our previous observation by stating that elementary row operations applied to the augmented matrix do not change the solutions of the associated linear system. So our solution strategy will be to simplify the augmented matrix by successive elementary row operations until the solution can be easily “read off”. Before we formulate this explicitly, we note a useful criterion which helps us to decide whether or not  $\mathbf{b} \in \text{Im}(A)$ , that is, whether or not the linear system has solutions.

**Lemma 4.2.**  $\mathbf{b} \in \text{Im}(A) \iff \text{rk}(A) = \text{rk}(A')$

*Proof.* “ $\Rightarrow$ ”: If  $\mathbf{b} \in \text{Im}(A)$  it is a linear combination of the column vectors of  $A$  and adding it to the matrix does not increase the rank.

“ $\Leftarrow$ ”: If  $\text{rk}(A) = \text{rk}(A')$  the rank does not increase when  $\mathbf{b}$  is added to the matrix. Therefore,  $\mathbf{b} \in \text{Span}(\mathbf{A}^1, \dots, \mathbf{A}^n) = \text{Im}(A)$ .  $\square$

Let us now describe the general algorithm.

1. Apply row operations, as described in Section (3.3), to the augmented matrix  $A'$  until the matrix  $A$  within  $A'$  is in upper echelon form. Then, the resulting matrix has the form

$$A' \rightarrow \left( \begin{array}{cccc|c} \dots & \boxed{a_{1j_1}} & & * & b'_1 \\ & & \boxed{a_{2j_2}} & \ddots & \vdots \\ & & & \ddots & \vdots \\ & & & & \vdots \\ & & & & \vdots \\ & & & & \vdots \\ 0 & \ddots & & \boxed{a_{rj_r}} \dots & b'_r \\ & & & & b'_{r+1} \\ & & & & \vdots \\ & & & & b'_n \end{array} \right)$$

where  $a_{ij_i} \neq 0$  for  $i = 1, \dots, r$  so that  $A$  has rank  $r$ . In this form it is easy to apply the criterion, Lemma 4.2. If  $b'_i \neq 0$  for any  $i > r$  then  $\text{rk}(A') > \text{rk}(A)$  and the linear system has no solutions. In this case we can stop. On the other hand, if  $b'_i = 0$  for all  $i > r$  which we assume from hereon, then  $\text{rk}(A') = \text{rk}(A)$  and the system has a solution.

- As explained we assume that  $b'_i = 0$  for all  $i > r$ . For ease of notation we also permute the columns of  $A$  (this corresponds to a permutation of the variables that we will have to keep track of) so that the columns with the non-zero entries  $a_{ij_i}$  become the first  $r$  of the matrix. The result is

$$A' \rightarrow \left( \begin{array}{cccc|cc} a_{1j_1} & & & & b'_1 & \\ & a_{2j_2} & & * & * & \vdots \\ & & \ddots & & & \vdots \\ 0 & & & a_{rj_r} & & b'_r \\ & & & & 0 & 0 \\ & & & & & \vdots \\ & & & & & 0 \end{array} \right)$$

- By further row operations we can convert the  $r \times r$  matrix in the upper left corner of the previous matrix into a unit matrix  $\mathbb{1}_r$ . Schematically, the result is

$$A'_{\text{fin}} = \left( \begin{array}{c|c|c} \mathbb{1}_r & B & \mathbf{c} \\ \hline 0 & 0 & 0 \end{array} \right) \quad (4.15)$$

where  $B$  is an  $r \times (n - r)$  matrix and  $\mathbf{c}$  is an  $r$ -dimensional column vector.

- Recall that  $r = \text{rk}(A)$  is the rank and  $n - r = \dim \text{Ker}(A)$  is the number of free parameters of the solution. For this reason it makes sense to split our variables as

$$\mathbf{x} = \begin{pmatrix} \boldsymbol{\xi} \\ \mathbf{t} \end{pmatrix} \quad (4.16)$$

into an  $r$ -dimensional vector  $\boldsymbol{\xi}$  and an  $(n - r)$ -dimensional vector  $\mathbf{t}$ . Note that this split is adapted to the form of the matrix  $A'_{\text{fin}}$  so that the associated linear system takes the simple form

$$\boldsymbol{\xi} + B\mathbf{t} = \mathbf{c}. \quad (4.17)$$

The point is that this system can be easily solved for  $\boldsymbol{\xi}$  in terms of  $\mathbf{t}$ . This leads to the general solution

$$\mathbf{x} = \begin{pmatrix} \mathbf{c} - B\mathbf{t} \\ \mathbf{t} \end{pmatrix}, \quad (4.18)$$

which depends on  $n - r$  free parameters  $\mathbf{t}$ , as expected.

Let us see how this works for an explicit example.

**Example 4.2:** Solving linear systems with row reduction of the augmented matrix

Consider the following system of linear equations and its augmented matrix

$$\begin{array}{rcl} x + y - 2z & = & 1 \\ 2x - y + 3z & = & 0 \\ -x - 4y + 9z & = & b \end{array} \quad A' = \left( \begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 2 & -1 & 3 & 0 \\ -1 & -4 & 9 & b \end{array} \right), \quad (4.19)$$

where  $b \in \mathbb{R}$  is an arbitrary real parameter. We proceed in the four steps outlined above.

1. First we bring  $A$  within  $A'$  into upper echelon form which results in

$$A' \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -3 & 7 & -2 \\ 0 & 0 & 0 & b+3 \end{array} \right)$$

For  $b \neq -3$  we have  $\text{rk}(A') = 3 > 2 = \text{rk}(A)$  so there are no solutions. So we assume from hereon that  $b = -3$ .

2. Setting  $b = -3$  we have

$$\left( \begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -3 & 7 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

In this case, we do not have to permute columns since the (two) steps of the upper echelon form already arise in the first two columns.

3. By further elementary row operations we convert the  $2 \times 2$  matrix in the upper left corner into a unit matrix.

$$A'_{\text{fin}} = \left( \begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{7}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array} \right)$$

4. We have  $r = \text{rk}(A) = 2$  and  $\dim \text{Ker}(A) = 3 - \text{rk}(A) = 1$  so we expect a solution with one free variable  $t$  (a line). Accordingly, we split the variables as

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ t \end{pmatrix}, \quad (4.20)$$

where  $\boldsymbol{\xi} = (x, y)^T$  in our general notation. Writing the linear system for  $A'_{\text{fin}}$  in those variables results in

$$x + \frac{1}{3}t = \frac{1}{3} \quad (4.21)$$

$$y - \frac{7}{3}t = \frac{2}{3} \quad (4.22)$$

This can be easily solved for  $x, y$  in terms of  $t$  which was really the point of the exercise. The result is  $x = \frac{1}{3} - \frac{1}{3}t$  and  $y = \frac{2}{3} + \frac{7}{3}t$  and, inserting into Eq. (4.20), this results in the vector form

$$\mathbf{x} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{1}{3} \\ \frac{7}{3} \\ 1 \end{pmatrix}$$

for the line of solutions.

### Application: Linear algebra and circuits

Electrical circuits with batteries and resistors, such as the circuit in Fig. 21, can be described using methods





For example, for resistances  $(R_1, \dots, R_6) = (3, 10, 4, 2, 5, 1)$  (in units of Ohm) we have the resistance matrix

$$R = \begin{pmatrix} 17 & -10 & -4 \\ -10 & 13 & -1 \\ -4 & -1 & 10 \end{pmatrix}. \quad (4.27)$$

For a battery voltage  $V = 12$  (in units of volt) we can write down the augmented matrix

$$R' = \left( \begin{array}{ccc|c} 17 & -10 & -4 & 12 \\ -10 & 13 & -1 & 0 \\ -4 & -1 & 10 & 0 \end{array} \right), \quad (4.28)$$

and solve the linear system by row reduction. This leads to the solution

$$\mathbf{I} = \frac{1}{905} \begin{pmatrix} 1548 \\ 1248 \\ 744 \end{pmatrix} \quad (4.29)$$

for the currents (in units of Ampere).

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## 5 Determinants

Determinants are multi-linear objects and are a useful tool in linear algebra. In Section 2 we have introduced the three-dimensional determinant as the triple product of three vectors. Here we will study the generalization to arbitrary dimensions and verify that the three-dimensional case coincides with our previous definition. As with the other general concepts, we first define the determinant by its properties before we derive its explicit form and study a few applications. In our discussion in Section 2 we have observed that the three-dimensional determinant is linear in each of its vector arguments (see Eq. (2.44)), it changes sign when two vector arguments are swapped (see Eq. (2.46)) and the determinant of the three standard units vector is one (see Eq. (2.48)). We will now use these properties to define the determinant in arbitrary dimensions.

### 5.1 Definition of a determinant

**Definition 5.1.** A determinant maps  $n$  vectors  $\mathbf{a}_1, \dots, \mathbf{a}_n \in F^n$  to a number, denoted  $\det(\mathbf{a}_1, \dots, \mathbf{a}_n) \in F$ , such that the following properties are satisfied:

- (D1)  $\det(\dots, \alpha \mathbf{a} + \beta \mathbf{b}, \dots) = \alpha \det(\dots, \mathbf{a}, \dots) + \beta \det(\dots, \mathbf{b}, \dots)$   
This means the determinant is linear in each argument.
- (D2)  $\det(\dots, \mathbf{a}, \dots, \mathbf{b}, \dots) = -\det(\dots, \mathbf{b}, \dots, \mathbf{a}, \dots)$   
This means the determinant is completely anti-symmetric.
- (D3)  $\det(\mathbf{e}_1, \dots, \mathbf{e}_n) = 1$   
The determinant of the standard unit vectors is one.

The determinant of an  $n \times n$  matrix  $A$  is defined as the determinant of its column vectors, so  $\det(A) := \det(\mathbf{A}^1, \dots, \mathbf{A}^n)$ .

An easy but important conclusion from these properties is that a determinant with two same arguments must vanish. Indeed, from the anti-symmetry property (D2) it follows that  $\det(\dots, \mathbf{a}, \dots, \mathbf{a}, \dots) = -\det(\dots, \mathbf{a}, \dots, \mathbf{a}, \dots)$ , which means that

$$\det(\dots, \mathbf{a}, \dots, \mathbf{a}, \dots) = 0. \quad (5.1)$$

We know that an object with these properties exists for  $n = 3$  but not yet in other dimensions. To address this problem we first need to understand a few basic facts about permutations. Here, we will just present a brief account of the relevant facts. For the formal-minded, Appendix B contains a more complete treatment which includes the relevant proofs.

#### Permutations

You probably have an intuitive understanding of a permutation as an operation which changes the order of a certain set of  $n$  objects. Here, we take this set to be the numbers  $\{1, \dots, n\}$ . Mathematically, a permutation is defined as a bijective map from this set to itself. So the set of all permutations of  $n$  objects is given by

$$S_n := \{\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\} \mid \sigma \text{ is bijective}\}, \quad (5.2)$$

and this set has  $n!$  elements. The basic idea is that, under a permutation  $\sigma \in S_n$ , a number  $i \in \{1, \dots, n\}$  is permuted to its image  $\sigma(i)$ . A useful notation for a permutation mapping  $1 \rightarrow \sigma(1), \dots, n \rightarrow \sigma(n)$  is <sup>3</sup>

$$\sigma = \begin{pmatrix} 1 & \dots & n \\ \sigma(1) & \dots & \sigma(n) \end{pmatrix}. \quad (5.3)$$

---

<sup>3</sup>Note, despite the similar notation, this is not a matrix in the sense introduced earlier.

For example, for  $n = 3$ , a permutation which swaps 2 and 3 is written as

$$\tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}. \quad (5.4)$$

Carrying out two permutations, one after the other, simply corresponds to composition of maps in this formalism. For example, consider a second permutation

$$\tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad (5.5)$$

of three objects which swaps the numbers 1 and 2. Permuting first with  $\tau_2$  and then with  $\tau_1$  corresponds to the permutation  $\sigma := \tau_1 \circ \tau_2$  which is given by

$$\sigma = \tau_1 \circ \tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad (5.6)$$

a cyclic permutation of the three numbers. A further advantage of describing permutations as bijective maps is that the inverse of a permutation  $\sigma$ , that is, the permutation which “undoes” the effect of the original permutation, is simple described by the inverse map  $\sigma^{-1}$ .

The specific permutations which only swap two numbers and leave all other numbers unchanged are called *transpositions*. For example, the permutations (5.4) and (5.5) are transpositions. A basic and important fact about permutations, proved in Appendix B, is that every permutation can be written as a composition of transpositions, so any  $\sigma \in S_n$  can be written as  $\sigma = \tau_1 \circ \dots \circ \tau_k$ , where  $\tau_1, \dots, \tau_k \in S_n$  are transpositions. Eq. (5.6) is an illustration of this general fact.

Writing permutations as a composition of transpositions is not unique, that is, two different such compositions can lead to the same permutation. Not even the number of transpositions required to generate a given permutation is fixed. For example, the permutation  $\sigma$  in Eq. (5.6) can also be written as  $\sigma = \tau_1 \circ \tau_2 \circ \tau_1 \circ \tau_1$ , that is, as a composition of four transpositions. However, it can be shown (see Appendix B) that the number of transpositions required is always either even or odd for a given permutation. For a permutation  $\sigma = \tau_1 \circ \dots \circ \tau_k$ , written as a composition of  $k$  transpositions, it, therefore, makes sense to define the *sign of the permutation* as

$$\text{sgn}(\sigma) := (-1)^k = \begin{cases} +1 & : \text{“even” permutation} \\ -1 & : \text{“odd” permutation} \end{cases}. \quad (5.7)$$

From this definition, transpositions  $\tau$  are odd permutations, so  $\text{sgn}(\tau) = -1$ . For the permutation  $\sigma$  in Eq. (5.6) we have  $\text{sgn}(\sigma) = 1$  since it can be built from two transpositions. It is, therefore, even as we would expect from a cyclic permutation of three objects. In essence, the definition (5.7) provides the correct mathematical way to distinguish even and odd permutations.

When two permutations, each written in terms of transpositions, are composed with each other the number of transpositions simply adds up. From the definition (5.7) this means that

$$\text{sgn}(\sigma_1 \circ \sigma_2) = \text{sgn}(\sigma_1)\text{sgn}(\sigma_2). \quad (5.8)$$

A direct consequence of this rule is that  $1 = \text{sgn}(\sigma \circ \sigma^{-1}) = \text{sgn}(\sigma)\text{sgn}(\sigma^{-1})$  and, hence,

$$\text{sgn}(\sigma^{-1}) = \text{sgn}(\sigma). \quad (5.9)$$

In other words, a permutation and its inverse have the same sign.

We are now ready to return to determinants and derive an explicit formula. We start with an  $n \times n$  matrix  $A$  with entries  $a_{ij}$  whose column vectors we write as linear combinations of the standard unit vectors:

$$\mathbf{A}^i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix} = \sum_j a_{ji} \mathbf{e}_j. \quad (5.10)$$

By using the properties of the determinant from Def. (5.1) we can then attempt to work out the determinant of  $A$ . We find

$$\begin{aligned} \det(A) &= \det(\mathbf{A}^1, \dots, \mathbf{A}^n) \stackrel{(5.10)}{=} \det \left( \sum_{j_1=1}^n a_{j_1 1} \mathbf{e}_{j_1}, \dots, \sum_{j_n=1}^n a_{j_n n} \mathbf{e}_{j_n} \right) \stackrel{(D1)}{=} \sum_{j_1, \dots, j_n} a_{j_1 1} \cdots a_{j_n n} \det(\mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_n}) \\ &\stackrel{(5.1), j_a = \sigma(a)}{=} \sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n} \det(\mathbf{e}_{\sigma(1)}, \dots, \mathbf{e}_{\sigma(n)}) \stackrel{(D2)}{=} \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n} \det(\mathbf{e}_1, \dots, \mathbf{e}_n) \\ &\stackrel{(D3)}{=} \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n} \end{aligned}$$

Hence, having just used the general properties of determinants and some facts about permutations, we have arrived at a unique expression for the determinant. Conversely, it is straightforward to show that this expression satisfies all the requirements of Def. 5.1. In summary, we conclude that the determinant, as defined in Def. 5.1, is unique and explicitly given by

$$\det(A) = \det(\mathbf{A}^1, \dots, \mathbf{A}^n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n}, \quad (5.11)$$

where  $a_{ij}$  are the entries of the  $n \times n$  matrix  $A$ . Note that the sum on the RHS runs over all permutations in  $S_n$  and, therefore, has  $n!$  terms. A useful way to think about this sum is as follows. From each column of the matrix  $A$ , choose one entry such that no two entries lie in the same row. A term in Eq. (5.11) consists of the product of these  $n$  entries (times the sign of the permutation involved) and the sum amounts to all possible ways of making this choice.

Another useful way to write the determinant which is often employed in physics involves the  $n$ -dimensional generalization of the Levi-Civita tensor, defined by

$$\epsilon_{i_1 \dots i_n} = \begin{cases} +1 & \text{if } i_1, \dots, i_n \text{ is an even permutation of } 1, \dots, n \\ -1 & \text{if } i_1, \dots, i_n \text{ is an odd permutation of } 1, \dots, n \\ 0 & \text{otherwise} \end{cases}. \quad (5.12)$$

Essentially, the Levi-Civita tensor plays the same role as the sign of the permutation (plus it vanishes if it has an index appearing twice when  $i_1, \dots, i_n$  is not actually a permutation of  $1, \dots, n$ ) so that Eq. (5.11) can alternatively be written as

$$\det(A) = \epsilon_{i_1 \dots i_n} a_{i_1 1} \cdots a_{i_n n}, \quad (5.13)$$

with a sum over the  $n$  indices  $i_1, \dots, i_n$  implied.

### Low dimensions and some special cases

To get a better feel for the determinant it is useful to look at low dimensions first. For  $n = 2$  we have

$$\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \epsilon_{ij} a_i b_j = \epsilon_{12} a_1 b_2 + \epsilon_{21} a_2 b_1 = a_1 b_2 - a_2 b_1. \quad (5.14)$$

The two terms on the right-hand side correspond to the two permutations of  $\{1, 2\}$ . In three dimensions we find

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \epsilon_{ijk} a_i b_j c_k = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1 - a_1 b_3 c_2 \quad (5.15)$$

$$= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \quad (5.16)$$

The last line follows by comparison with Eq. (2.43). Hence, the three-dimensional determinant as from our general definition is indeed the triple product and coincides with our earlier definition of the determinant. The six terms in the right-hand side of Eq. (5.15) correspond to the six permutations of  $\{1, 2, 3\}$  and we recall from Eq. (2.50) that they can be explicitly computed by multiplying the terms along the diagonals of the matrix.

**Example 5.1:** Computing determinants for  $2 \times 2$  and  $3 \times 3$  matrices

(a) For a  $2 \times 2$  matrix

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -5 \end{pmatrix} \quad (5.17)$$

we have from Eq. (5.14)

$$\det(A) = \det \begin{pmatrix} 3 & -2 \\ 4 & -5 \end{pmatrix} = 3 \cdot (-5) - (-2) \cdot 4 = -7 \quad (5.18)$$

(b) To compute the determinant of a  $3 \times 3$  matrix

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ 4 & 2 & 5 \end{pmatrix} \quad (5.19)$$

we need to write down the six terms in Eq. (5.15) which can be obtained by multiplying the terms along the diagonals of  $A$ , following the rule (2.50). Explicitly

$$\begin{aligned} \det(A) &= \det \begin{pmatrix} 1 & -2 & 0 \\ \times & 3 & \times \\ \times & 4 & \times \end{pmatrix} \begin{pmatrix} -2 & \times & 0 \\ \times & 2 & \times \\ \times & 2 & \times \end{pmatrix} \begin{pmatrix} 0 & \times & \times \\ -1 & \times & \times \\ 5 & \times & \times \end{pmatrix} \\ &= 1 \cdot 2 \cdot 5 + (-2) \cdot (-1) \cdot 4 + 0 \cdot 3 \cdot 2 - 0 \cdot 2 \cdot 4 - (-2) \cdot 3 \cdot 5 - 1 \cdot (-1) \cdot 2 \\ &= 10 + 8 + 30 + 2 = 50. \end{aligned} \quad (5.20)$$

Recall that each of the six terms are obtained by multiplying three entries along a diagonal as indicated by the lines (where corresponding lines at the left and right edge of the matrix should be identified to collect all factors). The three  $\setminus$  diagonals correspond to the three cyclic permutations which appear with a positive sign while the three  $/$  diagonals lead to the anti-cyclic terms which come with a negative sign.

The determinant of a  $4 \times 4$  matrix has  $4! = 24$  terms and it is  $n!$  terms for an  $n \times n$  matrix, so this becomes complicated quickly. An interesting class of matrices for which the determinant is simple consists of *upper*

triangular matrices, that is, matrices with all entries below the diagonal vanishing. In this case

$$\det \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} = a_1 \cdots a_n, \quad (5.21)$$

so the determinant is simply the product of the diagonal elements<sup>4</sup>. This can be seen from Eq. (5.11). We should consider all ways of choosing one entry per column such that no two entries appear in the same row. For an upper triangular matrix, the only non-zero choice in the first column is the first entry, so that the first row is “occupied”. In the second column the only available non-trivial choice is, therefore, the entry in the second row etc. In conclusion, from the  $n!$  terms in Eq. (5.11) only the term which corresponds to the product of the diagonal elements is non-zero. An easy conclusion from Eq. (5.21) is that

$$\det(\mathbb{1}_n) = 1, \quad (5.22)$$

as must be the case from property (D3) in Def. 5.1.

## 5.2 Properties of the determinant and calculation

As we have seen from the previous discussion, the explicit expression for the determinant becomes complicated quickly as the dimension increases. To be able to work with determinants in general we, therefore, need to explore some of their more sophisticated properties. We begin with the relation between the determinant and the transposition of matrices.

**Lemma 5.1.** *The determinant of a matrix and its transpose are the same, so  $\det(A) = \det(A^T)$ .*

*Proof.* By setting  $j_a = \sigma(a)$ , for a permutation  $\sigma \in S_n$  we can re-write a term in the sum (5.11) for the determinant as  $A_{\sigma(1)1} \cdots A_{\sigma(n)n} = A_{j_1\sigma^{-1}(j_1)} \cdots A_{j_n\sigma^{-1}(j_n)} = A_{1\sigma^{-1}(1)} \cdots A_{n\sigma^{-1}(n)}$ , where the last equality follows simply by re-ordering the factors, given that  $j_1, \dots, j_n$  is a permutation of  $1, \dots, n$ . From this observation the determinant (5.11) can be written as

$$\begin{aligned} \det(A) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A_{1\sigma^{-1}(1)} \cdots A_{n\sigma^{-1}(n)} \stackrel{(5.9)}{=} \sum_{\sigma^{-1} \in S_n} \operatorname{sgn}(\sigma^{-1}) A_{1\sigma^{-1}(1)} \cdots A_{n\sigma^{-1}(n)} \\ &\stackrel{\rho = \sigma^{-1}}{=} \sum_{\rho \in S_n} \operatorname{sgn}(\rho) (A^T)_{\rho(1)1} \cdots (A^T)_{\rho(n)n} = \det(A^T). \end{aligned}$$

□

Another obvious question is about the relation between the determinant and matrix multiplication. Fortunately, there is a simple and beautiful answer.

**Theorem 5.1.**  $\det(AB) = \det(A) \det(B)$ , for any two  $n \times n$  matrices  $A, B$ .

*Proof.* Recall from Eq. (3.52) the index form of matrix multiplication

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}.$$

By focusing on a particular value of  $j$  in this expression we can write the  $j^{\text{th}}$  column of  $AB$  as

$$(AB)^j = \sum_k B_{kj} \mathbf{A}^k, \quad (5.23)$$

---

<sup>4</sup>Of course an analogous statement holds for *lower triangular matrices*.

where  $\mathbf{A}^k$  are the columns of  $A$ . Hence,

$$\begin{aligned} \det(AB) &= \det((AB)^1, \dots, (AB)^n) \stackrel{(5.23)}{=} \det\left(\sum_{k_1} B_{k_1 1} \mathbf{A}^{k_1}, \dots, \sum_{k_n} B_{k_n n} \mathbf{A}^{k_n}\right) \\ &\stackrel{(D1)}{=} \sum_{k_1, \dots, k_n} B_{k_1 1} \cdots B_{k_n n} \det(\mathbf{A}^{k_1}, \dots, \mathbf{A}^{k_n}) \stackrel{k_a = \sigma(a)}{=} \sum_{\sigma \in S_n} B_{\sigma(1)1} \cdots B_{\sigma(n)n} \det(\mathbf{A}^{\sigma(1)}, \dots, \mathbf{A}^{\sigma(n)}) \\ &\stackrel{(D2)}{=} \underbrace{\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) B_{\sigma(1)1} \cdots B_{\sigma(n)n}}_{\det(B)} \underbrace{\det(\mathbf{A}^1, \dots, \mathbf{A}^n)}_{\det(A)} = \det(A) \det(B) \end{aligned}$$

□

This simple multiplication rule for determinants of matrix products has a number of profound consequences. First, we can prove a criterion for invertibility of a matrix, based on the determinant, essentially a more complete version of Claim 2.1.

**Corollary 5.1.** *For an  $n \times n$  matrix  $A$  we have:*

$$A \text{ is bijective (that is, } A \text{ has an inverse)} \iff \det(A) \neq 0 \quad (5.24)$$

If  $A$  is invertible then  $\det(A^{-1}) = (\det(A))^{-1}$ .

*Proof.* “ $\Rightarrow$ ”: If  $A$  is bijective it has an inverse  $A^{-1}$  and  $1 = \det(\mathbb{1}_n) = \det(AA^{-1}) = \det(A) \det(A^{-1})$ . This implies that  $\det(A) \neq 0$  and that  $\det(A^{-1}) = (\det(A))^{-1}$  which is the second part of our assertion. “ $\Leftarrow$ ”: We prove this indirectly, so we start by assuming that  $A$  is not bijective. From Lemma 3.4 this means that  $\operatorname{rk}(A) < n$ , so the rank of  $A$  is less than maximal. Hence, at least one of the column vectors of  $A$ , say  $\mathbf{A}^1$  for definiteness, can be expressed as a linear combination of the others, so that

$$\mathbf{A}^1 = \sum_{i=2}^n \alpha_i \mathbf{A}^i$$

for some coefficients  $\alpha_i$ . For the determinant of  $A$  this means

$$\det(A) = \det(\mathbf{A}^1, \mathbf{A}^2, \dots, \mathbf{A}^n) = \det\left(\sum_{i=2}^n \alpha_i \mathbf{A}^i, \mathbf{A}^2, \dots, \mathbf{A}^n\right) \stackrel{(D1)}{=} \sum_{i=2}^n \alpha_i \det(\mathbf{A}^i, \mathbf{A}^2, \dots, \mathbf{A}^n) \stackrel{(5.1)}{=} 0$$

□

Note that, for invertible matrices  $A$ , this provides us with a useful way to calculate the determinant of the inverse matrix by

$$\det(A^{-1}) = (\det(A))^{-1}. \quad (5.25)$$

Combining this rule and Theorem 5.1 implies that  $\det(PAP^{-1}) = \det(P) \det(A) (\det(P))^{-1} = \det(A)$ , so, in short

$$\det(PAP^{-1}) = \det(A). \quad (5.26)$$

This equation says that the determinant remains unchanged under basis transformations (3.96) and, as a result, the determinant is the same for every matrix representing a given linear map. The determinant is, therefore, a genuine property of the linear map and we can talk about the determinant of a linear map, defined as the determinant of any of its representing matrices.



**Example 5.2:** Using the determinant to check if a matrix is invertible

The above Corollary is useful to check if (small) matrices are invertible. Consider, for example, the family of  $3 \times 3$  matrices

$$A = \begin{pmatrix} 1 & -1 & a \\ 0 & a & -3 \\ -2 & 0 & 1 \end{pmatrix} \quad (5.27)$$

where  $a \in \mathbb{R}$  is a real parameter. We can ask for which values of  $a$  the matrix  $A$  is invertible. Computing the determinant is straightforward and leads to

$$\det(A) = 2a^2 + a - 6. \quad (5.28)$$

This vanishes precisely when  $a = -2$  or  $a = 3/2$  and, hence, for these values of  $a$  the matrix  $A$  is not invertible. For all other values it is invertible.

Our next goal is to find a recursive method to calculate the determinant, essentially by writing the determinant of a matrix in terms of determinants of sub-matrices. To this end, for an  $n \times n$  matrix  $A$ , we define the associated  $n \times n$  matrices

$$\tilde{A}_{(i,j)} = \begin{pmatrix} & & 0 & \leftarrow j^{\text{th}} \text{ col} & & \\ & \text{"A"} & \vdots & & \text{"A"} & \\ & & 0 & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & 0 & & & & \\ & \text{"A"} & \vdots & & \text{"A"} & \\ & & 0 & & & & \end{pmatrix} \leftarrow i^{\text{th}} \text{ row} \quad (5.29)$$

They are obtained from  $A$  by setting the  $(i, j)$  entry to 1, the other entries in row  $i$  and column  $j$  to zero and keeping the rest of the matrix unchanged. Note that the subscripts  $(i, j)$  indicate the row and column which have been changed rather than specific entries of the matrix (hence the bracket notation). With the so-defined matrices we define the *co-factor matrix*, an  $n \times n$  matrix  $C$  with entries

$$C_{ij} := \det(\tilde{A}_{(i,j)}). \quad (5.30)$$

To find a more elegant expression for the co-factor matrix, we also introduce the  $(n-1) \times (n-1)$  matrices  $A_{(i,j)}$  which are obtained from  $A$  by simply removing the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column. It takes  $i-1$  swaps of neighbouring rows in (5.29) to move row  $i$  to the first row (without changing the order of any other rows) and a further  $j-1$  swaps to move column  $j$  to the first column. After these swaps the matrix  $\tilde{A}_{(i,j)}$  becomes

$$B_{(i,j)} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A_{(i,j)} & \\ 0 & & & \end{pmatrix}, \quad (5.31)$$

From Def. (5.1) (D2) and Lemma 5.1 it is clear that  $\det(\tilde{A}_{(i,j)}) = (-1)^{i+j} \det(B_{(i,j)})$ , since we need a total of  $i+j-2$  swaps of rows and columns to convert one matrix into the other. Further, the explicit form of the determinant (5.11) implies that  $\det(B_{(i,j)}) = \det(A_{(i,j)})$  (as the only non-trivial choice of entry in the first column of  $B_{(i,j)}$  is the 1 in the first row). Combining these observations means the co-factor matrix is given by

$$C_{ij} = \det(\tilde{A}_{(i,j)}) = (-1)^{i+j} \det(A_{(i,j)}). \quad (5.32)$$

Hence, the co-factor matrix contains, up to signs, the determinants of the  $(n - 1) \times (n - 1)$  sub-matrices of  $A$ , obtained by deleting one row and one column from  $A$ . As we will see, for explicit calculations, it is useful to note that the signs in Eq. (5.32) follow a “chess board pattern”, that is, the matrix with entries  $(-1)^{i+j}$  has the form

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad (5.33)$$

Our goal is to relate the determinant of  $A$  to the determinants of sub-matrices, that is to the entries of the co-factor matrix. This is accomplished by

**Lemma 5.2.** *For an  $n \times n$  matrix  $A$  with associated co-factor matrix  $C$ , defined by Eq. (5.32), we have*

$$C^T A = \det(A) \mathbb{1}_n \quad (5.34)$$

*Proof.* This follows from the definition of the co-factor matrix, more or less by direct calculation.

$$\begin{aligned} (C^T A)_{ij} &\stackrel{(3.52)}{=} \sum_k (C^T)_{ik} A_{kj} = \sum_k A_{kj} C_{ki} \stackrel{(5.32)}{=} \sum_k A_{kj} \det(\tilde{A}_{(k,i)}) \\ &\stackrel{(5.29)}{=} \sum_k A_{kj} \det(\mathbf{A}^1, \dots, \mathbf{A}^{i-1}, \mathbf{e}_k, \mathbf{A}^{i+1}, \dots, \mathbf{A}^n) \\ &\stackrel{(D1)}{=} \det\left(\mathbf{A}^1, \dots, \mathbf{A}^{i-1}, \sum_k A_{kj} \mathbf{e}_k, \mathbf{A}^{i+1}, \dots, \mathbf{A}^n\right) \\ &= \det(\mathbf{A}^1, \dots, \mathbf{A}^{i-1}, \mathbf{A}^j, \mathbf{A}^{i+1}, \dots, \mathbf{A}^n) \stackrel{(5.1)}{=} \delta_{ij} \det(A) = (\det(A) \mathbb{1}_n)_{ij} \end{aligned}$$

□

An immediate conclusion from Lemma 5.2 is

$$\det(A) = (C^T A)_{jj} = \sum_i (C^T)_{ji} A_{ij} = \sum_i C_{ij} A_{ij} = \sum_i (-1)^{i+j} A_{ij} \det(A_{(i,j)}).$$

so,

$$\det(A) = \sum_i (-1)^{i+j} A_{ij} \det(A_{(i,j)}). \quad (5.35)$$

This identity is referred to as *Laplace expansion* of the determinant. It realizes our goal of expressing the determinant of  $A$  in terms of determinants of the sub-matrices  $A_{(i,j)}$ . More specifically, in Eq. (5.35) we can choose any column  $j$  and compute the determinant of  $A$  by summing over the entries  $i$  in this column times the determinants of the corresponding sub-matrices  $A_{(i,j)}$  (taking into account the sign). This is also referred to as expanding the determinant “along the  $j^{\text{th}}$  column”. Since the determinant remains unchanged under transposition it can also be computed in a similar way by expanding “along the  $i^{\text{th}}$  row”. To see how this works in practice we consider the following

**Example 5.3:** Laplace expansion of determinant

We would like to compute the determinant of the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & -2 \\ 0 & 3 & 4 \end{pmatrix} \quad (5.36)$$

by expanding along its 1<sup>st</sup> column. From Eq. (5.35), taking into account the signs as indicated in (5.33), we find

$$\begin{aligned} \det(A) &= A_{11} \det(A_{(1,1)}) - A_{21} \det(A_{(2,1)}) + A_{31} \det(A_{(3,1)}) \\ &= 2 \cdot \det \begin{pmatrix} 2 & -2 \\ 3 & 4 \end{pmatrix} - 1 \cdot \det \begin{pmatrix} -1 & 0 \\ 3 & 4 \end{pmatrix} + 0 \cdot \det \begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix} = 2 \cdot 14 - 1 \cdot (-4) + 0 \cdot 2 = 32 \end{aligned}$$

Note that the efficiency of the calculation can be improved by choosing the row or column with the most zeros.

A by-product of Lemma 5.2 is a new method to compute the inverse of a matrix. If  $A$  is invertible then, from Cor. 5.1,  $\det(A) \neq 0$  and we can divide by  $\det(A)$  to get

$$\frac{1}{\det(A)} C^T A = \mathbb{1}_n .$$

Hence, the inverse of  $A$  is given by

$$A^{-1} = \frac{1}{\det(A)} C^T . \quad (5.37)$$

Again, it is worth applying this to an example.

**Example 5.4:** Inverse of a matrix using the co-factor method

(a) We would like to find the inverse of a general  $2 \times 2$  matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (5.38)$$

using the co-factor method. The co-factor matrix of  $A$  is easily obtained by switching around the diagonal and non-diagonal entries and inverting the signs of the latter:

$$C = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} . \quad (5.39)$$

With  $\det(A) = ad - cb$  (which should be different from zero for the inverse to exist) we have for the inverse

$$A^{-1} = \frac{1}{\det(A)} C^T = \frac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} . \quad (5.40)$$

Note that this provides a rule for inverting  $2 \times 2$  matrices which is relatively easy to remember: Exchange the diagonal elements, invert the signs of the off-diagonal elements and divide by the determinant.

(b) We consider the matrix  $A$ , Eq. (5.36), from the previous example. From Eq. (5.32) we find for the associated co-factor matrix

$$C = \begin{pmatrix} 14 & -4 & 3 \\ 4 & 8 & -6 \\ 2 & 4 & 5 \end{pmatrix} .$$

With  $\det(A) = 32$  the inverse is

$$A^{-1} = \frac{1}{\det(A)} C^T = \frac{1}{32} \begin{pmatrix} 14 & 4 & 2 \\ -4 & 8 & 4 \\ 3 & -6 & 5 \end{pmatrix} .$$

We note that, for larger matrices, the row reduction method discussed in Section 3.3 is a more efficient way of computing the inverse than the co-factor method. Indeed, for an  $n \times n$  matrix the number of operations required for a row reduction grows roughly as  $n^3$  while computing a determinant requires  $\sim n!$  operations.

Despite our improved methods, the calculation of determinants of large matrices remains a problem, essentially because the aforementioned  $n!$  growth of the number of terms in Eq. (5.11). Using a Laplace expansion will improve matters only if the matrix in question has many zeros. However, by using elementary row operations, we can get to an efficient way of computing large determinants. The key observation is that, from the general properties of the determinant in Def. 5.1, row operations of type (R1) (see Def. 3.7) only change the sign of the determinant and row operations of type (R2) leave the determinant unchanged. A given matrix  $A$  can be brought into upper echelon form,  $A'$ , by a succession of these row operations and, hence,  $\det(A) = (-1)^p \det(A')$ , where  $p$  is the number of row swaps used in the process. The matrix  $A'$  is in fact in upper triangular form

$$A' = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix}.$$

and, as discussed earlier, the determinant of such a matrix is simply the product of its diagonal entries. It follows that  $\det(A) = (-1)^p a_1 \cdots a_n$ .

### 5.3 Cramer's Rule

We have already seen how the determinant of a matrix can be used to decide if an  $n \times n$  matrix  $A$  is invertible, and how to compute the inverse of a matrix. Here we introduce *Cramer's Rule*, which uses determinants to solve systems of linear equations  $\mathbf{Ax} = \mathbf{b}$  for the case of quadratic and invertible  $n \times n$  matrices  $A$ . Recall from our general discussion in Section 4.1 that, in this case, the linear system has a unique solution,  $\mathbf{x} = A^{-1}\mathbf{b}$ , for any vector  $\mathbf{b}$ .

To derive Cramer's rule we first define the matrices

$$B_{(i)} := (\mathbf{A}^1, \dots, \mathbf{A}^{i-1}, \mathbf{b}, \mathbf{A}^{i+1}, \dots, \mathbf{A}^n), \quad (5.41)$$

which are obtained from  $A$  by replacing the  $i^{\text{th}}$  column with  $\mathbf{b}$  and keeping all other columns unchanged. We also note that, in terms of the column vectors  $\mathbf{A}^j$  of  $A$  the linear system  $\mathbf{Ax} = \mathbf{b}$  can be written as (see, for example, Eq. (3.36))

$$\sum_j x_j \mathbf{A}^j = \mathbf{b}, \quad (5.42)$$

where  $\mathbf{x} = (x_1, \dots, x_n)^T$ . Then we find

$$\begin{aligned} \det(B_{(i)}) &= \det(\mathbf{A}^1, \dots, \mathbf{A}^{i-1}, \mathbf{b}, \mathbf{A}^{i+1}, \dots, \mathbf{A}^n) \stackrel{(5.42)}{=} \det(\mathbf{A}^1, \dots, \mathbf{A}^{i-1}, \sum_j x_j \mathbf{A}^j, \mathbf{A}^{i+1}, \dots, \mathbf{A}^n) \\ &\stackrel{(D1)}{=} \sum_j x_j \det(\mathbf{A}^1, \dots, \mathbf{A}^{i-1}, \mathbf{A}^j, \mathbf{A}^{i+1}, \dots, \mathbf{A}^n) \stackrel{(D2)}{=} x_i \det(\mathbf{A}^1, \dots, \mathbf{A}^{i-1}, \mathbf{A}^i, \mathbf{A}^{i+1}, \dots, \mathbf{A}^n) \\ &= x_i \det(A). \end{aligned}$$

Solving for  $x_i$  we find Cramer's rule

$$x_i = \frac{\det(B_{(i)})}{\det(A)} = \frac{\det(\mathbf{A}^1, \dots, \mathbf{A}^{i-1}, \mathbf{b}, \mathbf{A}^{i+1}, \dots, \mathbf{A}^n)}{\det(A)} \quad (5.43)$$

for the solution  $\mathbf{x} = (x_1, \dots, x_n)^T$  of the linear system  $A\mathbf{x} = \mathbf{b}$ , where  $A$  is an invertible  $n \times n$  matrix. To solve linear systems explicitly, Cramer's rule is only useful for relatively small systems, due to the  $n!$  growth of the determinant. For larger linear systems the row reduction method introduced in Section (4.3) should be used.

**Example 5.5:** Cramer's rule

Let us apply Cramer's rule to a linear system  $A\mathbf{x} = \mathbf{b}$  with

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & -2 \\ 0 & 3 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}. \quad (5.44)$$

From Eq. (5.41), that is by replacing one column of  $A$  with the vector  $\mathbf{b}$ , we find the three matrices

$$B_{(1)} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & -2 \\ 0 & 3 & 4 \end{pmatrix}, \quad B_{(2)} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & 0 & 4 \end{pmatrix}, \quad B_{(3)} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 0 & 3 & 0 \end{pmatrix}. \quad (5.45)$$

By straightforward computation, for example using a Laplace expansion, it follows that  $\det(A) = 32$ ,  $\det(B_{(1)}) = 22$ ,  $\det(B_{(2)}) = 12$  and  $\det(B_{(3)}) = -9$ . From Eq. (5.43) this leads to the solution

$$\mathbf{x} = \frac{1}{32} \begin{pmatrix} 22 \\ 12 \\ -9 \end{pmatrix}.$$

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## 6 Scalar products

In Section 2 we have introduced the standard scalar product on  $\mathbb{R}^n$  (the dot product) and we have seen its usefulness, particularly for geometrical applications. Here, we study its generalizations to arbitrary real and complex vector spaces.

### 6.1 Real and hermitian scalar products

**Definition 6.1.** A real (hermitian) scalar product on a vector space  $V$  over  $F = \mathbb{R}$  ( $F = \mathbb{C}$ ) is a map  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$  ( $\mathbb{C}$ ) satisfying

- (S1)  $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$ , for a real scalar product,  $F = \mathbb{R}$   
 $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle^*$ , for a hermitian scalar product,  $F = \mathbb{C}$   
(S2)  $\langle \mathbf{v}, \alpha \mathbf{u} + \beta \mathbf{w} \rangle = \alpha \langle \mathbf{v}, \mathbf{u} \rangle + \beta \langle \mathbf{v}, \mathbf{w} \rangle$   
(S3)  $\langle \mathbf{v}, \mathbf{v} \rangle > 0$  if  $\mathbf{v} \neq \mathbf{0}$

for all vectors  $\mathbf{v}, \mathbf{u}, \mathbf{w} \in V$  and all scalars  $\alpha, \beta \in F$ .

If (S1) and (S2), but not necessarily (S3) are satisfied, then  $\langle \cdot, \cdot \rangle$  is called a bi-linear form (in the real case  $F = \mathbb{R}$ ) or a sesqui-linear form (in the complex case  $F = \mathbb{C}$ ).

Let us discuss this definition, beginning with the case of a real scalar product. The condition (S2) says that a scalar product is linear in the second argument, in precisely the same sense that a linear map is linear (see Def. 3.5). For the real case, the scalar product is symmetric in the two arguments from condition (S1) and, together with (S2), this implies linearity in the first argument, so

$$\langle \alpha \mathbf{v} + \beta \mathbf{u}, \mathbf{w} \rangle = \alpha \langle \mathbf{v}, \mathbf{w} \rangle + \beta \langle \mathbf{u}, \mathbf{w} \rangle . \quad (6.1)$$

So, in the real case, the scalar product is *bi-linear*. In this sense, we should think of the above definition as natural, extending our notion of linearity to maps with two vectorial arguments.

The situation is somewhat more complicated in the hermitian case. Here, the complex conjugation in (S1) together with (S2) leads to

$$\langle \alpha \mathbf{v} + \beta \mathbf{u}, \mathbf{w} \rangle = \alpha^* \langle \mathbf{v}, \mathbf{w} \rangle + \beta^* \langle \mathbf{u}, \mathbf{w} \rangle . \quad (6.2)$$

Hence, sums in the first argument of a hermitian scalar product can still be pulled apart, but scalars are pulled out with a complex conjugation. This property, together with the linearity in the second argument<sup>5</sup> is also called *sesqui-linearity*.

The property (S3) ensures that we can sensibly define the *norm* (or length) of a vector as

$$|\mathbf{v}| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} . \quad (6.3)$$

Note that in the hermitian case, (S1) implies that  $\langle \mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle^*$  so that  $\langle \mathbf{v}, \mathbf{v} \rangle$  is real. For this reason, the condition (S3) actually makes sense in the hermitian case (if  $\langle \mathbf{v}, \mathbf{v} \rangle$  was complex there would be no well-defined sense in which we could demand it to be positive) and this explains the need for including the complex conjugation in (S1).

The Cauchy-Schwarz inequality can be shown as in Lemma 2.1 (taking care to include complex conjugation in the hermitian case), so we have in general

$$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq |\mathbf{v}| |\mathbf{w}| . \quad (6.4)$$

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<sup>5</sup>In some parts of the mathematics literature a hermitian scalar product is defined to be linear in the first argument. Our definition based on linearity in the second argument is the usual convention in the physics literature.

The proof of the triangle inequality in Lemma 2.2 also goes through in general, so for the norm (6.3) of a general scalar product we have

$$|\mathbf{v} + \mathbf{w}| \leq |\mathbf{v}| + |\mathbf{w}| . \quad (6.5)$$

For a real scalar product, in analogy with Eq. (2.10), the Cauchy-Schwarz inequality allows the definition of the angle  $\sphericalangle(\mathbf{v}, \mathbf{w}) \in [0, \pi]$  between two non-zero vectors  $\mathbf{v}, \mathbf{w}$  by

$$\cos(\sphericalangle(\mathbf{v}, \mathbf{w})) := \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{|\mathbf{v}| |\mathbf{w}|} . \quad (6.6)$$

For any scalar product, two vectors  $\mathbf{v}$  and  $\mathbf{w}$  are called *orthogonal* iff  $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ . Hence, for a real scalar product, the non-zero vectors  $\mathbf{v}$  and  $\mathbf{w}$  are orthogonal precisely when they form an angle  $\sphericalangle(\mathbf{v}, \mathbf{w}) = \pi/2$ . We should now discuss some examples of scalar products.

**Example 6.1:** Examples of scalar products

(a) Standard scalar product in  $\mathbb{R}^n$

This is the dot product introduced earlier. For two vectors  $\mathbf{v} = (v_1, \dots, v_n)^T$  and  $\mathbf{w} = (w_1, \dots, w_n)^T$  in  $\mathbb{R}^n$  it is defined as

$$\langle \mathbf{v}, \mathbf{w} \rangle := \mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w} = \sum_{i=1}^n v_i w_i . \quad (6.7)$$

We already know from Eq. (2.3) that it satisfies all the requirements in Def. 6.1 for a real scalar product.

(b) Standard scalar product in  $\mathbb{C}^n$

For two vectors  $\mathbf{v} = (v_1, \dots, v_n)^T$  and  $\mathbf{w} = (w_1, \dots, w_n)^T$  in  $\mathbb{C}^n$  the standard scalar product in  $\mathbb{C}^n$  is defined as

$$\langle \mathbf{v}, \mathbf{w} \rangle := \mathbf{v}^\dagger \mathbf{w} = \sum_{i=1}^n v_i^* w_i . \quad (6.8)$$

It is easy to check that it satisfies the requirements in Def. 6.1 for a hermitian scalar product. In particular, the associated norm is given by

$$|\mathbf{v}|^2 = \langle \mathbf{v}, \mathbf{v} \rangle = \sum_{i=1}^n |v_i|^2 , \quad (6.9)$$

where  $|v_i|$  denotes the modulus of the complex number  $v_i$ . This is indeed real and positive, as it must, but note that the inclusion of the complex conjugate in Eq. (6.8) is crucial.

(c) Minkowski product in  $\mathbb{R}^4$

For two four-vectors  $\mathbf{v} = (v_0, v_1, v_2, v_3)^T$  and  $\mathbf{w} = (w_0, w_1, w_2, w_3)^T$  in  $\mathbb{R}^4$ , the Minkowski product is defined as

$$\langle \mathbf{v}, \mathbf{w} \rangle := \mathbf{v}^T \eta \mathbf{w} = -v_0 w_0 + v_1 w_1 + v_2 w_2 + v_3 w_3 \quad \text{where} \quad \eta = \text{diag}(-1, 1, 1, 1) . \quad (6.10)$$

It is easy to show that it satisfies conditions (S1) and (S2) but not condition (S3). For example, for  $\mathbf{v} = (1, 0, 0, 0)^T$  we have

$$\langle \mathbf{v}, \mathbf{v} \rangle = -1 , \quad (6.11)$$

which contradicts (S3). Therefore, the Minkowski product is not a scalar product but merely a bi-linear form. Nevertheless, it plays an important role in physics, specifically in the context of special (and general) relativity.

(d) Scalar product for function vector spaces

Def. 6.1 applies to arbitrary vector spaces so we should discuss at least one example of a more abstract vector space. Consider the vector space of continuous (real- or complex-valued) functions  $f : [a, b] \rightarrow \mathbb{R}$  or  $\mathbb{C}$  on an interval  $[a, b] \subset \mathbb{R}$ . A scalar product for such functions can be defined by the integral

$$\langle f, g \rangle := \int_a^b dx f(x)^* g(x) . \quad (6.12)$$

It is easily checked that the conditions (S1)–(S3) are satisfied. Scalar products of this kind are of great importance in physics, particularly in quantum mechanics.

(e) Scalar product for real matrices

The real  $n \times n$  matrices form a vector space  $V$  with vector addition and scalar multiplication defined component-wise as in Example 1.4 (e). The dimension of this space is  $n^2$  with a basis given by the matrices  $E_{(ij)}$ , defined in Eq. (1.54). On this space, we can introduce a scalar product by

$$\langle A, B \rangle := \text{tr}(A^T B) = \sum_{i,j} A_{ij} B_{ij} . \quad (6.13)$$

where the symbol  $\text{tr}$  denotes the *trace* of a matrix  $A$ , defined as the sum over its diagonal entries, so  $\text{tr}(A) := \sum_i A_{ii}$ . The sum on the RHS of Eq. (6.13) shows that this definition is in complete analogy with the dot product for real vectors, but with the summation running over two indices instead of just one. It is, therefore, clear that all requirements for a scalar product are satisfied. For complex matrices a hermitian scalar product can be defined analogously simply by replacing the transpose in Eq. (6.13) with a hermitian conjugate.

We conclude our introduction to scalar products with a simple but important observation about orthogonal vectors.

**Lemma 6.1.** *Pairwise orthogonal and non-zero vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  are linearly independent.*

*Proof.* Start with

$$\sum_{i=1}^k \alpha_i \mathbf{v}_i = 0$$

and take the scalar product of this equation with one of the vectors,  $\mathbf{v}_j$ . Since  $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$  for  $i \neq j$  it follows that  $\alpha_j |\mathbf{v}_j|^2 = 0$ . Since  $\mathbf{v}_j \neq 0$  its norm is positive,  $|\mathbf{v}_j| > 0$ , so  $\alpha_j = 0$ .  $\square$

## 6.2 Orthonormal basis, Gram-Schmidt procedure

From the previous Lemma,  $n$  pairwise orthogonal, non-zero vectors in an  $n$ -dimensional vector space form a basis. This motivates the following

**Definition 6.2.** *A basis  $\epsilon_1, \dots, \epsilon_n$  of a vector space  $V$  with a scalar product is called ortho-normal iff*

$$\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij} , \quad (6.14)$$

*that is, if the basis vectors are pairwise orthogonal and have length one.*

**Example 6.2:** Examples of ortho-normal basis

(a) The basis of standard unit vectors,  $\mathbf{e}_1, \dots, \mathbf{e}_n$  of  $\mathbb{R}^n$  ( $\mathbb{C}^n$ ) is an ortho-normal basis with respect to the



standard scalar product on  $\mathbb{R}^n$  ( $\mathbb{C}^n$ ), as defined in Example 6.1.

(b) The vectors

$$\boldsymbol{\epsilon}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{\epsilon}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (6.15)$$

form an orthonormal basis on  $\mathbb{R}^2$  with respect to the standard scalar product, that is,  $\boldsymbol{\epsilon}_i^T \boldsymbol{\epsilon}_j = \delta_{ij}$ .

(c) The vectors

$$\boldsymbol{\epsilon}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ i \end{pmatrix}, \quad \boldsymbol{\epsilon}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2i \end{pmatrix} \quad (6.16)$$

form an orthonormal basis on  $\mathbb{C}^2$  with respect to the standard scalar product, that is,  $\boldsymbol{\epsilon}_i^\dagger \boldsymbol{\epsilon}_j = \delta_{ij}$ . Note, it is crucial to use the proper standard scalar product 6.1 (b) for the complex case which involves the hermitian conjugate rather than the transpose.

(d) For the vector space of real  $n \times n$  matrices, the matrices  $E_{(ij)}$ , defined in Eq. (1.54), form an orthonormal basis with respect to the scalar product (6.13).

An ortho-normal basis has many advantages compared to an arbitrary basis of a vector space. For example, consider the coordinates of a vector  $\mathbf{v} \in V$  relative to an ortho-normal basis  $\{\boldsymbol{\epsilon}_1, \dots, \boldsymbol{\epsilon}_n\}$ . Of course, we can write  $\mathbf{v}$  as a linear combination  $\mathbf{v} = \sum_{i=1}^n \alpha_i \boldsymbol{\epsilon}_i$  with some coordinates  $\alpha_i$  but, in the general case, these coefficients need to be determined by solving a system of linear equations. For an ortho-normal basis, we can just take the scalar product of this equation with  $\boldsymbol{\epsilon}_j$ , leading to

$$\langle \boldsymbol{\epsilon}_j, \mathbf{v} \rangle = \langle \boldsymbol{\epsilon}_j, \sum_{i=1}^n \alpha_i \boldsymbol{\epsilon}_i \rangle = \sum_{i=1}^n \alpha_i \underbrace{\langle \boldsymbol{\epsilon}_j, \boldsymbol{\epsilon}_i \rangle}_{=\delta_{ij}} = \alpha_j$$

So in summary, the coordinates of a vector  $\mathbf{v}$  relative to an ortho-normal basis  $\{\boldsymbol{\epsilon}_1, \dots, \boldsymbol{\epsilon}_n\}$  can be computed as

$$\mathbf{v} = \sum_{i=1}^n \alpha_i \boldsymbol{\epsilon}_i \iff \alpha_i = \langle \boldsymbol{\epsilon}_i, \mathbf{v} \rangle. \quad (6.17)$$

**Example 6.3:** Coordinates relative to an ortho-normal basis in  $\mathbb{R}^2$  and  $\mathbb{C}^2$

As before, we have to be careful to distinguish the real and the complex case, since the respective standard scalar products differ by a complex conjugation. We begin with the real case.

(a) Consider  $\mathbb{R}^2$  with ortho-normal basis  $\{\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2\}$  as in Eq. (6.15) and the vector  $\mathbf{v} = (2, -3)^T$ . We would like to write this vector as a linear combination  $\mathbf{v} = \alpha_1 \boldsymbol{\epsilon}_1 + \alpha_2 \boldsymbol{\epsilon}_2$ . Then, the coordinates  $\alpha_1, \alpha_2$  are given by

$$\alpha_1 = \boldsymbol{\epsilon}_1^T \mathbf{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 2 \\ -3 \end{pmatrix} = -\frac{1}{\sqrt{2}}, \quad \alpha_2 = \boldsymbol{\epsilon}_2^T \mathbf{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}^T \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \frac{5}{\sqrt{2}}.$$

(b) For  $\mathbb{C}^2$  we use the ortho-normal basis  $\{\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2\}$  from (6.16) and the same vector  $\mathbf{v} = (2, -3)^T$  which we would like to write as a linear combination  $\mathbf{v} = \beta_1 \boldsymbol{\epsilon}_1 + \beta_2 \boldsymbol{\epsilon}_2$ . Then,

$$\beta_1 = \boldsymbol{\epsilon}_1^\dagger \mathbf{v} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ i \end{pmatrix}^\dagger \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \frac{4+3i}{\sqrt{5}}, \quad \beta_2 = \boldsymbol{\epsilon}_2^\dagger \mathbf{v} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2i \end{pmatrix}^\dagger \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \frac{2-6i}{\sqrt{5}}.$$

Note it is crucial to use the hermitian conjugate, rather than the transpose in this calculation.

Does every (finite-dimensional) vector space have an ortho-normal basis and, if so, how can it be determined? The *Gram-Schmidt* procedure answers both of these questions.

**Theorem 6.1.** (*Gram-Schmidt procedure*) *If  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is a basis of the vector space  $V$ , then there exists an ortho-normal basis  $\{\epsilon_1, \dots, \epsilon_n\}$  of  $V$  such that  $\text{Span}(\epsilon_1, \dots, \epsilon_k) = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$  for all  $k = 1, \dots, n$ .*

*Proof.* The proof is constructive. The first vector of our prospective ortho-normal basis is obtained by simply normalizing  $\mathbf{v}_1$ , that is,

$$\epsilon_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|}. \quad (6.18)$$

Clearly,  $|\epsilon_1| = 1$  and  $\text{Span}(\epsilon_1) = \text{Span}(\mathbf{v}_1)$ . Suppose we have already constructed the first  $k - 1$  vectors  $\epsilon_1, \dots, \epsilon_{k-1}$ , mutually orthogonal, normalized and such that  $\text{Span}(\epsilon_1, \dots, \epsilon_j) = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_j)$  for all  $j = 1, \dots, k - 1$ . The next vector,  $\epsilon_k$ , is then constructed by first subtracting from  $\mathbf{v}_k$  its projections onto  $\epsilon_1, \dots, \epsilon_{k-1}$  and then normalizing, so

$$\mathbf{v}'_k = \mathbf{v}_k - \sum_{i=1}^{k-1} \langle \epsilon_i, \mathbf{v}_k \rangle \epsilon_i, \quad \epsilon_k = \frac{\mathbf{v}'_k}{|\mathbf{v}'_k|}. \quad (6.19)$$

Obviously,  $|\epsilon_k| = 1$  and for any vector  $\epsilon_j$  with  $j < k$  we have

$$\langle \epsilon_j, \mathbf{v}'_k \rangle = \langle \epsilon_j, \mathbf{v}_k \rangle - \sum_{i=1}^{k-1} \langle \epsilon_i, \mathbf{v}_k \rangle \underbrace{\langle \epsilon_j, \epsilon_i \rangle}_{=\delta_{ij}} = \langle \epsilon_j, \mathbf{v}_k \rangle - \langle \epsilon_j, \mathbf{v}_k \rangle = 0.$$

Hence,  $\epsilon_k$  is orthogonal to all vectors  $\epsilon_1, \dots, \epsilon_{k-1}$ . Moreover, since  $\text{Span}(\epsilon_1, \dots, \epsilon_{k-1}) = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_{k-1})$  and  $\mathbf{v}_k$  and  $\epsilon_k$  only differ by a re-scaling and terms proportional to  $\epsilon_1, \dots, \epsilon_{k-1}$  it follows that  $\text{Span}(\epsilon_1, \dots, \epsilon_k) = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ .  $\square$

We have seen that every finitely spanned vector space has a basis. The above theorem, therefore, shows that every finitely spanned vector space with a scalar product also has an ortho-normal basis. Note that the proof provides a practical method, summarized by Eqs. (6.18), (6.19), to compute an ortho-normal basis from a given basis. Let us apply this method to some explicit examples.

**Example 6.4:** Gram-Schmidt procedure

(a) Start with the basis

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

of  $\mathbb{R}^3$ . We would like to construct the associated ortho-normal basis with respect to the standard scalar product (the dot product).

1) To find  $\epsilon_1$  use Eq. (6.18):

$$\epsilon_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

2) To find  $\epsilon_2$  use Eq. (6.19) for  $k = 2$ :

$$\mathbf{v}'_2 = \mathbf{v}_2 - \langle \epsilon_1, \mathbf{v}_2 \rangle \epsilon_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \epsilon_2 = \frac{\mathbf{v}'_2}{|\mathbf{v}'_2|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

3) To find  $\epsilon_3$  use Eq. (6.19) for  $k = 3$ :

$$\mathbf{v}'_3 = \mathbf{v}_3 - \langle \epsilon_1, \mathbf{v}_3 \rangle \epsilon_1 - \langle \epsilon_2, \mathbf{v}_3 \rangle \epsilon_2 = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{7}{6} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \quad \epsilon_3 = \frac{\mathbf{v}'_3}{|\mathbf{v}'_3|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}.$$

So, in summary, the ortho-normal basis is

$$\epsilon_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \epsilon_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \epsilon_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}.$$

It is easy (and always advisable) to check that indeed  $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$ .

(b) For a somewhat more adventurous application of the Gram-Schmidt procedure consider the vector space of quadratic polynomials in one variable  $x \in [-1, 1]$  with real coefficients and a scalar product defined by

$$\langle f, g \rangle = \int_{-1}^1 dx f(x)g(x).$$

We would like to find the ortho-normal basis associated to the standard monomial basis  $\mathbf{v}_1 = 1$ ,  $\mathbf{v}_2 = x$ ,  $\mathbf{v}_3 = x^2$  of this space.

1) To find  $\epsilon_1$ :

$$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \int_{-1}^1 dx = 2, \quad \epsilon_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|} = \frac{1}{\sqrt{2}}.$$

2) To find  $\epsilon_2$  first compute  $\mathbf{v}'_2$

$$\langle \epsilon_1, \mathbf{v}_2 \rangle = \int_{-1}^1 dx \frac{x}{\sqrt{2}} = 0, \quad \mathbf{v}'_2 = \mathbf{v}_2 - \langle \epsilon_1, \mathbf{v}_2 \rangle \epsilon_1 = x,$$

and then normalize

$$\langle \mathbf{v}'_2, \mathbf{v}'_2 \rangle = \int_{-1}^1 dx x^2 = \frac{2}{3}, \quad \epsilon_2 = \frac{\mathbf{v}'_2}{|\mathbf{v}'_2|} = \sqrt{\frac{3}{2}} x.$$

3) To find  $\epsilon_3$  first compute  $\mathbf{v}'_3$

$$\langle \epsilon_1, \mathbf{v}_3 \rangle = \frac{1}{\sqrt{2}} \int_{-1}^1 dx x^2 = \frac{\sqrt{2}}{3}, \quad \langle \epsilon_2, \mathbf{v}_3 \rangle = \sqrt{\frac{3}{2}} \int_{-1}^1 dx x^3 = 0, \quad \mathbf{v}'_3 = \mathbf{v}_3 - \langle \epsilon_1, \mathbf{v}_3 \rangle \epsilon_1 - \langle \epsilon_2, \mathbf{v}_3 \rangle \epsilon_2 = x^2 - \frac{1}{3},$$

and normalize

$$\langle \mathbf{v}'_3, \mathbf{v}'_3 \rangle = \int_{-1}^1 dx \left( x^2 - \frac{1}{3} \right)^2 = \frac{8}{45}, \quad \epsilon_3 = \frac{\mathbf{v}'_3}{|\mathbf{v}'_3|} = \sqrt{\frac{5}{8}} (3x^2 - 1).$$

So, in summary, the ortho-normal polynomial basis is

$$\epsilon_1 = \frac{1}{\sqrt{2}}, \quad \epsilon_2 = \sqrt{\frac{3}{2}} x, \quad \epsilon_3 = \sqrt{\frac{5}{8}} (3x^2 - 1).$$

These are the first three of an infinite family of ortho-normal polynomials, referred to as *Legendre polynomials*, which play an important role in mathematical physics.

We have already seen in Eq. (6.17) that the coordinates of a vector relative to an ortho-normal basis are easily computed from the scalar product. There are a few more helpful simplifications which arise for an ortho-normal basis. For their derivation, we start with two vectors

$$\mathbf{v} = \sum_i \alpha_i \boldsymbol{\epsilon}_i, \quad \alpha_i = \langle \boldsymbol{\epsilon}_i, \mathbf{v} \rangle \quad (6.20)$$

$$\mathbf{w} = \sum_i \beta_i \boldsymbol{\epsilon}_i, \quad \beta_i = \langle \boldsymbol{\epsilon}_i, \mathbf{w} \rangle \quad (6.21)$$

and compute their scalar product

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i,j} \alpha_i^* \beta_j \underbrace{\langle \boldsymbol{\epsilon}_i, \boldsymbol{\epsilon}_j \rangle}_{=\delta_{ij}} = \sum_i \alpha_i^* \beta_i = \sum_i \langle \mathbf{v}, \boldsymbol{\epsilon}_i \rangle \langle \boldsymbol{\epsilon}_i, \mathbf{w} \rangle. \quad (6.22)$$

This shows that, relative to an ortho-normal basis, a scalar product can be expressed in terms of the standard scalar product on  $\mathbb{R}^n$  or  $\mathbb{C}^n$ . Suppose we would like to compute the representing matrix  $A$  of a linear map  $f : V \rightarrow V$  relative to an ortho-normal basis  $\{\boldsymbol{\epsilon}_1, \dots, \boldsymbol{\epsilon}_n\}$  of  $V$ . In general, following Lemma 3.3, the entries  $A_{ij}$  of the matrix  $A$  can be obtained from

$$f(\boldsymbol{\epsilon}_j) = \sum_i A_{ij} \boldsymbol{\epsilon}_i. \quad (6.23)$$

Taking the scalar product of this equation with  $\boldsymbol{\epsilon}_k$  results in the simple formula

$$A_{ij} = \langle \boldsymbol{\epsilon}_i, f(\boldsymbol{\epsilon}_j) \rangle. \quad (6.24)$$

In physics, the RHS of this expression is often referred to as a *matrix element* of the map  $f$ . It is worth noting that a linear map is uniquely determined by its matrix elements.

**Lemma 6.2.** *If two linear maps  $f : V \rightarrow V$  and  $g : V \rightarrow V$  satisfy  $\langle \mathbf{v}, f(\mathbf{w}) \rangle = \langle \mathbf{v}, g(\mathbf{w}) \rangle$  (or  $\langle f(\mathbf{v}), \mathbf{w} \rangle = \langle g(\mathbf{v}), \mathbf{w} \rangle$ ) for all  $\mathbf{v}, \mathbf{w} \in V$  then  $f = g$ .*

*Proof.* By linearity of the scalar product in the second argument the assumption implies that  $\langle \mathbf{v}, f(\mathbf{w}) - g(\mathbf{w}) \rangle = 0$  for all  $\mathbf{v}, \mathbf{w} \in V$ . In particular, if we choose  $\mathbf{v} = f(\mathbf{w}) - g(\mathbf{w})$ , it follows from Def. 6.1 (S3) that  $f(\mathbf{w}) - g(\mathbf{w}) = 0$ . Since this holds for all  $\mathbf{w}$  it follows that  $f = g$ . The alternative statement follows simply by applying Def. 6.1 (S1).  $\square$

**Example 6.5:** Calculating the matrix representing a linear map relative to an ortho-normal basis

For a fixed vector  $\mathbf{n} \in \mathbb{R}^3$ , we consider the linear map  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by

$$f(\mathbf{v}) = (\mathbf{n} \cdot \mathbf{v}) \mathbf{n}. \quad (6.25)$$

Evidently, this map projects vectors into the direction of  $\mathbf{n}$ . We would like to compute the matrix  $A$  representing this linear map relative to the ortho-normal basis given by the three standard unit vectors  $\mathbf{e}_i$ , using Eq. (6.24). We find

$$A_{ij} = \mathbf{e}_i \cdot f(\mathbf{e}_j) = (\mathbf{n} \cdot \mathbf{e}_i)(\mathbf{n} \cdot \mathbf{e}_j) = n_i n_j. \quad (6.26)$$

and, hence,  $A_{ij} = n_i n_j$  or, in matrix notation

$$A = \begin{pmatrix} n_1^2 & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & n_2^2 & n_2 n_3 \\ n_1 n_3 & n_2 n_3 & n_3^2 \end{pmatrix} \quad (6.27)$$

We end this discussion of orthogonality with a result on perpendicular spaces. For a sub vector space  $W \subset V$  the perpendicular space  $W^\perp$  is defined as

$$W^\perp = \{\mathbf{v} \in V \mid \langle \mathbf{w}, \mathbf{v} \rangle = 0 \text{ for all } \mathbf{w} \in W\}. \quad (6.28)$$

In other words,  $W^\perp$  consists of all vectors which are orthogonal to all vector in  $W$ . For example, if  $W \subset \mathbb{R}^3$  is a plane through the origin then  $W^\perp$  is the line through the origin perpendicular to this plane. The following statements are intuitive and will be helpful for our treatment of eigenvectors and eigenvalues in the next section.

**Lemma 6.3.** *For a sub vector space  $W \subset V$  of a finite dimensional vector space  $V$  with a scalar product the following holds:*

- (i)  $W^\perp$  is a sub vector space of  $V$ .
- (ii)  $W \cap W^\perp = \{\mathbf{0}\}$
- (iii)  $\dim(W) + \dim(W^\perp) = \dim(V)$

*Proof.* (i) If  $\mathbf{v}_1, \mathbf{v}_2 \in W^\perp$  then clearly  $\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 \in W^\perp$  so from Def. 1.2  $W^\perp$  is a sub vector space.

(ii) If  $\mathbf{v} \in W \cap W^\perp$  then  $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ , but from Def. 6.1 (S3) this implies that  $\mathbf{v} = \mathbf{0}$ .

(iii) Choose an ortho-normal basis  $\{\epsilon_1, \dots, \epsilon_k\}$  of  $W$  and define the linear map  $f : V \rightarrow V$  by  $f(\mathbf{v}) = \sum_{i=1}^k \langle \epsilon_i, \mathbf{v} \rangle \epsilon_i$  (a projection onto  $W$ ). Clearly  $\text{Im}(f) \subset W$ . For  $\mathbf{w} \in W$  it follows from Eq. (6.17) that  $f(\mathbf{w}) = \mathbf{w}$  so that  $\text{Im}(f) = W$ . Moreover,  $\text{Ker}(f) = W^\perp$  and the claim follows from the dimension formula (3.4) applied to the map  $f$ .  $\square$

### 6.3 Adjoint linear map

A common theme in mathematics is to explore the new structures which arise from consistency requirements when two mathematical ideas are combined. In the present case, we have combined vector spaces and scalar products. Since vector spaces are equipped with linear maps it is, therefore, natural to ask about the relation between linear maps and scalar product. Specifically, we would like to study, in this sub-section and the next, specific classes of linear maps which relate to a given scalar product in an interesting way. We begin by defining *adjoint linear maps*.

**Definition 6.3.** *For a linear map  $f : V \rightarrow V$  on a vector space  $V$  with scalar product, an adjoint linear map,  $f^\dagger : V \rightarrow V$  is a map satisfying*

$$\langle \mathbf{v}, f\mathbf{w} \rangle = \langle f^\dagger \mathbf{v}, \mathbf{w} \rangle \quad (6.29)$$

for all  $\mathbf{v}, \mathbf{w} \in V$ .

In other words, a linear map can be “moved” into the other argument of the scalar product by taking its adjoint. The following properties of the adjoint map are relatively easy to show.

**Lemma 6.4.** *(Properties of the adjoint)*

- (i) For a given linear map  $f$  the adjoint  $f^\dagger$  is uniquely determined.
- (ii)  $(f^\dagger)^\dagger = f$
- (iii)  $(f + g)^\dagger = f^\dagger + g^\dagger$
- (iv)  $(\alpha f)^\dagger = \alpha^* f^\dagger$
- (v)  $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$
- (vi)  $(f^{-1})^\dagger = (f^\dagger)^{-1}$ , if  $f$  is invertible.

*Proof.* (i) For two adjoints  $f_1, f_2$  for  $f$  we have  $\langle f_1(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, f(\mathbf{w}) \rangle = \langle f_2(\mathbf{v}), \mathbf{w} \rangle$  for all  $\mathbf{v}, \mathbf{w} \in V$ . Then Lemma 6.2 implies that  $f_1 = f_2$ .

(ii)  $\langle (f^\dagger)^\dagger(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, f^\dagger(\mathbf{w}) \rangle = \langle f(\mathbf{v}), \mathbf{w} \rangle$ . Comparing the LHS and RHS together with Lemma 6.2 shows that  $(f^\dagger)^\dagger = f$ .

(iii)  $\langle (f+g)^\dagger(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, (f+g)(\mathbf{w}) \rangle = \langle \mathbf{v}, f(\mathbf{w}) \rangle + \langle \mathbf{v}, g(\mathbf{w}) \rangle = \langle f^\dagger(\mathbf{v}), \mathbf{w} \rangle + \langle g^\dagger(\mathbf{v}), \mathbf{w} \rangle = \langle (f^\dagger+g^\dagger)(\mathbf{v}), \mathbf{w} \rangle$  and the claim follows from Lemma 6.2.

(iv)  $\langle (\alpha f)^\dagger(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, (\alpha f)(\mathbf{w}) \rangle = \alpha \langle \mathbf{v}, f(\mathbf{w}) \rangle = \alpha \langle f^\dagger(\mathbf{v}), \mathbf{w} \rangle = \langle (\alpha^* f^\dagger)(\mathbf{v}), \mathbf{w} \rangle$  and Lemma 6.2 leads to the stated result.

(v)  $\langle (f \circ g)^\dagger(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, (f \circ g)(\mathbf{w}) \rangle = \langle f^\dagger(\mathbf{v}), g(\mathbf{w}) \rangle = \langle g^\dagger \circ f^\dagger(\mathbf{v}), \mathbf{w} \rangle$ .

(vi) From (v) we have  $\text{id}_V = (f \circ f^{-1})^\dagger = f^\dagger \circ (f^{-1})^\dagger$ . This means  $(f^{-1})^\dagger$  is the inverse of  $f^\dagger$  and, hence,  $(f^\dagger)^{-1} = (f^{-1})^\dagger$ .  $\square$

Let us now proceed in a more practical way and understand the adjoint map relative to an orthonormal basis  $\epsilon_1, \dots, \epsilon_n$  of  $V$ . From Eq. (6.24) the matrices  $A, B$  describing  $f$  and  $f^\dagger$  relative to this basis are given by

$$A_{ij} = \langle \epsilon_i, f(\epsilon_j) \rangle, \quad B_{ij} = \langle \epsilon_i, f^\dagger(\epsilon_j) \rangle. \quad (6.30)$$

Using the scalar product property (S1) in Def. (6.1) these matrices are related by

$$B_{ij} = \langle \epsilon_i, f^\dagger(\epsilon_j) \rangle = \langle f^\dagger(\epsilon_j), \epsilon_i \rangle^* = \langle \epsilon_j, f(\epsilon_i) \rangle^* = A_{ji}^* \implies B = A^\dagger, \quad (6.31)$$

that is, if  $A$  represents  $f$  then the hermitian conjugate  $A^\dagger$  represents  $f^\dagger$ . This also shows that, by reversing the above argument and defining  $f^\dagger$  as the linear map associated to  $A^\dagger$ , that the adjoint always exists - this was not immediately clear from the definition.

Previously, we have introduced hermitian conjugation merely as a “mechanical” operation to be carried out for matrices. Now we understand its proper mathematical context - it leads to the matrix which describes the adjoint linear map. In the case of a real scalar product we can of course drop the complex conjugation in Eq. (6.31) and the matrix describing the adjoint becomes  $A^T$ , the transpose of  $A$ . Hence, we have also found a mathematical interpretation for the transposition of matrices.

We have seen in Eq. (6.22) that, with respect to an orthonormal basis, a scalar product is described by the standard (real or complex) scalar product on  $\mathbb{R}^n$  or  $\mathbb{C}^n$ . It is, therefore, clear that the adjoint of a matrix  $A$  with respect to the standard scalar product is given by its hermitian conjugate,  $A^\dagger$  (or  $A^T$  in the real case). This is easy to verify explicitly from the definition of the standard scalar product in Example 6.1.

$$\langle \mathbf{v}, A\mathbf{w} \rangle = \mathbf{v}^\dagger A\mathbf{w} = (A^\dagger \mathbf{v})^\dagger \mathbf{w} = \langle A^\dagger \mathbf{v}, \mathbf{w} \rangle. \quad (6.32)$$

A particularly important class of linear maps are those which coincide with their adjoint.

**Definition 6.4.** A linear map  $f : V \rightarrow V$  on a vector space  $V$  with scalar product is called *self-adjoint* (or *hermitian*) iff  $f = f^\dagger$ .

In other words, self-adjoint maps can be moved from one argument of the scalar product into another, so

$$\langle \mathbf{v}, f(\mathbf{w}) \rangle = \langle f(\mathbf{v}), \mathbf{w} \rangle \iff \langle \mathbf{v}, f(\mathbf{w}) \rangle = \langle \mathbf{w}, f(\mathbf{v}) \rangle^*. \quad (6.33)$$

Clearly, relative to an orthonormal basis, a self-adjoint linear map is described by a hermitian matrix (or a symmetric matrix in the real case). Further, the self-adjoint linear maps on  $\mathbb{R}^n$  ( $\mathbb{C}^n$ ) with respect to the standard scalar product are the symmetric (hermitian) matrices.

**Example 6.6:** A self-adjoint derivative map

For a more abstract example of a self-adjoint linear map, consider the vector space of (infinitely many times) differentiable functions  $\varphi : [a, b] \rightarrow \mathbb{C}$ , satisfying  $\varphi(a) = \varphi(b)$ , with scalar product

$$\langle \varphi, \psi \rangle = \int_a^b dx \varphi(x)^* \psi(x) .$$

The derivative operator

$$D = -i \frac{d}{dx}$$

defines a linear map on this space and we would like to check that it is self-adjoint. Performing an integration by parts we find

$$\begin{aligned} \langle \varphi, D\psi \rangle &= -i \int_a^b dx \varphi(x)^* \frac{d\psi}{dx}(x) = -i [\varphi(x)^* \psi(x)]_a^b + i \int_a^b dx \frac{d\varphi}{dx}(x)^* \psi(x) = \int_a^b dx (D\varphi)(x)^* \psi(x) \\ &= \langle D\varphi, \psi \rangle . \end{aligned}$$

Hence,  $D$  is indeed hermitian. Note that the boundary term vanishes due to the boundary condition on our functions and that including the factor of  $i$  in the definition of  $D$  is crucial for the sign to work out correctly. In quantum mechanics physical quantities are represented by hermitian operators. In this context, the present operator  $D$  plays an important role as it corresponds to linear momentum.

## 6.4 Orthogonal and unitary maps

Another important class of linear maps which relate to scalar product in a particular way are *orthogonal and unitary maps*. They are the linear maps which leave a scalar product unchanged in the sense of

**Definition 6.5.** *Let  $V$  be a vector space with a real (hermitian) scalar product. A linear map  $f : V \rightarrow V$  is called orthogonal (unitary) iff*

$$\langle f(\mathbf{v}), f(\mathbf{w}) \rangle = \langle \mathbf{v}, \mathbf{w} \rangle \tag{6.34}$$

for all  $\mathbf{v}, \mathbf{w} \in V$ .

In particular, orthogonal or unitary maps  $f$  leave lengths of vectors unchanged, so  $|f(\mathbf{v})| = |\mathbf{v}|$ . In the real case, we can use the scalar product to define angles between vectors as in Eq. (6.6), so orthogonal maps  $f$  leave such angles unchanged, that is,  $\sphericalangle(\mathbf{v}, \mathbf{w}) = \sphericalangle(f(\mathbf{v}), f(\mathbf{w}))$ .

**Lemma 6.5.** *(Properties of unitary maps)*

- (i) Unitary maps  $f$  can also be characterized by  $f^\dagger \circ f = \text{id}_V$ .
- (ii) Unitary maps  $f$  are invertible and  $f^{-1} = f^\dagger$ .
- (iii) The composition of unitary maps is a unitary map.
- (iv) The inverse,  $f^\dagger$ , of a unitary map  $f$  is unitary.

*Proof.* (i) Using the adjoint map, the condition (6.34) can be re-written as  $\langle \mathbf{v}, f^\dagger \circ f(\mathbf{w}) \rangle = \langle \mathbf{v}, \text{id}_V(\mathbf{w}) \rangle$ . From Lemma 6.2 a function is uniquely determined by its matrix elements so that orthogonal and unitary operators can also be defined by the condition

$$f^\dagger \circ f = \text{id}_V . \tag{6.35}$$

(ii) A direct consequence of (i).

(iii) For two unitary maps  $f, g$ , satisfying  $\langle f(\mathbf{v}), f(\mathbf{w}) \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$  and  $\langle g(\mathbf{v}), g(\mathbf{w}) \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ , it follows that  $\langle f \circ g(\mathbf{v}), f \circ g(\mathbf{w}) \rangle = \langle f(\mathbf{v}), f(\mathbf{w}) \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$  and hence, that  $f \circ g$  is unitary.

(iv) From  $\langle f(\mathbf{v}), f(\mathbf{w}) \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ , writing  $\mathbf{v}' = f(\mathbf{v})$ ,  $\mathbf{w}' = f(\mathbf{w})$  it follows that  $\langle \mathbf{v}', \mathbf{w}' \rangle = \langle f^{-1}(\mathbf{v}'), f^{-1}(\mathbf{w}') \rangle$  so that  $f^{-1} = f^\dagger$  is unitary.  $\square$

As before, it is useful to work out what this means relative to an ortho-normal basis. If  $f$  is described by a matrix  $A$  relative to this basis then we already know that  $f^\dagger$  is described by the hermitian conjugate  $A^\dagger$  in the complex case or by the transpose  $A^T$  in the real case.

We begin with the real case where the condition (6.35) turns into

$$A^T A = \mathbb{1} \iff A^{-1} = A^T \iff \mathbf{A}^i \cdot \mathbf{A}^j = \delta_{ij}. \quad (6.36)$$

Matrices  $A$  satisfying this condition are called *orthogonal matrices* and they can be characterized, equivalently, by either one of the three conditions above. The simplest way to check if a given matrix is orthogonal is usually to verify the condition on the LHS. The condition in the middle tells us it is easy to compute the inverse of an orthogonal matrix - it is simply the transpose. And, finally, the condition on the RHS says that the column vectors of an orthogonal matrix form an ortho-normal basis with respect to the standard scalar product (the dot product). In fact, since a real scalar product, written in terms of an ortho-normal basis, corresponds to the dot product, see Eq. (6.22), we expect that orthogonal matrices are precisely those matrices which leave the dot product invariant. Indeed, we have

$$A^T A = \mathbb{1}_n \iff (A\mathbf{v})^T (A\mathbf{w}) = \mathbf{v}^T \mathbf{w} \quad \text{for all } \mathbf{v}, \mathbf{w} \in \mathbb{R}^n. \quad (6.37)$$

The set of all  $n \times n$  orthogonal matrices is also denoted by  $O(n)$ . Taking the determinant of the LHS condition in (6.36) and using Lemma 5.1 and Theorem 5.1 gives  $1 = \det(\mathbb{1}) = \det(AA^T) = \det(A) \det(A^T) = \det(A)^2$  so that

$$\det(A) = \pm 1 \quad (6.38)$$

for any orthogonal matrix. The subset of  $n \times n$  orthogonal matrices  $A$  with determinant  $\det(A) = +1$  is called *special orthogonal matrices* or *rotations* and denoted by  $SO(n)$ . Note that the term “rotation” is indeed appropriate for those matrices. Since they leave the dot product invariant they do not change lengths of vectors and angles between vectors and the  $\det(A) = +1$  conditions excludes orthogonal matrices which contain reflections. The relation between orthogonal matrices with positive and negative determinants is easy to understand. Consider an orthogonal matrix  $A$  with  $\det(A) = -1$  and the specific orthogonal matrix  $F = \text{diag}(-1, 1, \dots, 1)$  with  $\det(F) = -1$  which corresponds to a reflection in the first coordinate direction. Then the matrix  $R = FA$  is a rotation since  $\det(R) = \det(F) \det(A) = (-1)^2 = 1$ . This means every orthogonal matrix  $A$  can be written as a product

$$A = FR \quad (6.39)$$

of a rotation  $R$  and a reflection  $F$ . To get a better feeling for rotations we should look at some low-dimensional examples.

### Example 6.7: Rotations in two and three dimensions

#### (a) Two dimensions

To find the explicit form of two-dimensional rotation matrices we start with a general  $2 \times 2$  matrix

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where  $a, b, c, d$  are real numbers and impose the conditions  $R^T R = \mathbb{1}_2$  and  $\det(R) = 1$ . This gives

$$R^T R = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \det(R) = ad - bc \stackrel{!}{=} 1,$$



and, hence, the equations  $a^2 + c^2 = b^2 + d^2 = 1$ ,  $ab + cd = 0$  and  $ad - bc = 1$ . It is easy to show that a solution to these equations can always be written as  $a = d = \cos(\theta)$ ,  $c = -b = \sin(\theta)$ , for some angle  $\theta$  so that two-dimensional rotation matrices can be written in the form

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \quad (6.40)$$

For the rotation of an arbitrary vector  $\mathbf{x} = (x, y)^T$  we get

$$\mathbf{x}' = R\mathbf{x} = \begin{pmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{pmatrix}. \quad (6.41)$$

It is easy to verify explicitly that  $|\mathbf{x}'| = |\mathbf{x}|$ , as must be the case, and that the cosine of the angle between  $\mathbf{x}$  and  $\mathbf{x}'$  is given by

$$\cos(\angle(\mathbf{x}', \mathbf{x})) = \frac{\mathbf{x}' \cdot \mathbf{x}}{|\mathbf{x}'||\mathbf{x}|} = \frac{(x \cos \theta - y \sin \theta)x + (x \sin \theta + y \cos \theta)y}{|\mathbf{x}|^2} = \cos \theta. \quad (6.42)$$

This result means we should interpret  $R(\theta)$  as a rotation by an angle  $\theta$ . From the addition theorems of sin and cos it also follows easily that

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2), \quad (6.43)$$

that is, the rotation angle adds up for subsequent rotations, as one would expect. Note, Eq. (6.43) also implies that two-dimensional rotations commute, since  $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2) = R(\theta_2 + \theta_1) = R(\theta_2)R(\theta_1)$ , again a property intuitively expected.

### (b) Three dimensions

To find the explicit form for three-dimensional rotations we could, in principle, use the same approach as in two dimensions and impose all relevant constraints on an arbitrary  $3 \times 3$  matrix. However, this leads to a set of equations in 9 variables and is much more complicated. However, it is easy to obtain special three-dimensional rotations from two-dimensional ones. For example, the matrices

$$R_1(\theta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & -\sin \theta_1 \\ 0 & \sin \theta_1 & \cos \theta_1 \end{pmatrix} \quad (6.44)$$

clearly satisfy  $R_1(\theta_1)^T R_1(\theta_1) = \mathbb{1}_3$  and  $\det(R_1(\theta_1)) = 1$  and are, hence, rotation matrices. They describe a rotation by an angle  $\theta_1$  around the first coordinate axis. Analogously, rotation matrices around the other two coordinate axis can be written as

$$R_2(\theta_2) = \begin{pmatrix} \cos \theta_2 & 0 & -\sin \theta_2 \\ 0 & 1 & 0 \\ \sin \theta_2 & 0 & \cos \theta_2 \end{pmatrix}, \quad R_3(\theta_3) = \begin{pmatrix} \cos \theta_3 & -\sin \theta_3 & 0 \\ \sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6.45)$$

It turns out that general three-dimensional rotation matrices can be obtained as products of the above three special types. For example, we can write a three-dimensional rotation matrix as  $R(\theta_1, \theta_2, \theta_3) = R_1(\theta_1)R_2(\theta_2)R_3(\theta_3)$ , that is, as subsequent rotations around the three coordinate axis. Of course, there are different ways of doing this, another choice frequently used in physics being  $R(\psi, \theta, \phi) = R_3(\psi)R_1(\theta)R_3(\phi)$ . The angles  $\psi, \theta, \phi$  in this parametrization are also called the *Euler angles* and in this case, the rotation is combined from a rotation by  $\phi$  around the  $z$ -axis, then a rotation by  $\theta$  around the  $x$ -axis and finally another rotation by  $\psi$  around the (new)  $z$ -axis. The Euler angles are particularly useful to describe the motion of tops in classical mechanics.

Finally, we note that, unlike their two-dimensional counterparts, three-dimensional rotations do not, in general, commute. For example, apart from special choices for the angles  $R_1(\theta_1)R_2(\theta_2) \neq R_2(\theta_2)R_1(\theta_1)$ .

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### Application: Rotating physical systems

Suppose we have a stationary coordinate system with coordinates  $\mathbf{x} \in \mathbb{R}^3$  and another coordinate system with coordinates  $\mathbf{y} \in \mathbb{R}^3$ , which is rotating relative to the first one. Such a set-up can be used to describe the mechanics of objects in rotating systems and has many applications, for example to the physics of tops or the laws of motion in rotating systems such as the earth (see below). Mathematically, the relation between these two coordinate system can be described by the equation

$$\mathbf{x} = R(t)\mathbf{y} , \quad (6.46)$$

where  $R(t)$  are time-dependent rotation matrices. This means the matrices  $R(t)$  satisfy

$$R(t)^T R(t) = \mathbb{1}_3 , \quad (6.47)$$

(as well as  $\det(R(t)) = 1$ ) for all times  $t$ . In practice, we can write rotation matrices in terms of rotation angles, as we have done in Example 6.7. The time-dependence of  $R(t)$  then means that the rotation angles are functions of time. For example, a rotation around the  $z$ -axis with constant angular speed  $\omega$  can be written as

$$R(t) = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} . \quad (6.48)$$

In physics, a rotation is often described by the angular velocity  $\boldsymbol{\omega}$ , a vector whose direction indicates the axis of rotation and whose length gives the angular speed. It is very useful to understand the relation between  $R(t)$  and  $\boldsymbol{\omega}$ . To do this, define the matrix

$$W = R^T \dot{R} , \quad (6.49)$$

where the dot denotes the time derivative and observe, by differentiating Eq. (6.47) with respect to time, that

$$\underbrace{R^T \dot{R}}_{=W} + \underbrace{\dot{R}^T R}_{=W^T} = 0 . \quad (6.50)$$

Hence,  $W$  is an anti-symmetric matrix and can be written in the form

$$W = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad \text{or} \quad W_{ij} = \epsilon_{ikj} \omega_k . \quad (6.51)$$

The three independent entries  $\omega_i$  of this matrix define the angular velocity  $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T$ . To see that this makes sense consider the example (6.48) and work out the matrix  $W$ .

$$W = \omega \begin{pmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin(\omega t) & -\cos(\omega t) & 0 \\ \cos(\omega t) & -\sin(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \quad (6.52)$$

Comparison with the general form (6.51) of  $W$  then shows that the angular velocity for this case is given by  $\boldsymbol{\omega} = (0, 0, \omega)$ , indicating a rotation with angular speed  $\omega$  around the  $z$ -axis, as expected.

In Example 3.8 we have seen that the multiplication of an anti-symmetric  $3 \times 3$  matrix with a vector can be written as a cross-product, so that

$$W\mathbf{b} = \boldsymbol{\omega} \times \mathbf{b} \quad (6.53)$$

for any vector  $\mathbf{b} = (b_1, b_2, b_3)^T$ . This can also be directly verified using the matrix form of  $W$  together with the definition (2.17) of the cross product or, more elegantly, by the index calculation  $W_{ij}b_j = \epsilon_{ikj}\omega_k b_j = (\boldsymbol{\omega} \times \mathbf{b})_i$ , using the index form (2.29) of the cross product. This relation can be used to re-write expressions involving  $W$  in terms of the angular velocity  $\boldsymbol{\omega}$ .

For a simple application of this formalism, consider an object moving with velocity  $\dot{\mathbf{y}}$  relative to the rotating system. What is its velocity relative to the stationary coordinate system? Differentiating Eq. (6.46) gives

$$\dot{\mathbf{x}} = R\dot{\mathbf{y}} + \dot{R}\mathbf{y} = R(\dot{\mathbf{y}} + W\mathbf{y}) = R(\dot{\mathbf{y}} + \boldsymbol{\omega} \times \mathbf{y}) . \quad (6.54)$$

The velocity  $\dot{\mathbf{x}}$  in the stationary system has, therefore, two contribution, namely the velocity  $\dot{\mathbf{y}}$  relative to the rotating system and the velocity  $\boldsymbol{\omega} \times \mathbf{y}$  due to the rotation itself.

We now turn to the complex case. In this case, from Eq. (6.35), (complex) matrices  $A$  describing unitary maps relative to an ortho-normal basis are characterized by the three equivalent conditions

$$A^\dagger A = \mathbb{1} \quad \iff \quad A^{-1} = A^\dagger \quad \iff \quad (\mathbf{A}^i)^\dagger \mathbf{A}^j = \delta_{ij} . \quad (6.55)$$

Matrices satisfying these conditions are called *unitary*. As for orthogonal matrices, checking whether a given matrix is unitary is usually easiest accomplished using the condition on the LHS. The condition in the middle states that the inverse of a unitary matrix is simply its hermitian conjugate and the condition on the RHS says that the column vectors of a unitary matrix form an ortho-normal basis under the standard hermitian scalar product on  $\mathbb{C}^n$ . Unitary matrices are precisely those matrices which leave the standard hermitian scalar product invariant, explicitly

$$A^\dagger A = \mathbb{1}_n \quad \iff \quad (A\mathbf{v})^\dagger (A\mathbf{w}) = \mathbf{v}^\dagger \mathbf{w} \quad \text{for all } \mathbf{v}, \mathbf{w} \in \mathbb{C}^n . \quad (6.56)$$

The set of all  $n \times n$  unitary matrices is denoted by  $U(n)$ . Orthogonal matrices (being real) also satisfy the condition for unitary matrices so  $O(n) \subset U(n)$ . For the determinant of unitary matrices we conclude that  $1 = \det(\mathbb{1}) = \det(A^\dagger A) = \det(A)^* \det(A) = |\det(A)|^2$ . Hence, the determinant of unitary matrices has complex modulus 1, so

$$|\det(A)| = 1 . \quad (6.57)$$

The unitary matrices  $U$  with  $\det(U) = 1$  are called *special unitary matrices*, and the set of these matrices is denoted by  $SU(n)$ . Clearly, rotations are also special unitary so  $SO(n) \subset SU(n)$ . For an arbitrary unitary  $n \times n$  matrix  $A$  we can always find a complex phase  $\zeta$  such that  $\zeta^n = \det(A)$ . Then, the matrix  $U = \zeta^{-1}A$  is special unitary since  $\det(U) = \det(\zeta^{-1}A) = \zeta^{-n} \det(A) = 1$ . This means every unitary matrix  $A$  can be written as a product

$$A = \zeta U \quad (6.58)$$

of a special unitary matrix  $U$  and a complex phase  $\zeta$ .

### Example 6.8: Special unitary matrices in two dimensions

We can find all two-dimensional special unitary matrices by using the same method as for two-dimensional rotations. We start with an arbitrary complex  $2 \times 2$  matrix

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} ,$$

where  $\alpha, \beta, \gamma, \delta$  are complex numbers and impose the conditions  $U^\dagger U = \mathbb{1}_2$  and  $\det(U) = 1$ . After a short calculation we find that every two-dimensional special unitary matrix can be written in terms of

two complex numbers  $\alpha, \beta$  as

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad \text{where} \quad |\alpha|^2 + |\beta|^2 = 1 . \quad (6.59)$$

This shows that two-dimensional special unitary matrices depend on two complex parameters  $\alpha, \beta$  subject to the (real) constraint  $|\alpha|^2 + |\beta|^2 = 1$  and, hence, on three real parameters. Inserting the special choice  $\alpha = \cos \theta, \beta = -\sin \theta$  into (6.59) we recover the two-dimensional rotation matrices (6.7), so that  $SO(2) \subset SU(2)$ , as expected from our general discussion.

The general study of orthogonal and unitary matrices is part of the theory of *Lie groups*, a more advanced mathematical discipline which is beyond the scope of this introductory text.

Orthogonal and unitary matrices have numerous applications in physics which we would like to illustrate with an example from classical mechanics.

### Application: Newton's law in a rotating system

Newton's law for the motion  $\mathbf{x} = \mathbf{x}(t)$  of a mass point with mass  $m$  under the influence of a force  $\mathbf{F}$  reads

$$m\ddot{\mathbf{x}} = \mathbf{F} , \quad (6.60)$$

where the dot denotes the derivative with respect to time  $t$ . We would like to work out the form this law takes if we transform it to rotating coordinates  $\mathbf{y}$ , related to the original, non-rotating coordinates  $\mathbf{x}$  by

$$\mathbf{x} = R(t)\mathbf{y} . \quad (6.61)$$

Here  $R(t)$  is a (generally time-dependent) rotation, that is, a  $3 \times 3$  matrix satisfying

$$R(t)^T R(t) = \mathbb{1}_3 \quad (6.62)$$

for all times  $t$ . For example, such a version of Newton's law is relevant to describing mechanics on earth.

To re-write Eq. (6.60) in terms of  $\mathbf{y}$  we first multiply both sides with  $R^T = R^{-1}$  so that

$$mR^T \ddot{\mathbf{x}} = \mathbf{F}_R , \quad (6.63)$$

with  $\mathbf{F}_R := R^T \mathbf{F}$  the force in the rotating coordinate system. If the rotation matrix is time-independent it can be pulled through the time derivatives on the LHS of Eq. (6.63) and we get  $m\ddot{\mathbf{y}} = \mathbf{F}_R$ . This simply says that Newton's law keeps the same form in any rotated (but not rotating!) coordinate system.

If  $R$  is time-dependent so that the system with coordinates  $\mathbf{y}$  is indeed rotating relative to the coordinate system  $\mathbf{x}$  we have to be more careful. Taking two time derivatives of Eq. (6.61) gives

$$\dot{\mathbf{x}} = R\dot{\mathbf{y}} + \dot{R}\mathbf{y} , \quad \ddot{\mathbf{x}} = R\ddot{\mathbf{y}} + 2\dot{R}\dot{\mathbf{y}} + \ddot{R}\mathbf{y} . \quad (6.64)$$

Using the second of these equations to replace  $\ddot{\mathbf{x}}$  in Eq. (6.63) leads to

$$m\ddot{\mathbf{y}} = \mathbf{F}_R - 2mR^T \dot{R}\dot{\mathbf{y}} - mR^T \ddot{R}\mathbf{y} . \quad (6.65)$$

Compared to Newton's equation in the standard form (6.60) we have acquired the two additional terms on the RHS which we should work out further. From Eq. (6.49), recall the definition  $W = R^T \dot{R}$  and further note that  $\dot{W} = R^T \ddot{R} + \dot{R}^T \dot{R} = R^T \ddot{R} + (\dot{R}^T R)(R^T \dot{R}) = R^T \ddot{R} - W^2$ , so that

$$R^T \ddot{R} = \dot{W} + W^2 . \quad (6.66)$$

With these results we can re-write Newton's equation (6.65) as

$$m\ddot{\mathbf{y}} = \mathbf{F}_R - 2mW\dot{\mathbf{y}} - mW^2\mathbf{y} - m\dot{W}\mathbf{y} . \quad (6.67)$$

Also, recall that the matrix  $W$  is anti-symmetric, encodes the angular velocity  $\boldsymbol{\omega}$ , as in Eq. (6.51) and its action on vectors can be re-written as a cross product with the angular velocity  $\boldsymbol{\omega}$  (see Eq. (6.53)). Then, Newton's equation (6.67) in a rotating system can be written in its final form

$$m\ddot{\mathbf{y}} = \mathbf{F}_R \underbrace{-2m\boldsymbol{\omega} \times \dot{\mathbf{y}}}_{\text{Coriolis force}} \underbrace{-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{y})}_{\text{centrifugal force}} \underbrace{-2m\dot{\boldsymbol{\omega}} \times \mathbf{y}}_{\text{Euler force}} . \quad (6.68)$$

The three terms on the RHS represent the additional forces a mass point experiences in a rotating system. The centrifugal force is well-known. The Coriolis force is proportional to the velocity,  $\dot{\mathbf{y}}$ , and, hence, vanishes for mass points which rest in the rotating frame. It is, for example, responsible for the rotation of a Foucault pendulum. Finally, the Euler force is proportional to the angular acceleration,  $\dot{\boldsymbol{\omega}}$ . For the earth's rotation,  $\boldsymbol{\omega}$  is approximately constant so the Euler force is quite small in this case.

## 6.5 Dual vector space

We have seen in Lemma 3.2 that the linear maps  $f : V \rightarrow W$ , between two vector spaces  $V$  and  $W$  over  $F$ , form a vector space themselves. An important special case is the set of all linear maps  $\varphi : V \rightarrow F$ , where we consider the field  $F$  as a (trivial, one-dimensional) vector space. Such linear maps are also called *linear functionals* and the set of all linear functionals is called the *dual vector space*  $V^*$  of  $V$ .

**Definition 6.6.** For a vector space  $V$  over  $F$  the dual vector space  $V^*$  is the set of all linear maps  $V \rightarrow F$  (where  $F$  is seen as a one-dimensional vector space.) The elements of  $V^*$  are called linear functionals.

**Example 6.9:** Examples of linear functionals

(a) For  $V = \mathbb{R}^n$  and a fixed vector  $\mathbf{w} \in V$  we can define  $\varphi_{\mathbf{w}} \in (\mathbb{R}^n)^*$  by

$$\varphi_{\mathbf{w}}(\mathbf{v}) = \mathbf{w}^T \mathbf{v} \in \mathbb{R} . \quad (6.69)$$

It is clear that  $\varphi_{\mathbf{w}}$  is a linear functional. Indeed all linear functionals in  $(\mathbb{R}^n)^*$  are of this form. To see this, start with an arbitrary  $\varphi \in (\mathbb{R}^n)^*$  and define the vector  $\mathbf{w}$  with components  $w_i = \varphi(\mathbf{e}_i)$ . Then

$$\varphi(\mathbf{v}) = \varphi\left(\sum_i v_i \mathbf{e}_i\right) = \sum_i v_i \varphi(\mathbf{e}_i) = \sum_i w_i v_i = \mathbf{w}^T \mathbf{v} = \varphi_{\mathbf{w}}(\mathbf{v}) . \quad (6.70)$$

Hence,  $\varphi = \varphi_{\mathbf{w}}$  and we have written an arbitrary functional in the form (6.69). This result means we can think of the functionals on  $\mathbb{R}^n$  as  $n$ -dimensional row vectors.

(b) For the vector space of continuous functions  $h : [a, b] \rightarrow \mathbb{R}$  the integral

$$I(h) = \int_a^b dx h(x) \quad (6.71)$$

is a linear functional. Another interesting functional on the same vector space is

$$\delta_{x_0}(h) := h(x_0) , \quad (6.72)$$

where  $x_0 \in [a, b]$  is a fixed point. In the physics literature this functional is also called *Dirac delta function*.

We know that a linear map  $f : V \rightarrow W$ , with  $n = \dim(V)$  and  $m = \dim W$ , is described by an  $m \times n$  matrix relative to a choice of basis on  $V$  and  $W$ . For  $W = F$ , we have  $m = \dim(W) = \dim(F) = 1$ , so relative to a basis on  $V$ , linear functionals are described by  $1 \times n$  matrices, that is, by row vectors. So, for a choice of basis, we can think of the vector space  $V$  as consisting of column vectors and its dual  $V^*$  as consisting of row vectors. To make this more precise we prove the following

**Theorem 6.2.** *For a basis  $\epsilon_1, \dots, \epsilon_n$  of  $V$  there is a basis  $\epsilon_*^1, \dots, \epsilon_*^n$  of  $V^*$ , called the dual basis, such that*

$$\epsilon_*^i(\epsilon_j) = \delta_j^i. \quad (6.73)$$

In particular,  $\dim(V^*) = \dim(V)$ .

*Proof.* Recall from Example 3.4, that we can define a coordinate map  $\psi(\alpha) = \sum_i \alpha^i \epsilon_i$  which assigns to a coordinate vector  $\alpha = (\alpha^1, \dots, \alpha^n)^T$  the corresponding vector, relative to the chosen basis  $\epsilon_i$ . We define

$$\epsilon_*^i(\mathbf{v}) := \mathbf{e}_i^T \psi^{-1}(\mathbf{v}), \quad (6.74)$$

and claim that this provides the correct dual basis. First we check

$$\epsilon_*^i(\epsilon_j) = \mathbf{e}_i^T \psi^{-1}(\epsilon_j) = \mathbf{e}_i^T \mathbf{e}^j = \delta_i^j. \quad (6.75)$$

To verify that the  $\epsilon_*^i$  form a basis we first check linear independence. Applying  $\sum_i \beta_i \epsilon_*^i = 0$  to  $\epsilon_j$  and using Eq. (6.75) shows immediately that  $\beta_j = 0$ , so that the  $\epsilon_*^i$  are indeed linearly independent. To see that they span  $V^*$  start with an arbitrary functional  $\varphi \in V^*$  and a vector  $\mathbf{v} = \sum_i v^i \epsilon_i$ . Then

$$\varphi(\mathbf{v}) = \varphi\left(\sum_i v^i \epsilon_i\right) = \sum_i v^i \underbrace{\varphi(\epsilon_i)}_{:=\varphi_i \in F} = \sum_i \varphi_i v^i = \sum_i \varphi_i \epsilon_*^i(\mathbf{v}). \quad (6.76)$$

This means  $\varphi = \sum_i \varphi_i \epsilon_*^i$  so that we have written an arbitrary functional  $\varphi$  as a linear combination of the  $\epsilon_*^i$ .  $\square$

To summarize the discussion, for a basis  $\{\epsilon_i\}$  and its dual basis  $\{\epsilon_*^i\}$  we can write vectors and dual vectors as in the following table.

	vectors in $V$	dual vectors in $V^*$
vectors	$\mathbf{v} = v^i \epsilon_i$	$\varphi = \varphi_j \epsilon_*^j$
coordinates	$v^i$	$\varphi_j$

(6.77)

You have probably noticed that we have quietly refined our index convention. Vector space basis elements have lower indices and their coordinates have upper indices while the situation is reversed for dual vectors. For one, this allows us to decide the origin of coordinate vectors simply by the position of their index - for an upper index,  $v^i$ , we refer to vectors and for a lower index,  $\varphi_j$  to dual vectors. From Eq. (6.73), the action of dual vectors on vectors can be written as

$$\varphi(\mathbf{v}) = \sum_{i,j} \varphi_i v^j \underbrace{\epsilon_*^i(\epsilon_j)}_{=\delta_j^i} = \varphi_i v^i, \quad (6.78)$$

so, as a simple summation over their indices, also referred to as *contraction*. Note that this corresponds to a refined Einstein summation convention where the same lower and upper index are being summed over.

From here it is only a few steps to defining tensors. For the curious, a basic introduction into tensors can be found in Appendix C.

Finally, we would like to have a look at the relation of dual vector spaces and scalar products. In fact, for reasons which will become clear, we keep the discussion slightly more general and consider symmetric bi-linear forms with an additional property:

**Definition 6.7.** A symmetric bi-linear form  $\langle \cdot, \cdot \rangle$  on a (real) vector space  $V$  is called non-degenerate if  $\langle \mathbf{v}, \mathbf{w} \rangle = 0$  for all  $\mathbf{w} \in V$  implies that  $\mathbf{v} = \mathbf{0}$ .

Note that a real scalar product is non-degenerate since already  $\langle \mathbf{v}, \mathbf{v} \rangle = 0$  implies that  $\mathbf{v} = \mathbf{0}$ . Intuitively, non-degeneracy demands that there is no vector which is orthogonal to all other vectors. It turns out that a non-degenerate symmetric bi-linear form allows for a “natural” identification of a vector space and its dual. This is the content of the following

**Lemma 6.6.** Let  $V$  be a real vector space with a symmetric bi-linear form  $\langle \cdot, \cdot \rangle$  and define the map  $\iota : V \rightarrow V^*$  by  $\iota(\mathbf{v})(\mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$ . Then we have

$$\langle \cdot, \cdot \rangle \text{ non-degenerate} \iff \iota \text{ is an isomorphism.} \quad (6.79)$$

*Proof.* The map  $\iota$  is certainly linear, given the linearity of the bi-linear form in the first argument. Since  $\dim(V) = \dim(V^*)$  and from Claim 3.1,  $\iota$  is bijective precisely when  $\text{Ker}(\iota) = \{\mathbf{0}\}$ . This is the same as saying that  $\iota(\mathbf{v})(\mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle = 0$  for all  $\mathbf{w}$  implies that  $\mathbf{v} = \mathbf{0}$  which is indeed precisely the definition of non-degeneracy.  $\square$

It is useful to work this out in a basis  $\{\epsilon_i\}$  of  $V$  with dual basis  $\{\epsilon_*^i\}$  of  $V^*$ . To do this we first introduce the symmetric matrix  $g$ , also called the *metric tensor* or metric in short, with entries

$$g_{ij} = \langle \epsilon_i, \epsilon_j \rangle. \quad (6.80)$$

We would like to work out the matrix which represents the map  $\iota$  relative to our basis choice. This means we should look at the images of the basis vectors, so  $\iota(\epsilon_i)(\epsilon_j) = \langle \epsilon_i, \epsilon_j \rangle = g_{ij} = g_{ki} \epsilon_*^k(\epsilon_j)$ . Stripping off the basis vector  $\epsilon_j$  we have

$$\iota(\epsilon_i) = g_{ji} \epsilon_*^j, \quad (6.81)$$

and, by comparison with Eq. (3.80), we learn that  $\iota$  is represented by the metric  $g$ . If the bi-linear form is non-degenerate, so that  $\iota$  is bijective, then  $g$  is invertible. The components of  $g^{-1}$  are usually denoted by  $g^{ij}$ , so that

$$g^{ij} g_{jk} = \delta_k^i. \quad (6.82)$$

In the physics literature it is common to use the same symbol for the components of a vector and the dual vector, related under  $\iota$ . So if  $\mathbf{v} = \sum_i v^i \epsilon_i$  then we write  $\iota(\mathbf{v}) = \sum_i v_i \epsilon_*^i$ . Since  $g$  represents  $\iota$  this means

$$v_i = g_{ij} v^j, \quad v^i = g^{ij} v_j. \quad (6.83)$$

Physicists refer to these equations by saying that we can “lower and raise indices” with the metric  $g_{ij}$  and its inverse  $g^{ij}$ . Mathematically, they are simply a component version of the isomorphism  $\iota$  between  $V$  and  $V^*$  which is induced from the non-degenerate bi-linear form. With this notation, the bi-linear form on two vector  $\mathbf{v} = \sum_i v^i \epsilon_i$  and  $\mathbf{w} = \sum_j w^j \epsilon_j$  can be written as

$$\langle \mathbf{v}, \mathbf{w} \rangle = g_{ij} v^i w^j = v^i w_i = v_j w^j. \quad (6.84)$$

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**Application: Minkowski product in  $\mathbb{R}^4$**

The Minkowski product has already been introduced in Example 6.1 (c). For two four-vectors  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^4$  and  $\eta = \text{diag}(-1, 1, 1, 1)$  the symmetric bi-linear form is defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T \eta \mathbf{w} . \quad (6.85)$$

It is customary in this context to label coordinates by Greek indices as  $\mu, \nu, \dots = 0, 1, 2, 3$ . The metric tensor with respect to the basis of standard unit vectors in  $\mathbb{R}^4$ ,  $\mathbf{e}_\mu = \mathbf{e}_\mu$  is

$$g_{\mu\nu} = \langle \mathbf{e}_\mu, \mathbf{e}_\nu \rangle = \mathbf{e}_\mu^T \eta \mathbf{e}_\nu = \eta_{\mu\nu} , \quad (6.86)$$

so is simply given by  $\eta$ . Since  $\eta$  is invertible this also shows, from Lemma 6.6, that the Minkowski product is non-degenerate. From Eq. (6.83) lowering and raising of indices then takes the form

$$v_\mu = \eta_{\mu\nu} v^\nu , \quad v^\mu = \eta^{\mu\nu} v_\nu , \quad (6.87)$$

and, from Eq. (6.84), the Minkowski product can be written as

$$\mathbf{v}^T \eta \mathbf{w} = \eta_{\mu\nu} v^\mu w^\nu = v^\mu w_\mu = v_\nu w^\nu . \quad (6.88)$$

All these equations are part of the standard covariant formulation of special relativity.

We can go one step further and ask about the linear transformations  $\Lambda : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  which leave the Minkowski product invariant, that is, which satisfy

$$\langle \Lambda \mathbf{v}, \Lambda \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle \iff \Lambda^T \eta \Lambda = \eta \iff \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma} . \quad (6.89)$$

Note that these linear transformations, which are referred to as *Lorentz transformations*, relate to the Minkowski product in the same way that orthogonal linear maps relate to the standard scalar product on  $\mathbb{R}^n$  (see Section 6.4). In Special Relativity the linear transformation

$$\mathbf{x} \rightarrow \mathbf{x}' = \Lambda \mathbf{x} \iff x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad (6.90)$$

generated by  $\Lambda$  is interpreted as a transformation from one inertial system with space-time coordinates  $\mathbf{x} = (t, x, y, z)^T$  to another one with space-time coordinates  $\mathbf{x}' = (t', x', y', z')^T$ .

Lorentz transformations have a number of interesting properties which follow immediately from their definition (6.89). Taking the determinant of the middle equation (6.89) and using standard properties of the determinant implies that  $(\det(\Lambda))^2 = 1$  so that

$$\det(\Lambda) = \pm 1 \quad (6.91)$$

Further, the  $\rho = \sigma = 0$  component of the last Eq. (6.89) reads  $-(\Lambda^0{}_0)^2 + \sum_{i=1}^3 (\Lambda^i{}_0)^2 = -1$  so that

$$\Lambda^0{}_0 \geq 1 \quad \text{or} \quad \Lambda^0{}_0 \leq -1 . \quad (6.92)$$

Combining the two sign ambiguities in Eqs. (6.91) and (6.92) we see that there are four types of Lorentz transformations. The sign ambiguity in the determinant is analogous to what we have seen for orthogonal matrices and its interpretation is similar to the orthogonal case. Lorentz transformation with determinant 1 are called “proper” Lorentz transformations while Lorentz transformations with determinant  $-1$  can be seen as a combination of a proper Lorentz transformation and a reflection. More specifically, consider the special Lorentz transformation  $P = \text{diag}(1, -1, -1, -1)$  (note that this matrix indeed satisfies Eq. (6.89)) which is also referred to as “parity”. Then every Lorentz transformation  $\Lambda$  can be written as

$$\Lambda = P \Lambda_+ , \quad (6.93)$$



where  $\Lambda_+$  is a proper Lorentz transformation. The sign ambiguity (6.92) in  $\Lambda_0^0$  is new but has an obvious physical interpretation. Under a Lorentz transformations  $\Lambda$  with  $\Lambda_0^0 \geq 1$  the sign of the time component  $x^0 = t$  of a vector  $\mathbf{x}$  remains unchanged, so that the direction of time is unchanged. Correspondingly, such Lorentz transformation with positive  $\Lambda_0^0$  are called “ortho-chronous”. On the other hand, Lorentz transformations  $\Lambda$  with  $\Lambda_0^0 \leq -1$  change the direction of time. If we introduce the special Lorentz transformation  $T = \text{diag}(-1, 1, 1, 1)$ , also referred to as “time reversal”, then every Lorentz transformation  $\Lambda$  can be written as

$$\Lambda = T\Lambda^\dagger, \quad (6.94)$$

where  $\Lambda^\dagger$  is an ortho-chronous Lorentz transformation. Combining the above discussion, we see that every Lorentz transformation  $\Lambda$  can be written in one of four ways, namely

$$\Lambda = \begin{cases} \Lambda_+^\dagger & \text{for } \det(\Lambda) = 1 \quad \text{and } \Lambda_0^0 \geq 1 \\ P\Lambda_+^\dagger & \text{for } \det(\Lambda) = -1 \quad \text{and } \Lambda_0^0 \geq 1 \\ PT\Lambda_+^\dagger & \text{for } \det(\Lambda) = 1 \quad \text{and } \Lambda_0^0 \leq -1 \\ T\Lambda_+^\dagger & \text{for } \det(\Lambda) = -1 \quad \text{and } \Lambda_0^0 \leq -1 \end{cases}, \quad (6.95)$$

where  $\Lambda_+^\dagger$  is a proper, ortho-chronous Lorentz transformation. The Lorentz transformations normally used in Special Relativity are the proper, ortho-chronous Lorentz transformations. However, the other Lorentz transformations are relevant as well and it is an important question as to whether they constitute symmetries of nature in the same way that proper, ortho-chronous Lorentz transformations do. More to the point, the question is whether nature respects parity  $P$  and time-reversal  $T$ .

What do proper, ortho-chronous Lorentz transformations look like explicitly? To answer this question we basically have to solve Eq. (6.89) which is clearly difficult to do in full generality. However, some special Lorentz transformations are more easily obtained. First, we note that matrices of the type

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \quad (6.96)$$

where  $R$  is a three-dimensional rotation matrix are proper, ortho-chronous Lorentz transformations. Indeed, such matrices satisfy Eq. (6.89) by virtue of  $R^T R = \mathbb{1}_3$  and we have  $\det(\Lambda) = \det(R) = 1$  and  $\Lambda_0^0 = 1$ . In other words, regular three-dimensional rotations in the spatial directions are proper, ortho-chronous Lorentz transformations.

To find less trivial examples we start with the Ansatz

$$\Lambda = \begin{pmatrix} \Lambda_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (6.97)$$

of a two-dimensional Lorentz transformation which affects time and the  $x$ -coordinate, but leaves  $y$  and  $z$  unchanged. Inserting this Ansatz into Eq. (6.89) and, in addition, requiring that  $\det(\Lambda_2) = 1$  for proper Lorentz transformations, leads to

$$a^2 - c^2 = 1, \quad d^2 - b^2 = 1, \quad ab - cd = 0, \quad ad - cb = 1. \quad (6.98)$$

Also demanding  $\Lambda_0^0 = a \geq 1$ , so that  $\Lambda$  is ortho-chronous, this set of equations is solved by  $a = \cosh(\xi)$  and  $b = \sinh(\xi)$  for a real number  $\xi$ . Hence, our two-dimensional Lorentz transformation can be written as

$$\Lambda_2(\xi) = \begin{pmatrix} \cosh(\xi) & \sinh(\xi) \\ \sinh(\xi) & \cosh(\xi) \end{pmatrix}. \quad (6.99)$$

Note the close analogy of this form with two-dimensional rotations in Example 6.7 (a). The quantity  $\xi$  is also called “rapidity”. It follows from the addition theorems for hyperbolic functions that  $\Lambda(\xi_1)\Lambda(\xi_2) = \Lambda(\xi_1 + \xi_2)$ , so rapidities add up in the same way that two-dimensional rotation angles do. For a more common parametrisation introduce the parameter  $\beta = \tanh(\xi) \in [-1, 1]$  so that

$$\cosh(\xi) = \frac{1}{\sqrt{1 - \beta^2}} =: \gamma, \quad \sinh(\xi) = \beta\gamma. \quad (6.100)$$

In terms of  $\beta$  and  $\gamma$  the two-dimensional Lorentz transformations can then be written in the more familiar form

$$\Lambda_2 = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}. \quad (6.101)$$

Here,  $\beta$  is interpreted as the relative speed of the two inertial systems (in units of the speed of light).

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## 7 Eigenvectors and eigenvalues

In Section 3, we have seen that a linear map  $f : V \rightarrow V$  is represented by a matrix  $A$ , relative to a choice of basis on  $V$ . For a different basis, the same linear map is represented by another matrix  $A'$ , related to  $A$  by the basis transformation  $A' = PAP^{-1}$ . This suggests an obvious problem. How can we find a basis for which the representing matrix of the linear map is particularly simply, for example diagonal? As we will see, eigenvectors and eigenvalues are the key to solving this problem. Eigenvectors and eigenvalues have numerous applications in mathematics and physics some of which will be discussed towards the end of the section.

### 7.1 Basic ideas

Recall from Eq. (3.80) that the matrix representing a linear map is computed by writing the images of the basis vectors as linear combinations of the basis, with the coefficients from each image forming a column of the matrix. Suppose that the image of a basis vector  $\mathbf{v}$  is simply a multiple of itself, so  $f(\mathbf{v}) = \lambda\mathbf{v}$  for some number  $\lambda$ . In this case, the corresponding column of the representing matrix only has one non-zero entry,  $\lambda$ , in the diagonal. Hence, for such basis vectors, the representing matrix becomes simple. This observation motivates the following

**Definition 7.1.** For a linear map  $f : V \rightarrow V$  on a vector space  $V$  over  $F$  the number  $\lambda \in F$  is called an *eigenvalue* of  $f$  if there is a non-zero vector  $\mathbf{v}$  such that

$$f(\mathbf{v}) = \lambda\mathbf{v}. \quad (7.1)$$

In this case,  $\mathbf{v}$  is called an *eigenvector* of  $f$  with eigenvalue  $\lambda$ .

In short, an eigenvector is a vector which is just “scaled” by the action of a linear map.

How can we find eigenvalues and eigenvectors of a linear map? To discuss this we first introduce the idea of *eigenspaces*. The eigenspace for  $\lambda \in F$  is defined by

$$\text{Eig}_f(\lambda) := \text{Ker}(f - \lambda \text{id}_V), \quad (7.2)$$

and, hence, from Eq. (7.1) it “collects” all eigenvectors for  $\lambda$ . Being the kernel of a linear map, an eigenspace is of course a sub vector space of  $V$ . Evidently,  $\lambda$  is an eigenvalue of  $f$  precisely when  $\dim \text{Eig}_f(\lambda) > 0$ . If  $\dim \text{Eig}_f(\lambda) = 1$  the eigenvalue  $\lambda$  is called *non-degenerate* (up to re-scaling there

is only one eigenvector for  $\lambda$ ) and if  $\dim \text{Eig}_f(\lambda) > 1$  the eigenvalue  $\lambda$  is called *degenerate* (there are at least two linearly independent eigenvectors for  $\lambda$ ).

We see that  $\lambda$  is an eigenvalue of  $f$  precisely when  $\text{Ker}(f - \lambda \text{id}_V)$  is non-trivial. From Lemma 3.1 this is the same as saying that  $f - \lambda \text{id}_V$  is not invertible which is equivalent to  $\det(f - \lambda \text{id}_V) = 0$ , using Lemma 5.1. So in summary

$$\lambda \text{ eigenvalue of } f \iff \text{Ker}(f - \lambda \text{id}_V) \neq \{\mathbf{0}\} \iff \det(f - \lambda \text{id}_V) = 0. \quad (7.3)$$

This leads to an explicit method to calculate eigenvalues and eigenvectors which we develop in the next sub-section.

## 7.2 Characteristic polynomial

**Definition 7.2.** *The characteristic polynomial of a linear map  $f : V \rightarrow V$  is defined by*

$$\chi_f(\lambda) := \det(f - \lambda \text{id}_V). \quad (7.4)$$

For an  $n$ -dimensional vector space  $V$  the characteristic polynomial  $\chi_f(\lambda)$  is a polynomial of order  $n$  in  $\lambda$  whose coefficients depend on  $f$ . Clearly, from Eq. (7.3), the eigenvalues of  $f$  are precisely the zeros of its characteristic polynomial. So schematically, eigenvalues and eigenvectors of  $f$  can be computed as follows.

1. Compute the characteristic polynomial  $\chi_f(\lambda) = \det(f - \lambda \text{id}_V)$  of  $f$ .
2. Find the zeros,  $\lambda$ , of the characteristic polynomial. They are the eigenvalues of  $f$ .
3. For each eigenvalue  $\lambda$  compute the eigenspace  $\text{Eig}_f(\lambda) = \text{Ker}(f - \lambda \text{id}_V)$  by finding all vectors  $\mathbf{v}$  which solve the equation

$$(f - \lambda \text{id}_V)(\mathbf{v}) = \mathbf{0}. \quad (7.5)$$

**Example 7.1:** Computing eigenvalues and eigenvectors

For  $V = \mathbb{R}^3$ , we would like to compute the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}. \quad (7.6)$$

The characteristic polynomial is

$$\chi_A(\lambda) = \det \begin{pmatrix} 1 - \lambda & -1 & 0 \\ -1 & 2 - \lambda & -1 \\ 0 & -1 & 1 - \lambda \end{pmatrix} = \lambda(\lambda - 1)(\lambda - 3), \quad (7.7)$$

so we have three eigenvalues  $\lambda_1 = 0$ ,  $\lambda_2 = 1$  and  $\lambda_3 = 3$ . Writing  $\mathbf{v} = (x, y, z)^T$ , we compute the eigenvectors for each of these eigenvalues in turn.

$\lambda_1 = 0$ :

$$(A - 0\mathbf{1})\mathbf{v} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y \\ -x + 2y - z \\ -y + z \end{pmatrix} \stackrel{!}{=} \mathbf{0} \iff x = y = z$$

Hence, up to scaling, there is only one eigenvector so the eigenvalue is non-degenerate. Normalizing the eigenvector with respect to the dot product gives

$$\mathbf{v}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$\lambda_2 = 1$ :

$$(A - 1\mathbb{1})\mathbf{v} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ -x + y - z \\ -y \end{pmatrix} \stackrel{!}{=} \mathbf{0} \iff y = 0, \quad x = -z$$

Again, the eigenvalue is non-degenerate and the normalized eigenvector is

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

$\lambda_3 = 3$ :

$$(A - 3\mathbb{1})\mathbf{v} = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2x - y \\ -x - y - z \\ -y - 2z \end{pmatrix} \stackrel{!}{=} \mathbf{0} \iff y = -2x, \quad z = x$$

The eigenvalue is non-degenerate and the normalized eigenvector is

$$\mathbf{v}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Some general properties of the characteristic polynomial are given in the following

**Lemma 7.1.** (*Properties of characteristic polynomial*) The characteristic polynomial  $\chi_A(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0$  of an  $n \times n$  matrix  $A$  has the following properties:

- (i)  $\chi_{PAP^{-1}} = \chi_A$ , so the characteristic polynomial is basis-independent.
- (ii) The coefficients  $c_i$  of the characteristic polynomial are basis-independent.
- (iii)  $c_n = (-1)^n$ ,  $c_{n-1} = (-1)^{n-1} \sum_{i=1}^n A_{ii}$ ,  $c_0 = \det(A)$ .

*Proof.* (i)  $\chi_{PAP^{-1}}(\lambda) = \det(PAP^{-1} - \lambda\mathbb{1}) = \det(P(A - \lambda\mathbb{1})P^{-1}) = \det(A - \lambda\mathbb{1}) = \chi_A(\lambda)$ .

(ii) This is a direct consequence of (i).

(iii) First, it is clear that  $c_0 = \chi_A(0) = \det(A)$ . The expressions for the other two coefficients follow by carefully thinking about the order in  $\lambda$  of the terms in  $\det(A - \lambda\mathbb{1})$ , by using the general expression (5.11) for the determinant. Terms of order  $\lambda^n$  and  $\lambda^{n-1}$  only receive contributions from the product of the diagonal elements, so that

$$\chi_A(\lambda) = \prod_{i=1}^n (A_{ii} - \lambda) + \mathcal{O}(\lambda^{n-2}) = (-1)^n \lambda^n + (-1)^{n-1} \left( \sum_{i=1}^n A_{ii} \right) \lambda^{n-1} + \mathcal{O}(\lambda^{n-2}).$$

□

The above Lemma shows that the constant term in the characteristic polynomial equals  $\det(A)$  and that this is basis-independent. Of course, we have already shown the basis-independence of the determinant in Section (5.2). However, we do gain some new insight from the basis-independence of the coefficient  $c_{n-1}$  in the characteristic polynomial. We define the *trace* of a matrix  $A$  by

$$\operatorname{tr}(A) := \sum_{i=1}^n A_{ii} , \quad (7.8)$$

that is, by the sum of its diagonal entries. Since  $c_{n-1} = (-1)^{n-1}\operatorname{tr}(A)$  it follows that the trace is basis-independent. This can also be seen more directly. First, note that

$$\operatorname{tr}(AB) = \sum_{i,j} A_{ij}B_{ji} = \sum_{i,j} B_{ji}A_{ij} = \operatorname{tr}(BA) , \quad (7.9)$$

so matrices inside a trace can be commuted without changing the value of the trace. Hence,

$$\operatorname{tr}(PAP^{-1}) = \operatorname{tr}((PA)P^{-1}) = \operatorname{tr}(P^{-1}(PA)) = \operatorname{tr}(A) , \quad (7.10)$$

and we have another proof for the basis-independence of the trace.

### 7.3 Diagonalization of matrices

We now come back to our original question. How can we find a basis in which a linear map or a matrix has a particularly simple form, preferably diagonal? To be precise we start with

**Definition 7.3.** *We say a linear map  $f : V \rightarrow V$  can be diagonalised if there exist a basis of  $V$  such that the matrix which describes  $f$  relative to this basis is diagonal.*

*Further, we say an  $n \times n$  matrix  $A$  with entries in  $F$  can be diagonalized if there is an invertible  $n \times n$  matrix  $P$  with entries in  $F$  such that  $\hat{A} := P^{-1}AP$  is diagonal.*

The key statement relating eigenvectors and eigenvalues to diagonalization of a linear map is

**Lemma 7.2.** *A linear map  $f : V \rightarrow V$  can be diagonalised iff there exists a basis of  $V$  consisting of eigenvectors of  $f$ . Relative to such a basis of eigenvectors,  $f$  is described by a diagonal matrix with the eigenvalues along the diagonal.*

*Proof.* The entries of the matrix  $A$  which describes  $f$  relative to a basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$  of  $V$  are obtained from  $f(\mathbf{v}_j) = \sum_i A_{ij}\mathbf{v}_i$ , see the discussion around Eq. (3.80). From this equation, if  $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$  is diagonal, then  $f(\mathbf{v}_j) = \lambda_j\mathbf{v}_j$  and the basis vectors  $\mathbf{v}_j$  are eigenvectors with eigenvalues  $\lambda_i$ . Conversely, if  $f$  has a basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$  of eigenvectors with eigenvalues  $\lambda_i$ , then, from the eigenvalue equation (7.1), we have  $f(\mathbf{v}_i) = \lambda_i\mathbf{v}_i$  and, hence, the matrix describing  $f$  relative to this basis is  $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ .  $\square$

The analogous statement for matrices is

**Lemma 7.3.** *The  $n \times n$  matrix  $A$  with entries in  $F$  can be diagonalized iff  $A$  has  $n$  eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  which form a basis of  $F^n$ . In this case, if we define the matrix*

$$P = (\mathbf{v}_1, \dots, \mathbf{v}_n) \quad (7.11)$$

*whose columns are the eigenvectors of  $A$  it follows that*

$$P^{-1}AP = \operatorname{diag}(\lambda_1, \dots, \lambda_n) , \quad (7.12)$$

*where  $\lambda_i$  are the eigenvalues for  $\mathbf{v}_i$ .*

*Proof.* “ $\Leftarrow$ ”: We assume that we have a basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$  of eigenvectors with eigenvalues  $\lambda_i$  so that  $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$ . Define the matrix  $P = (\mathbf{v}_1, \dots, \mathbf{v}_n)$  whose columns are the eigenvectors of  $A$ . Since the eigenvectors form a basis of  $F^n$  the matrix  $P$  is invertible. Then

$$\begin{aligned} P^{-1}AP &= P^{-1}A(\mathbf{v}_1, \dots, \mathbf{v}_n) = P^{-1}(A\mathbf{v}_1, \dots, A\mathbf{v}_n) = P^{-1}(\lambda_1\mathbf{v}_1, \dots, \lambda_n\mathbf{v}_n) \\ &= P^{-1} \underbrace{(\mathbf{v}_1, \dots, \mathbf{v}_n)}_{=P} \text{diag}(\lambda_1, \dots, \lambda_n) = \text{diag}(\lambda_1, \dots, \lambda_n). \end{aligned}$$

“ $\Rightarrow$ ”: Assume that  $A$  can be diagonalized, so we have an invertible matrix  $P$  with  $P^{-1}AP = \hat{A} = \text{diag}(\lambda_1, \dots, \lambda_n)$ . Denote the column vectors of  $P$  by  $\mathbf{v}_i$  so that  $P\mathbf{e}_i = \mathbf{v}_i$ . Since  $P$  is invertible these column vectors form a basis of  $F^n$ . Then

$$\hat{A}\mathbf{e}_i = \lambda_i\mathbf{e}_i \implies P^{-1}A \underbrace{P\mathbf{e}_i}_{=\mathbf{v}_i} = \lambda_i\mathbf{e}_i \implies A\mathbf{v}_i = \lambda_i\mathbf{v}_i,$$

and, hence,  $\mathbf{v}_1, \dots, \mathbf{v}_n$  is a basis of eigenvectors of  $A$ . □

The requirement which is easily overlooked in the previous lemmas is that we are asking for a *basis* of eigenvectors. Once we have found all the eigenvectors of a linear map they might or might not form a basis of the underlying vector space. Only when they do can the linear map be diagonalized.

If a matrix  $A$  can be diagonalized, with eigenvalues  $\lambda_1, \dots, \lambda_n$ , so that  $P^{-1}AP = \text{diag}(\lambda_1, \dots, \lambda_n)$ , then the basis-independence of the determinant and the trace implies that

$$\det(A) = \prod_{i=1}^n \lambda_i, \quad \text{tr}(A) = \sum_{i=1}^n \lambda_i, \quad (7.13)$$

so, in this case, the determinant is the product of the eigenvalues and the trace is their sum.

### Example 7.2: Diagonalizing matrices

(a) We begin with the matrix (7.6) from our previous Example 7.1. We have already determined its eigenvalues and eigenvectors and the latter clearly form a basis of  $\mathbb{R}^3$ . Hence, this matrix can be diagonalized and the matrix

$$P = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}, \quad (7.14)$$

contains the three eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  from Example 7.1 as its columns. Note that these three columns form an ortho-normal system with respect to the dot product so the above matrix  $P$  is orthogonal. This means that its inverse is easily computed from  $P^{-1} = P^T$ . With the matrix  $A$  from Eq. (7.6) it can then be checked explicitly that

$$P^TAP = \text{diag}(0, 1, 3).$$

Note that the eigenvalues of  $A$  appear on the diagonal. It is not an accident that the eigenvectors of  $A$  are pairwise orthogonal and, as we will see shortly, this is related to  $A$  being a symmetric matrix.

(b) Consider the  $2 \times 2$  matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

whose characteristic polynomial is

$$\chi_A(\lambda) = \det \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} = \lambda^2.$$

Hence, there is only one eigenvalue,  $\lambda = 0$ . The associated eigenvectors are found by solving

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix} \stackrel{!}{=} \mathbf{0} \iff y = 0$$

so the eigenvalue is non-degenerate with eigenvectors proportional to  $(1, 0)^T$ . This amounts to only one eigenvector (up to re-scaling) so this matrix does not have a basis of eigenvectors (which requires two linearly independent vectors in  $\mathbb{R}^2$ ) and cannot be diagonalized.

(c) Our next example is for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with characteristic polynomial

$$\chi_A(\lambda) = \det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1.$$

At this point we have to be a bit more specific about the underlying vector space. If the vector space is  $\mathbb{R}^2$ , we have to work with real numbers and there are no eigenvalues since the characteristic polynomial has no real zeros. Hence, in this case, the matrix cannot be diagonalized. On the other hand, for  $\mathbb{C}^2$  and complex scalars, there are two eigenvalues,  $\lambda_{\pm} = \pm i$ . The corresponding eigenvectors  $\mathbf{v} = (x, y)^T$  are:

$$\underline{\lambda_+ = i}$$

$$(A - i\mathbb{1}_2)\mathbf{v} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -ix + y \\ -x - iy \end{pmatrix} \stackrel{!}{=} \mathbf{0} \iff y = ix.$$

The eigenvalue is non-degenerate and, as in Example (7.1) it is useful to normalize the eigenvector. However, since we are working over the complex numbers, we should be using the standard hermitian scalar product and demand that  $\mathbf{v}^\dagger \mathbf{v} = 1$ . Then

$$\mathbf{v}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

$$\underline{\lambda_- = -i}$$

$$(A + i\mathbb{1}_2)\mathbf{v} = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ix + y \\ -x + iy \end{pmatrix} \stackrel{!}{=} \mathbf{0} \iff y = -ix.$$

Again, this eigenvalue is non-degenerate with corresponding normalized eigenvector

$$\mathbf{v}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

The diagonalizing basis transformation is

$$P = (\mathbf{v}_+, \mathbf{v}_-) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},$$

and its column vectors form an ortho-normal system (under the standard hermitian scalar product on  $\mathbb{C}^2$ ). Therefore,  $P$  is a unitary matrix and  $P^{-1} = P^\dagger$ . Again, the orthogonality of the eigenvectors is not

an accident and is related to the matrix  $A$  being anti-symmetric. To check these results we verify that indeed

$$P^\dagger AP = \text{diag}(i, -i).$$

(d) Finally, we consider the hermitian matrix

$$A = \begin{pmatrix} 1 & 2i \\ -2i & 1 \end{pmatrix}$$

with characteristic polynomial

$$\chi_A(\lambda) = \det \begin{pmatrix} 1 - \lambda & 2i \\ -2i & 1 - \lambda \end{pmatrix} = (\lambda - 3)(\lambda + 1).$$

Hence, the eigenvalues are  $\lambda_1 = 3$  and  $\lambda_2 = -1$ .

$\lambda_1 = 3$

$$(A - 3\mathbb{1}_3)\mathbf{v} = \begin{pmatrix} -2 & 2i \\ -2i & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{!}{=} \mathbf{0} \iff x = iy.$$

The eigenvalue is non-degenerate and the corresponding eigenvector can be chosen as

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix},$$

so that it is properly normalized with respect to the  $\mathbb{C}^2$  standard scalar product,  $\mathbf{v}_1^\dagger \mathbf{v}_1 = 1$ .

$\lambda_2 = -1$

$$(A + \mathbb{1}_3)\mathbf{v} = \begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{!}{=} \mathbf{0} \iff x = -iy.$$

This eigenvalue is also non-degenerate and the normalized eigenvector, satisfying  $\mathbf{v}_2^\dagger \mathbf{v}_2 = 1$ , can be chosen as

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Note also that the two eigenvectors are orthogonal,  $\mathbf{v}_1^\dagger \mathbf{v}_2 = 0$ . Consequently, the diagonalizing matrix

$$U = (\mathbf{v}_1, \mathbf{v}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

is unitary,  $U^\dagger U = \mathbb{1}_2$ , and it is straightforward to verify that indeed  $U^\dagger A U = \text{diag}(3, 1)$ .

While Lemma 7.3 provides a general criterion for a matrix to be diagonalizable it requires calculation of all the eigenvectors and checking whether they form a basis. It would be helpful to have a simpler condition, at least for some classes of matrices, which can simply be “read off” from the matrix. To this end we prove

**Theorem 7.1.** *Let  $V$  be a vector space over  $\mathbb{R}$  ( $\mathbb{C}$ ) with real (hermitian) scalar product  $\langle \cdot, \cdot \rangle$ . If  $f : V \rightarrow V$  is self-adjoint then*

- (i) *All eigenvalues of  $f$  are real.*
- (ii) *Eigenvectors for different eigenvalues are orthogonal.*



*Proof.* (i) For the real case, the first part of the statement is of course trivial. For the complex case, we start with an eigenvector  $\mathbf{v} \neq \mathbf{0}$  of  $f$  with corresponding eigenvalue  $\lambda$ , so that  $f(\mathbf{v}) = \lambda\mathbf{v}$ . Then

$$\lambda\langle \mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{v}, \lambda\mathbf{v} \rangle = \langle \mathbf{v}, f(\mathbf{v}) \rangle = \langle f(\mathbf{v}), \mathbf{v} \rangle = \langle \lambda\mathbf{v}, \mathbf{v} \rangle = \lambda^*\langle \mathbf{v}, \mathbf{v} \rangle .$$

In the third step we have used the fact that  $f$  is self-adjoint and can, hence, be moved from one argument of the scalar product into the other. Since  $\mathbf{v} \neq \mathbf{0}$  and, hence,  $\langle \mathbf{v}, \mathbf{v} \rangle \neq 0$  it follows that  $\lambda = \lambda^*$ , so the eigenvalue is real.

(ii) Consider two eigenvectors  $\mathbf{v}_1, \mathbf{v}_2$ , so that  $f(\mathbf{v}_1) = \lambda_1\mathbf{v}_1$  and  $f(\mathbf{v}_2) = \lambda_2\mathbf{v}_2$ , with different eigenvalues,  $\lambda_1 \neq \lambda_2$ . Then

$$(\lambda_1 - \lambda_2)\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \lambda_1\mathbf{v}_1, \mathbf{v}_2 \rangle - \langle \mathbf{v}_1, \lambda_2\mathbf{v}_2 \rangle = \langle f(\mathbf{v}_1), \mathbf{v}_2 \rangle - \langle \mathbf{v}_1, f(\mathbf{v}_2) \rangle = \langle \mathbf{v}_1, f(\mathbf{v}_2) \rangle - \langle \mathbf{v}_1, f(\mathbf{v}_2) \rangle = 0 .$$

Since  $\lambda_1 - \lambda_2 \neq 0$  this means  $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$  and the two eigenvectors are orthogonal.  $\square$

**Theorem 7.2.** *Let  $V$  be a vector space over  $\mathbb{C}$  with hermitian scalar product  $\langle \cdot, \cdot \rangle$ . If  $f : V \rightarrow V$  is self-adjoint it has an ortho-normal basis,  $\epsilon_1, \dots, \epsilon_n$  of eigenvectors.*

*Proof.* The proof is by induction in  $n$ , the dimension of the vector space  $V$ . For  $n = 1$  the assertion is trivial. Assume that it is true for all dimensions  $k < n$ . We would like to show that it is true for dimension  $n$ . The characteristic polynomial  $\chi_f$  of  $f$  has at least one zero,  $\lambda$ , over the complex numbers. Since  $f$  is self-adjoint,  $\lambda$  is real from the previous theorem. Consider the eigenspace  $W = \text{Eig}_f(\lambda)$ . Since  $\lambda$  is an eigenvalue,  $\dim(W) > 0$ . Vectors  $\mathbf{v} \in W^\perp$  and  $\mathbf{w} \in W$  are perpendicular,  $\langle \mathbf{w}, \mathbf{v} \rangle = 0$ , so

$$\langle \mathbf{w}, f(\mathbf{v}) \rangle = \langle f(\mathbf{w}), \mathbf{v} \rangle = \langle \lambda\mathbf{w}, \mathbf{v} \rangle = \lambda\langle \mathbf{w}, \mathbf{v} \rangle = 0 .$$

This means that  $f(\mathbf{v})$  is perpendicular to  $\mathbf{w}$  so that, whenever  $\mathbf{v} \in W^\perp$ , then also  $f(\mathbf{v}) \in W^\perp$ . As a result,  $W^\perp$  is invariant under  $f$  and we can restrict  $f$  to  $W^\perp$ , that is, consider  $g = f|_{W^\perp}$ . Since  $\dim W^\perp < n$ , there is an ortho-normal basis  $\epsilon_1, \dots, \epsilon_k$  of  $W^\perp$  consisting of eigenvectors of  $g$  (which are also eigenvectors of  $f$ ) by the induction assumption. Add to this ortho-normal basis of  $W^\perp$  an ortho-normal basis of  $W$  (which, by definition of  $W$ , consists of eigenvectors of  $f$  with eigenvalue  $\lambda$ ). Since  $\dim(W) + \dim(W^\perp) = n$  (see Lemma 6.3) and pairwise orthogonal vector are linearly independent this list of vectors forms an ortho-normal basis of  $V$ , consisting of eigenvectors of  $f$ .  $\square$

In summary, these results mean that every real symmetric (hermitian) matrix can be diagonalized, has an ortho-normal basis  $\epsilon_1, \dots, \epsilon_n$  of eigenvectors with corresponding real eigenvalues  $\lambda_1, \dots, \lambda_n$  and the diagonalizing matrix  $P = (\epsilon_1, \dots, \epsilon_n)$  is orthogonal (unitary), such that

$$P^\dagger AP = \text{diag}(\lambda_1, \dots, \lambda_n) . \tag{7.15}$$

How can this ortho-normal basis of eigenvectors be found? First, the eigenvalues and eigenvectors have to be computed in the usual way, as outlined above. From Theorem 7.1 eigenvectors for different eigenvalues are orthogonal so if all eigenvalues are non-degenerate then the eigenvectors will be automatically pairwise orthogonal. What remains to be done in order to obtain an ortho-normal system is simply to normalize the eigenvectors. This is what has happened in Example 7.1 (and its continuation, Example 7.2 (a)) where all eigenvalues were indeed non-degenerate.

The situation is slightly more involved in the presence of degenerate eigenvalues. Of course eigenvectors for different eigenvalues are still automatically orthogonal. However, for a degenerate eigenvalue we have two or more linearly independent eigenvectors which are not guaranteed to be orthogonal. The point is that we can *choose* such eigenvectors to be orthogonal. To see how this works it is useful to think about

the eigenspaces,  $\text{Eig}_A(\lambda)$ , of the hermitian matrix  $A$ . Eigenspaces for different eigenvalues are of course orthogonal to one another (meaning that all vectors of one eigenspace are orthogonal to all vectors of the other), as a consequence of Theorem 7.1. For each eigenspace, we can find a basis of eigenvectors and, applying the Gram-Schmidt procedure to this basis, we can convert this into an ortho-normal basis. If the eigenvector is non-degenerate, so that  $\dim \text{Eig}_A(\lambda) = 1$ , this simply means normalizing the single basis vectors. For degenerate eigenvalues, when  $\dim \text{Eig}_A(\lambda) > 1$ , we have to follow the full Gram-Schmidt procedure as explained in Section 6.2. Combining the ortho-normal sets of basis vectors for each eigenspace into one list then gives the full basis of ortho-normal eigenvectors. To see how this works explicitly let us discuss a more complicated example with a degenerate eigenvalue.

**Example 7.3:** Diagonalizing with degenerate eigenvalues

In  $\mathbb{R}^3$ , we consider the matrix

$$A = \frac{1}{4} \begin{pmatrix} 2 & 3\sqrt{2} & 3\sqrt{2} \\ 3\sqrt{2} & -1 & 3 \\ 3\sqrt{2} & 3 & -1 \end{pmatrix}$$

with characteristic polynomial

$$\chi_A(\lambda) = \det \begin{pmatrix} \frac{1}{2} - \lambda & \frac{3}{2\sqrt{2}} & \frac{3}{2\sqrt{2}} \\ \frac{3}{2\sqrt{2}} & -\frac{1}{4} - \lambda & \frac{3}{4} \\ \frac{3}{2\sqrt{2}} & \frac{3}{4} & -\frac{1}{4} - \lambda \end{pmatrix} = -\lambda^3 + 3\lambda + 2 = (2 - \lambda)(1 + \lambda)^2.$$

Hence, there are two eigenvalues,  $\lambda_1 = 2$  and  $\lambda_2 = -1$ . For the eigenvectors  $\mathbf{v} = (x, y, z)^T$  we find:  
 $\lambda_1 = 2$ :

$$(A - 2\mathbb{1}_3)\mathbf{v} = \frac{3}{4} \begin{pmatrix} -2 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -3 & 1 \\ \sqrt{2} & 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{3}{4} \begin{pmatrix} -2x + \sqrt{2}y + \sqrt{2}z \\ \sqrt{2}x - 3y + z \\ \sqrt{2}x + y - 3z \end{pmatrix} \stackrel{!}{=} \mathbf{0} \implies y = z = \frac{x}{\sqrt{2}}.$$

Hence, this eigenvalue is non-degenerate and a suitable normalized eigenvector is

$$\epsilon_1 = \frac{1}{2} \begin{pmatrix} \sqrt{2} \\ 1 \\ 1 \end{pmatrix}.$$

$\lambda_1 = -1$ :

$$(A + \mathbb{1}_3)\mathbf{v} = \frac{3}{4} \begin{pmatrix} 2 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1 & 1 \\ \sqrt{2} & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{3}{4} \begin{pmatrix} 2x + \sqrt{2}y + \sqrt{2}z \\ \sqrt{2}x + y + z \\ \sqrt{2}x + y + z \end{pmatrix} \stackrel{!}{=} \mathbf{0} \implies z = -\sqrt{2}x - y.$$

Since we have found only one condition on  $x, y, z$  there are two linearly independent eigenvectors, so this eigenvalue has degeneracy 2. Obvious choices for the two eigenvectors are obtained by setting  $x = 1, y = 0$  and  $x = 0, y = 1$ , so

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -\sqrt{2} \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Both of these vectors are orthogonal to  $\epsilon_1$  above, as they must be, but they are not orthogonal to one another. However, they do form a basis of the two-dimensional eigenspace  $\text{Eig}_A(-1) = \text{Span}(\mathbf{v}_2, \mathbf{v}_3)$  so

that every linear combination of these vectors is also an eigenvector for the same eigenvalue  $-1$ . With this in mind we apply the Gram-Schmidt procedure to  $\mathbf{v}_2$  and  $\mathbf{v}_3$ . First normalizing  $\mathbf{v}_2$  leads to

$$\boldsymbol{\epsilon}_2 = \frac{\mathbf{v}_2}{|\mathbf{v}_2|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ -\sqrt{2} \end{pmatrix}.$$

Then, subtracting from  $\mathbf{v}_3$  its projection onto  $\boldsymbol{\epsilon}_2$  and normalizing results in

$$\mathbf{v}'_3 = \mathbf{v}_3 - (\boldsymbol{\epsilon}_2 \cdot \mathbf{v}_3)\boldsymbol{\epsilon}_2 = \frac{1}{3} \begin{pmatrix} -\sqrt{2} \\ 3 \\ -1 \end{pmatrix}, \quad \boldsymbol{\epsilon}_3 = \frac{\mathbf{v}'_3}{|\mathbf{v}'_3|} = \frac{1}{2\sqrt{3}} \begin{pmatrix} -\sqrt{2} \\ 3 \\ -1 \end{pmatrix}.$$

The system  $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3$  is now an ortho-normal basis of eigenvectors, so the matrix

$$P = (\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & -\sqrt{\frac{2}{3}} & -\frac{1}{2\sqrt{3}} \end{pmatrix}$$

indeed satisfies  $P^T P = \mathbb{1}_3$  and  $P^T A P = \text{diag}(2, -1, -1)$ .

## 7.4 Normal linear maps

We have seen above that symmetric and hermitian matrices can always be diagonalised by an orthogonal or unitary basis transformation, respectively. What about matrices which are neither symmetric nor hermitian? It turns out that a useful criterion for the existence of a diagonalising basis change can be formulated for the more general class of *normal* linear maps or matrices.

**Definition 7.4.** Let  $V$  be a vector space over  $\mathbb{C}$  with hermitian scalar product  $\langle \cdot, \cdot \rangle$ . A linear map  $f : V \rightarrow V$  is called *normal* if  $f \circ f^\dagger = f^\dagger \circ f$  (or, equivalently, iff the commutator of  $f$  and  $f^\dagger$  vanishes, that is,  $[f, f^\dagger] := f \circ f^\dagger - f^\dagger \circ f = 0$ ).

Recall from Section 6.3 that the adjoint map,  $f^\dagger$ , for a linear map  $f : V \rightarrow V$  is defined relative to a scalar product on  $V$ . Clearly hermitian and unitary linear maps are normal (since  $f = f^\dagger$  if  $f$  is hermitian and  $f^\dagger \circ f = f \circ f^\dagger = \text{id}$  if  $f$  is unitary), as are anti-hermitian maps, that is, maps satisfying  $f = -f^\dagger$ . If we consider the vector space  $V = \mathbb{C}^n$  over  $\mathbb{C}$  with the standard hermitian scalar product then (anti-) hermitian and unitary maps simply correspond to (anti-) hermitian and unitary matrices and we learn that these classes of matrices are normal.

A useful statement for normal linear maps is

**Lemma 7.4.** Let  $V$  be a vector space over  $\mathbb{C}$  with hermitian scalar product  $\langle \cdot, \cdot \rangle$  and  $f : V \rightarrow V$  be a normal linear map. If  $\lambda$  is an eigenvalue of  $f$  with eigenvector  $\mathbf{v}$  then  $\lambda^*$  is an eigenvalue of  $f^\dagger$  for the same eigenvector  $\mathbf{v}$ .

*Proof.* First, we show that the map  $g = f - \lambda \text{id}$  is also normal. This follows from the straightforward calculation

$$\begin{aligned} g \circ g^\dagger &= (f - \lambda \text{id}) \circ (f^\dagger - \lambda^* \text{id}) = f \circ f^\dagger - \lambda^* f - \lambda f^\dagger + |\lambda|^2 \text{id} \\ &= f^\dagger \circ f - \lambda^* f - \lambda f^\dagger + |\lambda|^2 \text{id} = g^\dagger \circ g. \end{aligned}$$

Note that some of the properties of the adjoint map in Lemma 6.4 have been used in this calculation. Now consider an eigenvalue  $\lambda$  of  $f$  with eigenvector  $\mathbf{v}$ , so that  $f(\mathbf{v}) = \lambda\mathbf{v}$  or, equivalently,  $g(\mathbf{v}) = 0$ . Then we have

$$0 = \langle g\mathbf{v}, g\mathbf{v} \rangle = \langle \mathbf{v}, g^\dagger \circ g\mathbf{v} \rangle = \langle \mathbf{v}, g \circ g^\dagger \mathbf{v} \rangle = \langle g^\dagger \mathbf{v}, g^\dagger \mathbf{v} \rangle ,$$

and, from the positivity property of the scalar product, (S3) in Def. 6.1, it follows that  $g^\dagger(\mathbf{v}) = \mathbf{0}$ . Since  $g^\dagger = f - \lambda^* \text{id}$  this, in turn means that  $f^\dagger(\mathbf{v}) = \lambda^* \mathbf{v}$ . Hence,  $\lambda^*$  is indeed an eigenvalue of  $f^\dagger$  with eigenvector  $\mathbf{v}$ .  $\square$

The key property of normal matrices relevant to matrix diagonalization is

**Theorem 7.3.** *Let  $V$  be a vector space over  $\mathbb{C}$  with hermitian scalar product  $\langle \cdot, \cdot \rangle$  and  $f : V \rightarrow V$  a linear map. Then we have:  $f$  is normal  $\iff f$  has an ortho-normal basis of eigenvectors*

*Proof.* “ $\Leftarrow$ ”: Start with an ortho-normal basis  $\epsilon_1, \dots, \epsilon_n$  of eigenvector of  $f$ , so that  $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$  and  $f(\epsilon_i) = \lambda_i \epsilon_i$ . Then

$$\langle \epsilon_j, f^\dagger(\epsilon_i) \rangle = \langle f(\epsilon_j), \epsilon_i \rangle = \lambda_j^* \delta_{ij}$$

which holds for all  $\epsilon_j$  and, hence, implies that  $f^\dagger(\epsilon_i) = \lambda_i^* \epsilon_i$ . From this result we have

$$f \circ f^\dagger(\epsilon_i) = |\lambda_i|^2 \epsilon_i = f^\dagger \circ f(\epsilon_i) ,$$

for all  $\epsilon_i$ . This means that  $f \circ f^\dagger = f^\dagger \circ f$  so that  $f$  is normal.

“ $\Rightarrow$ ”: Conversely, assume that  $f$  is normal. We will show that  $f$  has an ortho-normal basis of eigenvectors by induction in  $n = \dim(V)$ . For  $n = 1$  the statement is trivial. Assume that it is valid for all dimensions  $k < n$ . Since we are working over the complex numbers,  $f$  has at least one eigenvalue  $\lambda$  with eigenvector  $\mathbf{v}$ , so that  $f(\mathbf{v}) = \lambda(\mathbf{v})$ . Define the sub vector space  $W = \{\mathbf{w} \in V \mid \langle \mathbf{w}, \mathbf{v} \rangle = 0\}$ , that is the space perpendicular to  $\mathbf{v}$ . For any  $\mathbf{w} \in W$  it follows

$$\begin{aligned} \langle f(\mathbf{w}), \mathbf{v} \rangle &= \langle \mathbf{w}, f^\dagger(\mathbf{v}) \rangle = \langle \mathbf{w}, \lambda^* \mathbf{v} \rangle = \lambda^* \langle \mathbf{w}, \mathbf{v} \rangle = 0 \\ \langle f^\dagger(\mathbf{w}), \mathbf{v} \rangle &= \langle \mathbf{w}, f(\mathbf{v}) \rangle = \langle \mathbf{w}, \lambda \mathbf{v} \rangle = \lambda \langle \mathbf{w}, \mathbf{v} \rangle = 0 \end{aligned}$$

where the above Lemma has been used in the first line. This means that  $f(W) \subset W$  and  $f^\dagger(W) \subset W$ , so that sub vector space  $W$  is invariant under both  $f$  and  $f^\dagger$ . This implies immediately that the restriction,  $f|_W$  of  $f$  to  $W$  is normal as well. Since  $\dim(W) = n - 1$ , the induction assumption can be applied and we conclude that  $f|_W$  has an ortho-normal basis of eigenvectors. Combining this basis with  $\mathbf{v}/|\mathbf{v}|$  gives an ortho-normal basis of eigenvectors for  $f$ .  $\square$

As we have seen, unitary maps are normal so the theorem implies that they have an ortho-normal basis of eigenvectors. Focusing on  $V = \mathbb{C}^n$  with the standard hermitian scalar product this means that unitary matrices can be diagonalised. The eigenvalues of unitary maps are constrained by the following

**Lemma 7.5.** *Let  $V$  be a vector space over  $\mathbb{C}$  with hermitian scalar product  $\langle \cdot, \cdot \rangle$  and  $U : V \rightarrow V$  a unitary map. If  $\lambda$  is an eigenvalue of  $U$  then  $|\lambda| = 1$ .*

*Proof.* Let  $\lambda$  be an eigenvalue of  $U$  with eigenvector  $\mathbf{v}$ , so that  $U\mathbf{v} = \lambda\mathbf{v}$ . From unitarity of  $U$  it follows that

$$|\lambda|^2 \langle \mathbf{v}, \mathbf{v} \rangle = \langle \lambda\mathbf{v}, \lambda\mathbf{v} \rangle = \langle U\mathbf{v}, U\mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle ,$$

and, dividing by  $\langle \mathbf{v}, \mathbf{v} \rangle$  (which must be non-zero since  $\mathbf{v}$  is an eigenvector), gives  $|\lambda|^2 = 1$ .  $\square$

Combining these statements we learn that every unitary matrix  $U$  can be diagonalised, by means of a unitary coordinate transformation  $P$ , such that  $P^\dagger U P = \text{diag}(e^{i\phi_1}, \dots, e^{i\phi_n})$ . Since orthogonal matrices are also unitary they can be diagonalised in the same way, provided we are working over the complex numbers. In fact, we have already seen this explicitly in Example 7.2 (c) where the matrix  $A$  is a specific two-dimensional rotation matrix.

**Example 7.4:** Three-dimensional rotations – again

Some of our results on eigenvalues and eigenvectors can be used to extract useful properties of three dimensional rotations without much calculation. We know from the above discussion that, over the complex numbers, three-dimensional rotations  $R$  can be diagonalised (by a unitary basis transformation) to a matrix  $\text{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})$  with phases in the diagonal. However, rotations are matrices with real entries, acting on  $\mathbb{R}^3$ , and for many purposes it seems more appropriate to work over the real numbers.

Let us see how much we can say allowing for real basis transformation only. First, since a three-dimensional rotation matrix  $R$  has real entries, the coefficients of the characteristic polynomial,  $\chi_R$ , are also real. This means for any eigenvalue  $\lambda$  its complex conjugate  $\lambda^*$  is also an eigenvalue. Combine this observation with the fact that all eigenvalues of a rotation matrix have complex modulus one and that their product needs to be one (since  $\det(R) = 1$ ) and we learn that the eigenvalues of a three-dimensional rotation  $R$  must be of the form  $1, e^{i\phi}, e^{-i\phi}$ . In particular, for every three-dimensional rotation at least one of the eigenvalues equals 1 and the corresponding eigenvector  $\mathbf{n}$  (normalised, so that  $\mathbf{n} \cdot \mathbf{n} = 1$ ), which satisfies

$$R\mathbf{n} = \mathbf{n}, \tag{7.16}$$

is called the axis of rotation.

Let us consider an ortho-normal basis  $\{\mathbf{n}, \mathbf{u}_1, \mathbf{u}_2\}$  of  $\mathbb{R}^3$  with the axis of rotation as its first basis vector. What is the matrix,  $\tilde{R}$ , representing the rotation  $R$  relative to this basis? From Eq. (6.24) this can be worked out by computing the matrix elements

$$\mathbf{n} \cdot (R\mathbf{n}) = \mathbf{n} \cdot \mathbf{n} = 1, \quad \mathbf{u}_a \cdot (R\mathbf{n}) = \mathbf{u}_a \cdot \mathbf{n} = 0, \quad \mathbf{n} \cdot (R\mathbf{u}_a) = (R^{-1}\mathbf{n}) \cdot \mathbf{u}_a = \mathbf{n} \cdot \mathbf{u}_a = 0 \tag{7.17}$$

of  $R$ . These results show that the representing matrix is of the form

$$\tilde{R} = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & R_2 \end{pmatrix}, \tag{7.18}$$

where  $R_2$  is a  $2 \times 2$  matrix. However,  $\tilde{R}$  is also a rotation and, hence, needs to satisfy  $\tilde{R}^T \tilde{R} = \mathbb{1}_3$  and  $\det(\tilde{R}) = 1$ . This immediately implies that  $R_2^T R_2 = \mathbb{1}_2$  and  $\det(R_2) = 1$ , so that  $R_2$  must be a two-dimensional rotation,  $R_2 = R(\theta)$ , of the form given in Eq. (6.40).

In summary, we learn that for every three-dimensional rotation  $R$  we can find an orthonormal basis (where the first basis vector is the axis of rotation) where it takes the form

$$\tilde{R} = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & R(\theta) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{7.19}$$

The angle  $\theta$  which appears in this parametrisation is called the angle of rotation. Basis-independence of the trace means that  $\text{tr}(R) = \text{tr}(\tilde{R}) = 1 + 2\cos(\theta)$  which leads to the interesting and useful formula

$$\cos(\theta) = \frac{1}{2} (\text{tr}(R) - 1) \tag{7.20}$$

for the angle of rotation of a rotation matrix  $R$ . This formula allows for an easy computation of the angle of rotation, even if the rotation matrix is not in the simple form (7.19). The axis of rotation  $\mathbf{n}$ , on the other hand, can be found as the eigenvector for eigenvalue one, that is, by solving Eq. (7.16).

For example, consider the matrix

$$R = \frac{1}{2} \begin{pmatrix} \sqrt{2} & -1 & -1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 1 & 1 \end{pmatrix}. \quad (7.21)$$

It is easy to verify that  $R^T R = \mathbb{1}_3$  and  $\det(R) = 1$  so this is indeed a rotation. By solving Eq. (7.16) for this matrix (and normalising the eigenvector) we find for the axis of rotation

$$\mathbf{n} = \frac{1}{\sqrt{5 - 2\sqrt{2}}} (1, -1, \sqrt{2} - 1)^T. \quad (7.22)$$

Also, we have  $\text{tr}(R) = \sqrt{2} + 1/2$ , so from Eq. (7.20) the angle of rotation satisfies

$$\cos(\theta) = \frac{1}{4} (2\sqrt{2} - 1). \quad (7.23)$$

## 7.5 Simultaneous diagonalization

Frequently, one would like to know whether two  $n \times n$  matrices  $A, B$  can be *diagonalized simultaneously*, that is, whether a single basis transformation  $P$  can be found such that both  $PAP^{-1}$  and  $PBP^{-1}$  are diagonal. One way to solve this problem is to calculate the eigenvectors for both matrices and to check if a basis of common eigenvectors can be selected. However, this can be tedious. The following theorem gives a simple criterion which is easy to check.

**Theorem 7.4.** *Let  $A, B$  be two diagonalizable  $n \times n$  matrices. Then we have:*

$$A, B \text{ can be diagonalized simultaneously} \iff [A, B] = 0. \quad (7.24)$$

*Proof.* “ $\Rightarrow$ ”: This is the easy direction. Assume that  $A, B$  can be diagonalized simultaneously so that there is a basis transformation  $P$  such that both  $\hat{A} = P^{-1}AP$  and  $\hat{B} = P^{-1}BP$  are diagonal. Then

$$[A, B] = AB - BA = P\hat{A}\underbrace{P^{-1}P}_{=1}\hat{B}P^{-1} - P\hat{B}\underbrace{P^{-1}P}_{=1}\hat{A}P^{-1} = P(\hat{A}\hat{B} - \hat{B}\hat{A})P^{-1} = P[\hat{A}, \hat{B}]P^{-1} = 0,$$

since diagonal matrices commute.

“ $\Leftarrow$ ”: The converse is more difficult and to simplify matters we assume that the eigenvalues of  $A$  are non-degenerate. (Without this assumption the proof goes along similar lines but is more involved.) Since, by assumption,  $A$  can be diagonalized we have a basis,  $\mathbf{v}_1, \dots, \mathbf{v}_n$  of eigenvectors with eigenvalues  $\lambda_1, \dots, \lambda_n$  such that  $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$ . The two matrices commute so that

$$A(B\mathbf{v}_i) = B(A\mathbf{v}_i) = \lambda_i B\mathbf{v}_i.$$

This shows that  $B\mathbf{v}_i$  is also an eigenvector of  $A$  (or it is the zero vector), with eigenvalue  $\lambda_i$ . Since the eigenvalue  $\lambda_i$  is non-degenerate (and this is where our simplifying assumption enters) it follows that  $B\mathbf{v}_i$  must be a multiple of  $\mathbf{v}_i$ , so there should be scalars  $\mu_i$  such that

$$B\mathbf{v}_i = \mu_i\mathbf{v}_i.$$

This means the  $\mathbf{v}_i$  are also eigenvectors of  $B$ , although in general for different eigenvalues  $\mu_i$ . Hence,  $\mathbf{v}_1, \dots, \mathbf{v}_n$  is a basis of common eigenvectors, so  $A$  and  $B$  can be diagonalized simultaneously.  $\square$

**Example 7.5:** Simultaneous diagonalization

Can the three matrices

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

be diagonalized simultaneously? A straightforward explicit computation of the commutators shows that

$$[A, B] = 0, \quad [A, C] \neq 0, \quad [B, C] \neq 0.$$

Hence,  $A, B$  can be diagonalized simultaneously but not  $A, C$  or  $B, C$ .

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## 7.6 Applications

Eigenvectors and eigenvalues have a wide range of applications, both in mathematics and in physics. Here, we discuss a small selection of those applications.

### 7.6.1 Solving Newton-type differential equations with linear forces

We would like to find the solutions  $\mathbf{q}(t) = (q_1(t), \dots, q_n(t))^T$  to the differential equation

$$\frac{d^2 \mathbf{q}}{dt^2} = -M\mathbf{q}, \tag{7.25}$$

where  $M$  is a real symmetric  $n \times n$  matrix. In a physical context, this differential equation might describe a system of mass points connected by springs. The practical problem in solving this equation is that, for a non-diagonal matrix  $M$ , its various components are coupled. However, this coupling can be removed by diagonalizing the matrix  $M$ . To this end, we consider an orthogonal matrix  $P$  such that  $P^T M P = \hat{M} = \text{diag}(m_1, \dots, m_n)$  and introduce new coordinates  $\mathbf{Q}$  by setting  $\mathbf{q} = P\mathbf{Q}$ . By multiplying Eq. (7.25) with  $P^T$  this leads to

$$\frac{d^2 \mathbf{Q}}{dt^2} = -\underbrace{P^T M P}_{=\hat{M}} \mathbf{Q} \quad \text{or} \quad \frac{d^2 Q_i}{dt^2} = -m_i Q_i \quad \text{for } i = 1, \dots, n. \tag{7.26}$$

In terms of  $\mathbf{Q}$  the system decouples and the solutions can easily be written down as

$$Q_i(t) = \begin{cases} a_i \sin(w_i t) + b_i \cos(w_i t) & \text{for } m_i > 0 \\ a_i e^{w_i t} + b_i e^{-w_i t} & \text{for } m_i < 0 \\ a_i t + b_i & \text{for } m_i = 0 \end{cases} \quad \text{where } w_i = \sqrt{|m_i|}, \tag{7.27}$$

and  $a_i, b_i$  are arbitrary constants. In terms of the original coordinates, the solution is then obtained by inserting Eq. (7.27) into  $\mathbf{q} = P\mathbf{Q}$ . One interesting observation is that the nature of the solution depends on the signs of the eigenvalues  $m_i$  of the matrix  $M$ . For a positive eigenvalue, the solution is oscillatory, for a negative one exponential and for a vanishing one linear. Physically, a negative or vanishing eigenvalue  $m_i$  indicates an instability. In this case, the corresponding  $Q_i(t)$  becomes large at late times (except for special choices of the constants  $a_i, b_i$ ). The lesson is that stability of the system can be analyzed simply by looking at the eigenvalues of  $M$ . If they are all positive, the system is fully oscillatory and stable, if there are vanishing or negative eigenvalues the system generically “runs away” in some directions.

**Example 7.6:** As an explicit example, consider the differential equations

$$\begin{aligned}\frac{d^2q_1}{dt^2} &= -q_1 + q_2 \\ \frac{d^2q_2}{dt^2} &= q_1 - q_2 + q_3 \\ \frac{d^2q_3}{dt^2} &= q_2 - q_3.\end{aligned}$$

This system is indeed of the general form (7.25) with

$$M = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

This is the same matrix we have studied in Example 7.1 and it has eigenvalues  $m_1 = 0$ ,  $m_2 = 1$  and  $m_3 = 3$ . Due to the zero eigenvalue this system has a linear instability in one direction. Inserting into Eq. (7.27), the explicit solution reads

$$\mathbf{Q}(t) = \begin{pmatrix} a_1t + b_1 \\ a_2 \sin(t) + b_2 \cos(t) \\ a_3 \sin(\sqrt{3}t) + b_3 \cos(\sqrt{3}t) \end{pmatrix} \quad (7.28)$$

In terms of the original coordinates  $\mathbf{q}$ , the solution is obtained by inserting (7.28) into  $\mathbf{q} = P\mathbf{Q}$  using the diagonalizing matrix  $P$  given in Eq. (7.14).

## 7.6.2 Functions of matrices

Start with a real or complex function  $g(x)$ . We would like to “insert” an  $n \times n$  matrix  $A$  into this function, that is, we would like to make sense of the expression  $g(A)$ . This can be done whenever the function has a (suitably convergent) power series expansion

$$g(x) = a_0 + a_1x + a_2x^2 + \cdots. \quad (7.29)$$

In this case, we can define  $g(A)$  as

$$g(A) = a_0\mathbb{1}_n + a_1A + a_2A^2 + \cdots, \quad (7.30)$$

that is, by simply “replacing”  $x$  with  $A$  in the power series expansion. Note that, convergence assumed, the RHS of Eq. (7.30) is well-defined via addition and multiplication of matrices and the function “value”  $g(A)$  is a matrix of the same size as  $A$ .

**Example 7.7:** The matrix exponential is defined as

$$e^A = \mathbb{1} + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k. \quad (7.31)$$

Since the exponential series converges for all real (and complex)  $x$  it can be shown that the matrix exponential converges for all matrices.



Computing the function of a non-diagonal matrix can be complicated as it involves computing higher and higher powers of the matrix  $A$ . However, it is easily accomplished for a diagonal matrix  $\hat{A} = \text{diag}(a_1, \dots, a_n)$  since  $\hat{A}^k = \text{diag}(a_1^k, \dots, a_n^k)$  so that

$$g(\text{diag}(a_1, \dots, a_n)) = \text{diag}(g(a_1), \dots, g(a_n)) \quad (7.32)$$

for a function  $g$ . This suggests that we might be able to compute the function of a more general matrix by diagonalizing and then applying Eq. (7.32). To do this, we first observe that computing the function of a matrix “commutes” with a change of basis. Indeed from

$$(P^{-1}AP)^k = P^{-1}A \underbrace{PP^{-1}}_{=1} AP \dots P^{-1}AP = P^{-1}A^kP$$

it follows that

$$g(P^{-1}AP) = P^{-1}g(A)P. \quad (7.33)$$

Now suppose that  $A$  can be diagonalized and  $P^{-1}AP = \hat{A} = \text{diag}(\lambda_1, \dots, \lambda_n)$ . Then

$$g(A) = g(P\hat{A}P^{-1}) = P g(\hat{A})P^{-1} = P \text{diag}(g(\lambda_1), \dots, g(\lambda_n))P^{-1}. \quad (7.34)$$

That is, we can compute the function of the matrix  $A$  by first forming the diagonal matrix which contains the function values of the eigenvalues and then transforming this matrix back to the original basis. Let us see how this works explicitly.

**Example 7.8:** Computing functions of matrices

(a) Let us consider the hermitian matrix

$$A = \begin{pmatrix} 1 & 2i \\ -2i & 1 \end{pmatrix} \quad (7.35)$$

which we have already diagonalized in Example 7.2 (d). Recall that the eigenvalues of this matrix are 3,  $-1$  and the diagonalizing basis transformation is given by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

so that  $U^\dagger AU = \text{diag}(3, -1)$ . We would like to calculate  $g(A)$  for the function  $g(x) = x^n$ , where  $n$  is an arbitrary integer. Then, from Eq. (7.34), we have

$$g(A) = U \text{diag}(3^n, (-1)^n)U^\dagger = \frac{1}{2} \begin{pmatrix} (-1)^n + 3^n & -i((-1)^n - 3^n) \\ i((-1)^n - 3^n) & (-1)^n + 3^n \end{pmatrix}.$$

(b) For another example consider the matrix

$$A = \theta T, \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where  $\theta$  is an arbitrary real number. Apart from the overall  $\theta$  factor (which does not affect the eigenvectors and multiplies the eigenvalues) this is the matrix we have studied in Example 7.2 (c). Hence, we know that the eigenvalues are  $\pm i\theta$  and the diagonalizing basis transformation is

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},$$

so that  $P^\dagger AP = \text{diag}(i\theta, -i\theta)$ . From Eq. (7.34) we, therefore, find for the matrix exponential of  $A$

$$e^A = P \text{diag}(e^{i\theta}, e^{-i\theta}) P^\dagger = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

It is not an accident that this comes out as a two-dimensional rotation. The theory of Lie groups states that rotations (and special unitary matrices) in all dimensions can be obtained as matrix exponentials of certain, relative simple matrices, such as  $A$  in the present example. This fact is particularly useful in higher dimensions when the rotation matrices are not so easily written down explicitly. To explain this in detail is well beyond the scope of this lecture.

Sometimes functions of matrices can be computed more straightforwardly without resorting to diagonalizing the matrix. This is usually possible when the matrix in question is relatively simple so that its powers can be computed explicitly. Indeed, this works for the present example and leads to an alternative calculation of the matrix exponential. To carry this out we first observe that  $A^2 = -\theta^2 \mathbb{1}_2$  and, hence,

$$A^{2n} = (-1)^n \theta^{2n} \mathbb{1}_2, \quad A^{2n+1} = (-1)^n \theta^{2n+1} T.$$

With these results it is straightforward to work out the matrix exponential explicitly.

$$\begin{aligned} e^A &= \sum_{n=0}^{\infty} \frac{1}{n!} A^n = \sum_{n=0}^{\infty} \frac{1}{(2n)!} A^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} A^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} \mathbb{1}_2 + \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} T = \cos(\theta) \mathbb{1}_2 + \sin(\theta) T. \end{aligned}$$

This coincides with the earlier result, as it must.

(c) We can take the previous example somewhat further and consider the three *Pauli matrices*  $\sigma_i$ , defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (7.36)$$

The three-dimensional vector space  $\mathcal{L} := \text{Span}(\sigma_1, \sigma_2, \sigma_3)$  over  $\mathbb{R}$  spanned by the Pauli matrices consists of all  $2 \times 2$  hermitian, traceless matrices. The Pauli matrices have a number of nice algebraic properties which can be summarized by the relation

$$\sigma_i \sigma_j = \mathbb{1}_2 \delta_{ij} + i \epsilon_{ijk} \sigma_k, \quad (7.37)$$

which is easily verified by using the explicit matrices above. For example, this relation implies immediately that their commutator (defined by  $[A, B] := AB - BA$ ) and anti-commutator (defined by  $\{A, B\} := AB + BA$ ) are given by

$$[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k, \quad \{\sigma_i, \sigma_j\} = 2 \mathbb{1}_2 \delta_{ij}. \quad (7.38)$$

We would like to work out the matrix exponential of an arbitrary linear combination of the Pauli matrices. Introducing the formal vector  $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T$  we can write such a linear combination for a vector  $\mathbf{a}$  with components  $a_i$  as  $\mathbf{a} \cdot \boldsymbol{\sigma} = a_i \sigma_i$ . Multiplying Eq. (7.37) with  $a_i a_j$  shows that  $(\mathbf{a} \cdot \boldsymbol{\sigma})^2 = |\mathbf{a}|^2 \mathbb{1}_2$  and, hence, for an arbitrary positive integer  $n$ ,

$$(\mathbf{a} \cdot \boldsymbol{\sigma})^{2n} = |\mathbf{a}|^{2n} \mathbb{1}_2, \quad (\mathbf{a} \cdot \boldsymbol{\sigma})^{2n+1} = |\mathbf{a}|^{2n} \mathbf{a} \cdot \boldsymbol{\sigma}. \quad (7.39)$$

Thanks to these relations it is now easy to work out the matrix exponent of  $i\theta \mathbf{n} \cdot \boldsymbol{\sigma}$ , where  $\mathbf{n}$  is a unit vector and  $\theta$  a real number, even without prior diagonalization. Using the Eqs. (7.39) with  $\mathbf{a} = \mathbf{n}$  we find

$$U := \exp(i\theta \mathbf{n} \cdot \boldsymbol{\sigma}) = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} (\mathbf{n} \cdot \boldsymbol{\sigma})^n = \cos(\theta) \mathbb{1}_2 + i \sin(\theta) \mathbf{n} \cdot \boldsymbol{\sigma}. \quad (7.40)$$

Remembering that  $\sigma_i^\dagger = \sigma_i$ , it is easy to verify that

$$U^\dagger U = (\cos(\theta)\mathbb{1}_2 - i \sin(\theta)\mathbf{n} \cdot \boldsymbol{\sigma}) (\cos(\theta)\mathbb{1}_2 + i \sin(\theta)\mathbf{n} \cdot \boldsymbol{\sigma}) = \mathbb{1}_2, \quad (7.41)$$

and, hence, the matrix exponentials  $U$  are unitary. Writing  $U$  out explicitly, using the Pauli matrices (7.36) and Eq. (7.40), gives

$$U = \begin{pmatrix} \cos \theta + in_3 \sin \theta & (n_2 + in_1) \sin \theta \\ -(n_2 - in_1) \sin \theta & \cos \theta - in_3 \sin \theta \end{pmatrix} \quad (7.42)$$

This shows that  $\det(U) = |\mathbf{n}|^2 = 1$  so that  $U$  is, in fact, special unitary. It turns out that all  $2 \times 2$  special unitary matrices can be obtained as matrix exponentials of Pauli matrices in this way, another example of the general statement from the theory of Lie groups mentioned earlier. Indeed, the matrix (7.42) can be converted into our earlier general form for  $SU(2)$  matrices in Eq. (6.59) by setting  $\alpha = \cos \theta + in_3 \sin \theta$  and  $\beta = (n_2 + in_1) \sin \theta$ . In mathematical parlance, the vector space  $\mathcal{L}$  of  $2 \times 2$  hermitian, traceless matrices is referred to as the *Lie algebra* of the  $2 \times 2$  special unitary matrices  $SU(2)$ .

### Example 7.9: Solving differential equations with matrix exponentials

Consider the simple first order ordinary differential equation

$$\frac{dx}{dt} = ax$$

for a real function  $t \rightarrow x(t)$  and an arbitrary real constant  $a$ . The general solution to this equation is of course

$$x(t) = e^{at}c, \quad (7.43)$$

for an arbitrary “initial value”  $c$ . What about its multi-dimensional generalization

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad (7.44)$$

where  $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))^T$  is a vector of  $n$  real functions and  $A$  is a constant real  $n \times n$  matrix? The straightforward generalization of the solution (7.43) to the multi-dimensional case reads

$$\mathbf{x}(t) = e^{At}\mathbf{c}, \quad (7.45)$$

where  $\mathbf{c}$  is an arbitrary  $n$ -dimensional vector. Note that, given our definition of the matrix exponential, Eq. (7.45) makes perfect sense. But does it really solve the differential equation (7.44)? We verify this by simply inserting Eq. (7.45) into the differential equation (7.44), using the definition of the matrix exponential.

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}e^{At}\mathbf{c} = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{1}{n!} A^n t^n \mathbf{c} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} A^n t^{n-1} \mathbf{c} = A \sum_{n=0}^{\infty} \frac{1}{n!} A^n t^n \mathbf{c} = A\mathbf{x} \quad (7.46)$$

Hence, Eq. (7.45) is indeed a solution for arbitrary vectors  $\mathbf{c}$ .

### 7.6.3 Quadratic forms

A *quadratic form* in the (real) variables  $\mathbf{x} = (x_1, \dots, x_n)^T$  is an expression of the form

$$q(\mathbf{x}) := \sum_{i,j=1}^n Q_{ij}x_i x_j = \mathbf{x}^T Q \mathbf{x} . \quad (7.47)$$

where  $Q$  is a real symmetric  $n \times n$  matrix with entries  $Q_{ij}$ . We have already encountered examples of such quadratic forms in Eq. (6.84) and the comparison shows that they can be viewed as symmetric bi-linear forms on  $\mathbb{R}^n$ . Our present task is to simplify the quadratic form by diagonalizing the matrix  $Q$ . With the diagonalizing basis transformation  $P$  and  $P^T Q P = \hat{Q} = \text{diag}(\lambda_1, \dots, \lambda_n)$  and new coordinates defined by  $\mathbf{x} = P \mathbf{y}$  we have

$$q(\mathbf{x}) = \mathbf{x}^T P \hat{Q} P^T \mathbf{x} = \mathbf{y}^T \hat{Q} \mathbf{y} = \sum_{i=1}^n \lambda_i y_i^2 . \quad (7.48)$$

Hence, in the new coordinates  $\mathbf{y}$  the cross terms in the quadratic form have been removed and only the pure square terms,  $y_i^2$ , are present. Note that they are multiplied by the eigenvalues of the matrix  $Q$ .

#### Application: Kinetic energy of a rotating rigid body

In Section (2), we have shown that the kinetic energy of a rotating rigid body is given by

$$E_{\text{kin}} = \frac{1}{2} \sum_{i,j} I_{ij} \omega_i \omega_j = \frac{1}{2} \boldsymbol{\omega}^T I \boldsymbol{\omega} . \quad (7.49)$$

where  $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T$  is the angular velocity and  $I$  is the moment of inertia tensor of the rigid body. Clearly, this is a quadratic form and by diagonalizing the moment of inertia tensor,  $P I P^T = \text{diag}(I_1, I_2, I_3)$  and introducing  $\boldsymbol{\Omega} = P \boldsymbol{\omega}$  we can write

$$E_{\text{kin}} = \frac{1}{2} \sum_{i=1}^3 I_i \Omega_i^2 . \quad (7.50)$$

This simplification of the kinetic energy is an important step in understanding the dynamics of rigid bodies.

Quadratic forms can be used to define quadratic curves (in two dimensions) or quadratic surfaces (in three dimensions) by the set of all points  $\mathbf{x}$  satisfying

$$\mathbf{x}^T Q \mathbf{x} = c , \quad (7.51)$$

with a real constant  $c$ . By diagonalizing the quadratic form, as in Eq. (7.48), the nature of the quadratic curve or surface can be immediately read off from the eigenvalues  $\lambda_i$  of  $Q$  as indicated in the table below.

condition on eigenvalues $\lambda_i$	two dimensions	three dimensions
all $\lambda_i$ equal, same sign as $c$	circle	sphere
all $\lambda_i$ have same sign as $c$	ellipse	ellipsoid
$\lambda_i$ with both signs	hyperbola	hyperboloid

In terms of the coordinates  $\mathbf{y} = P\mathbf{x}$  which diagonalize the quadratic form as in Eq. (7.48), the curve or surface defined by Eq. (7.51) can be written as

$$\sum_i \lambda_i y_i^2 = c. \quad (7.52)$$

Focus on the case of an ellipse or ellipsoid. The standard form of the equation defining an ellipse or ellipsoid is given by

$$\sum_i \frac{y_i^2}{l_i^2} = 1, \quad (7.53)$$

where  $l_i$  can be interpreted as the lengths of the semi-axes. By comparison with Eq. (7.52) we see that these lengths can be computed from the eigenvalues of the matrix  $Q$  by

$$l_i = \sqrt{\frac{c}{\lambda_i}}. \quad (7.54)$$

In the basis with coordinates  $\mathbf{y}$  the semi-axes are in the directions of the standard unit vectors  $\mathbf{e}_i$ . Hence, in the original basis with coordinates  $\mathbf{x}$  the semi-axis are in the directions  $\mathbf{v}_i = P\mathbf{e}_i$ , that is, in the directions of the eigenvectors  $\mathbf{v}_i$  of  $Q$ .

**Example 7.10:** Quadratic curve in  $\mathbb{R}^2$

Consider a quadratic curve in  $\mathbb{R}^2$  which is defined by all points  $\mathbf{x} = (x_1, x_2)^T$  which satisfy the equation

$$q(\mathbf{x}) = 3x_1^2 + 2x_1x_2 + 2x_2^2 = 1$$

The quadratic form can also be written as  $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$  where

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

The characteristic polynomials for  $A$  is

$$\chi_A(\lambda) = \det \begin{pmatrix} 3 - \lambda & 1 \\ 1 & 2 - \lambda \end{pmatrix} = \lambda^2 - 5\lambda + 5,$$

which leads to eigenvalues  $\lambda_{\pm} = (5 \pm \sqrt{5})/2$ . The corresponding eigenvectors (not normalized) are given by

$$\mathbf{v}_+ = \begin{pmatrix} 1 + \sqrt{5} \\ 2 \end{pmatrix}, \quad \mathbf{v}_- = \begin{pmatrix} 1 - \sqrt{5} \\ 2 \end{pmatrix}. \quad (7.55)$$

Since both eigenvalues are positive, but different this curve is an ellipse. In the diagonalizing coordinates  $\mathbf{y} = (y_1, y_2)^T$  the equation for this ellipse can be written as

$$q(\mathbf{y}) = \lambda_+ y_1^2 + \lambda_- y_2^2 = 1.$$

Comparing with the standard form  $\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} = 1$  of an ellipse shows that the lengths of the two half axes are given by

$$a = \frac{1}{\sqrt{\lambda_+}} = \sqrt{\frac{2}{5 + \sqrt{5}}}, \quad b = \frac{1}{\sqrt{\lambda_-}} = \sqrt{\frac{2}{5 - \sqrt{5}}}.$$

The directions of these two half-axes with lengths  $1/\sqrt{\lambda_{\pm}}$  are given by the eigenvectors  $\mathbf{v}_{\pm}$  in Eq. (7.55).

## Literature

A large number of textbooks on the subject can be found, varying in style from “Vectors and Matrices for Dummies” to hugely abstract treatises. I suggest a trip to the library in order to pick one or two books in the middle ground that you feel comfortable with. Below is a small selection which have proved useful in preparing the course.

- *Mathematical Methods for Physics and Engineering*, K. F. Riley, M. P. Hobson and S. J. Bence, CUP 2002.

This is the recommended book for the first year physics course which covers vectors and matrices and much of the other basic mathematics required. As the title suggests it is a “hands-on” book, strong on explaining methods and concrete applications, rather weaker on presenting a coherent mathematical exposition.

- *Linear Algebra*, S. Lang, Springer, 3rd edition.  
A nice mathematics book, written by a famous mathematician and at a fairly informal level, but following the mathematical logic of the subject.
- *Linear Algebra. An Introductory Approach*, C. W. Curtis, Springer 1996.  
A useful mathematics book but, despite the understating title, more formal than Lang.
- *Linear Algebra*, K. Jänich, Springer 1994.  
A mathematics book but with an attempt at intuitive presentation (many figures) and some connections to physics.

## A Definition of groups and fields

**Definition A.1.** (*Definition of a group*) A group  $G$  is a set with an operation

$$\cdot : G \times G \rightarrow G, \quad (g, h) \rightarrow g \cdot h \quad \text{“group multiplication”}$$

satisfying:

- (G1)  $g \cdot (h \cdot k) = (g \cdot h) \cdot k$  for all  $g, h, k \in G$ . “associativity”
- (G2) There exists a  $1 \in G$  such that  $1 \cdot g = g$  for all  $g \in G$ . “neutral element”
- (G3) For all  $g \in G$ , there exists a  $g^{-1} \in G$ , such that  $g^{-1} \cdot g = 1$ . “inverse”

The group is called Abelian if in addition

$$(G4) \quad g \cdot h = h \cdot g \text{ for all } g, h \in G. \quad \text{“commutativity”}$$

**Definition A.2.** (*Definition of a field*) A field  $F$  is a set with two operations

- (i) addition:  $+: F \times F \rightarrow F, (a, b) \rightarrow a + b$
- (ii) multiplication:  $\cdot : F \times F \rightarrow F, (a, b) \rightarrow a \cdot b$

such that the following holds:

- (F1)  $F$  is an Abelian group with respect to addition.
- (F2)  $F \setminus \{0\}$  is an Abelian group with respect to multiplication.
- (F3)  $a \cdot (b + c) = a \cdot b + a \cdot c$  for all  $a, b, c \in F$ .

Standard examples of fields are the rational numbers,  $\mathbb{Q}$ , the real numbers,  $\mathbb{R}$  and the complex numbers,  $\mathbb{C}$ . Somewhat more exotic examples are the finite fields  $\mathbb{F}_p = \{0, 1, \dots, p-1\}$ , where  $p$  is a prime number and addition and multiplication are defined by regular addition and multiplication of integers modulo  $p$ , that is, by the remainder of a division by  $p$ . Hence, whenever the result of an addition or multiplication exceeds  $p-1$  it is “brought back” into the range  $\{0, 1, \dots, p-1\}$  by subtracting a suitable multiple of  $p$ . The smallest field is  $\mathbb{F}_2 = \{0, 1\}$  containing just the neutral elements of addition and multiplication which must exist in every field.

## B Some basics of permutations

*Permutations* of  $n$  objects, the numbers  $\{1, \dots, n\}$  to be specific, are mathematically described by bijective maps  $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$  and the set  $S_n$  of all such permutations is given by

$$S_n = \{\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\} \mid \sigma \text{ is bijective}\}. \quad (\text{B.1})$$

Clearly, this set has  $n!$  elements and it forms a group in the sense of Def. A.1, with composition of maps (which is associative) as the group operation, the identity map as the neutral element and the inverse map  $\sigma^{-1} \in S_n$  as the group inverse for  $\sigma \in S_n$ . Permutations are also sometimes written as

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}, \quad (\text{B.2})$$

so as a  $2 \times n$  array of numbers (not a matrix in the sense of linear algebra), indicating that a number in the first row is permuted into the number in the second row underneath. The permutation group  $S_2$  has only two elements

$$S_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}, \quad (\text{B.3})$$

the identity element and the permutation which swaps 1 and 2. Clearly,  $S_2$  is Abelian (that is, map composition commutes) but this is no longer true for  $n > 2$ . For example, in  $S_3$ , the two permutations

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad (\text{B.4})$$

do not commute since

$$\sigma_1 \circ \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{but} \quad \sigma_2 \circ \sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}. \quad (\text{B.5})$$

The special permutations which swap two numbers and leave all other numbers unchanged are called *transpositions*.

**Lemma B.1.** *Every permutation  $\sigma \in S_n$  (for  $n > 1$ ) can be written as  $\sigma = \tau_1 \circ \cdots \circ \tau_k$ , where  $\tau_i$  are transpositions.*

*Proof.* Suppose that  $\sigma$  maps the first  $k_1 - 1 \geq 0$  numbers into themselves, so  $\sigma(i) = i$  for all  $i = 1, \dots, k_1 - 1$  and this is the maximal such number, so that  $\sigma(k_1) \neq \sigma(k_1)$  and, indeed,  $\sigma(k_1) > k_1$ . Then define  $\tau_1$  as the transposition which swaps  $k_1$  and  $\sigma(k_1)$ . The permutation  $\sigma_1 = \tau_1 \circ \sigma$  then leaves the first  $k_2 - 1$  numbers unchanged and, crucially,  $k_2 > k_1$ . We can continue this process until, after at most  $n$  steps,  $\text{id} = \tau_k \circ \cdots \circ \tau_1 \circ \sigma$ . Since transpositions are their own inverse it follows that  $\sigma = \tau_1 \circ \cdots \circ \tau_k$ .  $\square$

We would like to distinguish between even and odd permutations. Formally, this is achieved by the following

**Definition B.1.** *The sign of a permutation  $\sigma \in S_n$  is defined as*

$$\text{sgn}(\sigma) = \prod_{i < j} \frac{\sigma(i) - \sigma(j)}{i - j}. \quad (\text{B.6})$$

*A permutation  $\sigma$  is called even if  $\text{sgn}(\sigma) = 1$  and it is called odd if  $\text{sgn}(\sigma) = -1$ .*

We note that the numerator and the denominator on the RHS of Eq. (B.6) have, up to signs, the same factors and, therefore,  $\text{sgn}(\sigma) \in \{\pm 1\}$ . The sign satisfies the following important property.

**Theorem B.1.**  $\text{sgn}(\sigma \circ \rho) = \text{sgn}(\sigma) \text{sgn}(\rho)$  for all  $\sigma, \rho \in S_n$ .

*Proof.*

$$\begin{aligned} \text{sgn}(\sigma \circ \rho) &= \prod_{i < j} \frac{\sigma(\rho(j)) - \sigma(\rho(i))}{j - i} = \prod_{i < j} \frac{\sigma(\rho(j)) - \sigma(\rho(i))}{\rho(j) - \rho(i)} \prod_{i < j} \frac{\rho(j) - \rho(i)}{j - i} \\ &= \prod_{\rho(i) < \rho(j)} \frac{\sigma(\rho(j)) - \sigma(\rho(i))}{\rho(j) - \rho(i)} \text{sgn}(\rho) = \text{sgn}(\sigma) \text{sgn}(\rho). \end{aligned}$$

$\square$

For a transposition  $\tau \in S_n$  we have from Eq. (B.6) that

$$\text{sgn}(\tau) = -1. \quad (\text{B.7})$$

If we write an arbitrary permutation  $\sigma \in S_n$  in terms of transpositions,  $\sigma = \tau_1 \circ \cdots \circ \tau_k$ , then, from Theorem B.1, we find

$$\text{sgn}(\sigma) = \text{sgn}(\tau_1) \cdots \text{sgn}(\tau_k) = (-1)^k. \quad (\text{B.8})$$

This means that even permutations are precisely those which are generated by combining an even number of transpositions and odd permutations those which are obtained from an odd number of transpositions.



## C Tensors for the curious

Tensors are part of more advanced linear algebra and for this reason they are often not considered in an introductory text. However, in many physics courses there is no room to return to the subject at a later time and, as a result, tensors are often not taught at all. To many physicists, they remain mysterious, despite their numerous applications in physics. If you do not want to remain perplexed, this appendix is for you. It provides a short, no-nonsense introduction into tensors, starting where the main text has left off.

We start with a vector space  $V$  over a field  $F$  with basis  $\epsilon_1, \dots, \epsilon_n$  and its dual  $V^*$ . Recall, that  $V^*$  is the vector space of linear functionals  $V \rightarrow F$ . From Section 6.5, we know that  $V^*$  has a dual basis  $\epsilon_*^1, \dots, \epsilon_*^n$  satisfying

$$\epsilon_*^i(\epsilon_j) = \delta_j^i. \quad (\text{C.1})$$

In particular,  $V$  and  $V^*$  have the same dimension. An obvious, but somewhat abstract problem which we need to clarify first has to do with the “double-dual” of a vector space. In other words, what is the dual,  $V^{**}$  of the dual vectors space  $V^*$ ? Our chosen terminology suggests the double-dual  $V^{**}$  should be the original vector space  $V$ . This is indeed the case, in the sense of the following

**Lemma C.1.** *The linear map  $j : V \rightarrow V^{**}$  defined by  $j(\mathbf{v})(\varphi) := \varphi(\mathbf{v})$  is bijective, that is, it is an isomorphism between  $V$  and  $V^{**}$ .*

*Proof.* From Lemma 3.1 and since  $\dim(V) = \dim(V^*) = \dim(V^{**})$  all we need to show is that  $\text{Ker}(j) = \{\mathbf{0}\}$ . Start with a vector  $\mathbf{v} = v^i \epsilon_i \in \text{Ker}(j)$ . Then, for all  $\varphi \in V^*$ , we have  $0 = j(\mathbf{v})(\varphi) = \varphi(\mathbf{v})$ . Choose  $\varphi = \epsilon_*^j$  and it follows that  $0 = \epsilon_*^j(\mathbf{v}) = v^j$ . Hence, all component  $v^j$  vanish and  $\mathbf{v} = \mathbf{0}$ .  $\square$

Note that the definition of the above map  $j$  does not depend on a choice of basis. For this reason it is also referred to as a *canonical isomorphism* between  $V$  or  $V^{**}$ . We should think of  $V$  and  $V^{**}$  as the same space by identifying vectors  $\mathbf{v} \in V$  with their images  $j(\mathbf{v}) \in V^{**}$  under  $j$ . Since  $V^{**} \cong V$  consists of linear functionals on  $V^*$  this means that the relation between  $V$  and  $V^*$  is “symmetric”. Not only can elements  $\varphi \in V^*$  act on vectors  $\mathbf{v} \in V$  but also the converse works. This is the essence of the relation  $j(\mathbf{v})(\varphi) := \varphi(\mathbf{v})$ , defining the map  $j$ , which, by abuse of notation, is often written as

$$\mathbf{v}(\varphi) = \varphi(\mathbf{v}). \quad (\text{C.2})$$

Having put the vector space and its dual on equal footing we can now proceed to define tensors. We consider two vector spaces  $V$  and  $W$  over  $F$  and define the tensor space

$$V^* \otimes W^* := \{\tau : V \times W \rightarrow F \mid \tau \text{ bi-linear}\}. \quad (\text{C.3})$$

In other words, the tensor space  $V^* \otimes W^*$  consists of all maps  $\tau$  which assign to their two vector arguments  $\mathbf{v} \in V$  and  $\mathbf{w} \in W$  a number  $\tau(\mathbf{v}, \mathbf{w})$  and are linear in each argument. Note that we can think of this as a generalization of a linear functional. While a linear functional assign a number to a single vector argument, the tensor  $\tau$  does the same for two vector arguments. This suggests that tensors might be “built up” from functionals. To this end, we introduce the tensor product  $\varphi \otimes \psi$  between two functionals  $\varphi \in V^*$  and  $\psi \in W^*$  by

$$(\varphi \otimes \psi)(\mathbf{v}, \mathbf{w}) := \varphi(\mathbf{v})\psi(\mathbf{w}). \quad (\text{C.4})$$

Clearly, the so-defined map  $\varphi \otimes \psi$  is an element of the tensor space  $V^* \otimes W^*$  since it takes two vector arguments and is linear in each of them (since  $\varphi$  and  $\psi$  are linear in their respective arguments).

Can we get all tensors from tensor products? In a certain sense, the answer is “yes” as explained in the following

**Lemma C.2.** For a basis  $\{\epsilon_*^i\}$ , where  $i = 1, \dots, n$ , of  $V^*$  and a basis  $\{\tilde{\epsilon}_*^a\}$ , where  $a = 1, \dots, m$ , of  $W^*$  the tensor products  $\{\epsilon_*^i \otimes \tilde{\epsilon}_*^a\}$  form a basis of  $V^* \otimes W^*$ . In particular,  $\dim(V^* \otimes W^*) = \dim(V^*) \dim(W^*)$ .

*Proof.* We introduce the dual basis  $\{\epsilon_j\}$  on  $V$  and  $\{\tilde{\epsilon}_b\}$  on  $W$  so that  $\epsilon_*^i(\epsilon_j) = \delta_j^i$  and  $\tilde{\epsilon}_*^a(\tilde{\epsilon}_b) = \delta_b^a$ . As usual, we need to prove that the tensors  $\epsilon_*^i \otimes \tilde{\epsilon}_*^a$  are linearly independent and span the tensor space  $V^* \otimes W^*$ . We begin with linear independence.

$$\sum_{i,a} \tau_{ia} \epsilon_*^i \otimes \tilde{\epsilon}_*^a = 0$$

Acting with this equation on the vector pair  $(\epsilon_j, \tilde{\epsilon}_b)$  gives

$$0 = \sum_{i,a} \tau_{ia} \epsilon_*^i \otimes \tilde{\epsilon}_*^a (\epsilon_j, \tilde{\epsilon}_b) = \sum_{i,a} \tau_{ia} \epsilon_*^i(\epsilon_j) \tilde{\epsilon}_*^a(\tilde{\epsilon}_b) = \sum_{i,a} \tau_{ia} \delta_j^i \delta_b^a = \tau_{jb} .$$

Hence, since all coefficients  $\tau_{jb} = 0$  linear independence follows.

To show that the space is spanned start with an arbitrary tensor  $\tau \in V^* \otimes W^*$ . We define its ‘‘components’’  $\tau_{ia} := \tau(\epsilon_i, \tilde{\epsilon}_a)$  and the tensor

$$\mu = \sum_{j,b} \tau_{jb} \epsilon_*^j \otimes \tilde{\epsilon}_*^b .$$

Since  $\mu(\epsilon_i, \tilde{\epsilon}_a) = \tau_{ia} = \tau(\epsilon_i, \tilde{\epsilon}_a)$ , the tensors  $\mu$  and  $\tau$  coincide on a basis and, hence,  $\mu = \tau$ . We have, therefore, written the arbitrary tensor  $\tau$  as a linear combination of the  $\epsilon_*^j \otimes \tilde{\epsilon}_*^b$ .  $\square$

The above Lemma provides us with a simple way to think about the tensors in  $V^* \otimes W^*$ . Given a basis  $\{\epsilon_j\}$  on  $V$  and  $\{\tilde{\epsilon}_b\}$  on  $W$  the tensors in  $V^* \otimes W^*$  are given by

$$\tau = \sum_{i,a} \tau_{ia} \epsilon_*^i \otimes \tilde{\epsilon}_*^a , \tag{C.5}$$

where  $\tau_{ia} \in F$  are arbitrary coefficients. Often, the basis elements are omitted and the set of components  $\tau_{ia}$ , labelled by two indices, are referred to as tensor. This can be viewed as a generalization of vectors whose components are labelled by one index.

Tensoring can of course be repeated with multiple vector spaces. A particularly important tensor space, on which we will focus from hereon, is

$$\underbrace{V \otimes \dots \otimes V}_p \otimes \underbrace{V^* \otimes \dots \otimes V^*}_q = \text{Span}\{\epsilon_{i_1} \otimes \dots \otimes \epsilon_{i_p} \otimes \epsilon_*^{j_1} \otimes \dots \otimes \epsilon_*^{j_q}\} , \tag{C.6}$$

formed from  $p$  factors of the vector space  $V$  and  $q$  factors of its dual  $V^*$ . Its dimension is  $\dim(V)^{p+q}$ . A general element of this space can be written as a linear combination

$$\tau = \sum_{i_1, \dots, i_p, j_1, \dots, j_q} \tau_{j_1 \dots j_q}^{i_1 \dots i_p} \epsilon_{i_1} \otimes \dots \otimes \epsilon_{i_p} \otimes \epsilon_*^{j_1} \otimes \dots \otimes \epsilon_*^{j_q} , \tag{C.7}$$

and is also referred to as a  $(p, q)$  tensor. It can also be represented by the components  $\tau_{j_1 \dots j_q}^{i_1 \dots i_p}$  which carry  $p$  upper and  $q$  lower indices and practical applications are often phrased in those terms. From a  $(p, q)$  tensor  $\tau_{j_1 \dots j_q}^{i_1 \dots i_p}$  and an  $(r, s)$  tensor  $\mu_{l_1 \dots l_s}^{k_1 \dots k_r}$  we can create a new tensor by multiplication and contraction (that is summation) over some (or all) of the upper indices of  $\tau$  and the lower indices of  $\mu$  and vice versa. Such a summation over same upper and lower indices is in line with the Einstein summation convention and corresponds to the action of dual vectors on vectors, as discussion in Section 6.5.

It is probably best to discuss this explicitly for a number of examples. It turns out that many of the objects we have introduced in the main part of the text can be phrased in the language of tensors.

**Example C.1:** Examples of tensors

(a) Vectors as tensors

A vector  $\mathbf{v} = v^i \boldsymbol{\epsilon}_i$  in a vector space  $V$  with basis  $\{\boldsymbol{\epsilon}_i\}$  is a  $(1, 0)$  tensor and, accordingly, its component form is  $v^i$ , an object with one upper index.

(b) Dual vectors as tensors

Linear functionals  $\varphi = \varphi_i \boldsymbol{\epsilon}_*^i$  in the dual vector space  $V^*$  with dual basis  $\{\boldsymbol{\epsilon}_*^i\}$  are  $(0, 1)$  tensors and are, hence, represented by components  $\varphi_i$  with one lower index. As already discussed in Section 6.5, the action of a linear functional on a vector is given by

$$\varphi(\mathbf{v}) = \varphi_i v^j \boldsymbol{\epsilon}_*^i(\boldsymbol{\epsilon}_j) = \varphi_i v^j \delta_j^i = \varphi^i v_i, \quad (\text{C.8})$$

which corresponds to the contraction of a  $(1, 0)$  and a  $(0, 1)$  tensor over their single index to produce a  $(0, 0)$  tensor, that is, a scalar.

(c) Linear maps and matrices as tensors

Consider a linear map  $f : V \rightarrow V$  which is represented by a matrix  $A$  relative to the basis  $\{\boldsymbol{\epsilon}_i\}$  of  $V$ . Then, from Lemma 3.3, the components  $A^i_j$  of  $A$  are obtained from the images of the basis vectors via  $f(\boldsymbol{\epsilon}_j) = \sum_i A^i_j \boldsymbol{\epsilon}_i$ . We can re-write this relation as  $f(\boldsymbol{\epsilon}_j) = A^i_k \boldsymbol{\epsilon}_i \boldsymbol{\epsilon}_*^k(\boldsymbol{\epsilon}_j)$  and, stripping off the basis vector  $\boldsymbol{\epsilon}_j$  on either side, this leads to

$$f = A^i_k \boldsymbol{\epsilon}_i \otimes \boldsymbol{\epsilon}_*^k. \quad (\text{C.9})$$

Hence, a linear map can be viewed as a  $(1, 1)$  tensor whose components  $A^i_k$  have one upper and one lower index (refining our notation from the main part of the text, where we have used two lower indices for matrices) and form the entries of the representing matrix for  $f$ . In components, the action of the linear map on the vector  $\mathbf{v} = v^k \boldsymbol{\epsilon}_k$  can, of course, be written as a matrix multiplication

$$v^i \rightarrow A^i_k v^k. \quad (\text{C.10})$$

In tensor language this amounts to the contraction of a  $(1, 1)$  tensor (the matrix  $A^i_k$ ) and a  $(1, 0)$  tensor (the vector  $v^k$ ) to produce another  $(1, 0)$  tensor (the resulting image vector  $A^i_k v^k$ ).

(d) The identity map as a tensor

The identity map,  $\text{id}_V$  is of course represented by the unit matrix,  $\mathbb{1}$ , whose entries are given by the Kronecker delta  $\delta_k^i$  so that

$$\text{id}_V = \delta_k^i \boldsymbol{\epsilon}_i \otimes \boldsymbol{\epsilon}_*^k. \quad (\text{C.11})$$

From this point of view we can interpret the Kronecker delta as a  $(1, 1)$  tensor.

(e) Bi-linear forms as tensors

For a bi-linear form  $\langle \cdot, \cdot \rangle$  on a vector space  $V$  with basis  $\{\boldsymbol{\epsilon}_i\}$  define the metric  $g_{ij} := \langle \boldsymbol{\epsilon}_i, \boldsymbol{\epsilon}_j \rangle$ , as we have done in Section 6.5. Then we can write

$$\langle \cdot, \cdot \rangle = g_{ij} \boldsymbol{\epsilon}_*^i \otimes \boldsymbol{\epsilon}_*^j, \quad (\text{C.12})$$

so that the bi-linear form can be viewed as a  $(0, 2)$  tensor with components  $g_{ij}$ . The scalar product of two vectors  $\mathbf{v} = v^i \boldsymbol{\epsilon}_i$  and  $\mathbf{w} = w^j \boldsymbol{\epsilon}_j$  is

$$\langle \mathbf{v}, \mathbf{w} \rangle = g_{ij} v^i w^j, \quad (\text{C.13})$$

and can, hence, be viewed as the contraction of a  $(0, 2)$  tensor (the metric  $g_{ij}$ ) with two  $(1, 0)$  tensors (the vectors  $v^i$  and  $w^j$ ) to form a scalar (the value  $g_{ij} v^i w^j$  of the scalar product).

If the bi-linear form is non-degenerate then, from Lemma 6.6, the metric  $g_{ij}$  is invertible. Its inverse is written as a  $(2, 0)$  tensor  $g^{ij}$  which satisfies

$$g^{ij} g_{jk} = \delta_k^i. \quad (\text{C.14})$$

As we have seen in Lemma 6.6, a non-degenerate bi-linear form leads to an isomorphism  $\iota : V \rightarrow V^*$  between the vector space and its dual and we can use this to define a scalar product  $\langle \cdot, \cdot \rangle_*$  on the dual vector space  $V^*$  by

$$\langle \varphi, \psi \rangle_* := \langle \iota^{-1}(\varphi), \iota^{-1}(\psi) \rangle. \quad (\text{C.15})$$

Since the representing matrix for  $\iota$  is  $g_{ij}$  its inverse,  $\iota^{-1}$ , is represented by  $g^{ij}$ . Hence, we can write the scalar product on the dual vector space as

$$\langle \varphi, \psi \rangle_* = g_{ij} g^{ik} \varphi_k g^{jl} \psi_l = g^{ij} \varphi_i \psi_j \quad \text{or} \quad \langle \cdot, \cdot \rangle_* = g^{ij} \epsilon_i \otimes \epsilon_j. \quad (\text{C.16})$$

This shows that the scalar product  $\langle \cdot, \cdot \rangle_*$  on  $V^*$  can be viewed as a  $(2, 0)$  tensor with component  $g^{ij}$ .

The identification of vector space and dual vector space by a non-degenerate bi-linear form (via the map  $\iota$  and its representing matrix  $g_{ij}$ ) can be extended to tensors and used to change their degree. We can use the metric  $g_{ij}$  to lower one of the (upper) indices of a  $(p, q)$  tensor, thereby converting it into a  $(p - 1, q + 1)$  tensor and the inverse metric  $g^{ij}$  to raise a (lower) index of a  $(p, q)$  tensor to produce a  $(p + 1, q - 1)$  tensor.

If the basis  $\{\epsilon_i\}$  is an ortho-normal basis of the scalar product  $\langle \cdot, \cdot \rangle$ , then the metric is  $g_{ij} = \delta_{ij}$  and its inverse  $g^{ij} = \delta^{ij}$  and, from this point of view, the Kronecker delta  $\delta_{ij}$  is a  $(0, 2)$  tensor and its upper-index version,  $\delta^{ij}$ , a  $(2, 0)$  tensor.

(f) Determinant as a tensor

We consider  $V = \mathbb{R}^n$  (or  $V = \mathbb{C}^n$ ) with the basis of standard unit vectors  $\{\mathbf{e}_i\}$  and the associated dual basis  $\{\mathbf{e}_*^i\}$ . The determinant is, by definition, linear in each of its  $n$  vectorial arguments and is, therefore, a tensor. To make this more explicit we start with  $n$  vectors  $\mathbf{v}_i = v_i^j \mathbf{e}_j$  and write

$$\det(\mathbf{v}_1, \dots, \mathbf{v}_n) = \epsilon_{i_1 \dots i_n} v_1^{i_1} \dots v_n^{i_n} = \epsilon_{i_1 \dots i_n} \mathbf{e}_*^{i_1} \otimes \dots \otimes \mathbf{e}_*^{i_n}(\mathbf{v}_1, \dots, \mathbf{v}_n). \quad (\text{C.17})$$

Stripping off the vectorial arguments results in

$$\det = \epsilon_{i_1 \dots i_n} \mathbf{e}_*^{i_1} \dots \mathbf{e}_*^{i_n}, \quad (\text{C.18})$$

which shows that the determinant can be viewed as a  $(0, n)$  tensor whose components are given by the Levi-Civita tensor  $\epsilon_{i_1 \dots i_n}$ . Computing the determinant amounts to contracting the Levi-Civita  $(0, n)$  tensor into  $n$   $(1, 0)$  tensors (that is, into  $n$  vectors), resulting in a scalar.