1

Introduction

The aim of these 8 lectures is to show how the ideas introduced earlier in the second section of the course in connection with electrostatic plasmas can be extended to stellar systems, with sometimes surprising results. This is rather a small, even niche, corner of stellar dynamics, but an intriguing one that fits neatly with the remainder of this course. A general introduction to stellar dynamics can be found in *Galactic Dynamics*, Binney & Tremaine, PUP (2008) – hereafter BT08 – while rather a different perspective is given in a review arXiv1309.2794 (NewAR, 57, 29). Any fan of recorded lectures can try the lectures at http://iactalks.iac.es/talks/view/329.

1.1 What differentiates stellar and electrostatic plasmas?

The key differences between a gravitational plasma and an electrostatic one are:

- Gravity, being an even-spin theory (e.g. A. Zee, *Quantum Field theory in a Nutshell*, Princeton University Press) endows all particles with the same sign of charge.
- Consequently, self-gravitating systems are globally inhomogeneous rather than inhomogeneous only on scales smaller than the Debye length $\lambda_{\rm D}$.

In a solid or liquid the forces on a particle are dominated by the contributions of near neighbours. In an electrostatic plasma the forces is dominated by contributions from particles less distant than $\lambda_{\rm D}$. A simple argument shows that in a self-gravitating system the forces are dominated by remote particles.

Consider the gravitational force on one particle in a system of $N \gg 1$ particles. On scales much bigger than the mean inter-particle distance λ we can characterise the system by a mean mass-density $\rho(\mathbf{x})$. We place the origin of the coordinates at our particle's location and consider the force due to mass near $\mathbf{x}$. The mass in the cell distance $|\mathbf{x}|$ away and subtending solid angle $\mathrm{d}^2\mathbf{\Omega}$ is $\delta M = \rho(\mathbf{x})|\mathbf{x}|^2\mathrm{d}|\mathbf{x}|\,\mathrm{d}^2\mathbf{\Omega}$ so by the inverse-square law, the force on our particle from this cell is $\delta\mathbf{F} = G\rho(\mathbf{x})\mathrm{d}|\mathbf{x}|\,\mathrm{d}^2\mathbf{\Omega}$. Crucially the factor $|\mathbf{x}|^2$ has disappeared, so as we increase $|\mathbf{x}|$, the contributions to $\mathbf{F}$ simply track $\rho(\mathbf{x})$ (Figure 1.1).

For every cell at $\mathbf{x}$ there is a corresponding one at $-\mathbf{x}$, which pulls in the opposite direction. Hence until $|\mathbf{x}|$ is not small compared to the scale on which $\rho(\mathbf{x})$ varies, the net force on our particle will be negligible. That is, the force on our particle is dominated by distant particles, not by near neighbours.

1.2 Virial theorem

We first obtain a result that links the mass of a self-gravitating system to its extents in real and velocity space. We have N particles of mass m moving in the mutually generated gravitational field. We dot the eqn of motion of one particle (α)

$$m\ddot{\mathbf{x}}_\alpha = Gm^2 \sum_\beta \frac{\mathbf{x}_\beta - \mathbf{x}_\alpha}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3} \tag{1.1}$$

by $\mathbf{x}_\alpha$ and sum over α:

$$m \sum_\alpha \ddot{\mathbf{x}}_\alpha \cdot \mathbf{x}_\alpha = Gm^2 \sum_{\alpha\beta} \frac{(\mathbf{x}_\beta - \mathbf{x}_\alpha)\cdot \mathbf{x}_\alpha}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3} \tag{1.2}$$

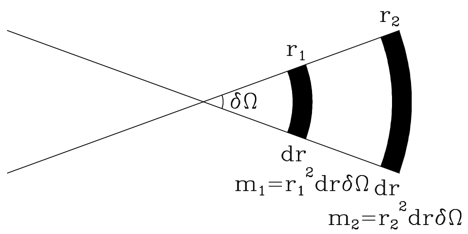

Figure 1.1 Each shaded portion of the cone contributes equally to the force on the star at its apex.

Adding to this the equation obtained by $\alpha \leftrightarrow \beta$ we get

$$m \sum_\alpha \ddot{\mathbf{x}}_\alpha \cdot \mathbf{x}_\alpha + m \sum_\beta \ddot{\mathbf{x}}_\beta \cdot \mathbf{x}_\beta = Gm^2 \sum_{\alpha\beta} \frac{(\mathbf{x}_\beta - \mathbf{x}_\alpha)\cdot(\mathbf{x}_\alpha - \mathbf{x}_\beta)}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3} \tag{1.3}$$

Simplifying each side

$$m \sum_\alpha \left(\frac{\mathrm{d}^2 |\mathbf{x}_\alpha|^2}{\mathrm{d}t^2} - 2|\dot{\mathbf{x}}_\alpha|^2 \right) = -Gm^2 \sum \frac{1}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|}. \tag{1.4}$$

If the derivative were significantly non-zero, the cluster would be secularly expanding or contracting. So we argue this term is negligible and conclude that $2K + W = 0$, where

$$K = \tfrac{1}{2} m \sum_{\alpha=1}^N |\dot{\mathbf{x}}_\alpha|^2; \quad W = -\tfrac{1}{2} Gm^2 \sum_{\alpha,\beta=1}^N \frac{1}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|}. \tag{1.5}$$

K is the cluster's kinetic energy while W is its potential energy. So we have an N-particle version of the 1-particle virial theorem, which should be familiar from quantum mechanics: that if the P.E. $V(\mathbf{x})$ scales as $V(s\mathbf{x}) = s^\alpha V(\mathbf{x})$, then $2\langle K \rangle = \alpha \langle V \rangle$, where $K = p^2/2m$ is the kinetic-energy operator.

Our Galaxy is thought to have a (largely dark) mass $M \sim 10^{12}\,\mathrm{M}_\odot$ distributed through a volume of characteristic radius $R \sim 100\,\mathrm{kpc}$. Taking $|\mathbf{x}_\alpha - \mathbf{x}_\beta| \sim R$ and summing N^2 terms $1/R$ we estimate

$$W \sim -\frac{GM^2}{2R}. \tag{1.6}$$

If the typical speed of a dark particle is σ, $2K \sim M\sigma^2$, so the virial theorem yields

$$\sigma^2 \sim \frac{GM}{2R}. \tag{1.7}$$

Putting in numbers

$$\sigma = \sqrt{\frac{6.7 \times 10^{-11} \times 10^{12} \times 1.6 \times 10^{30}}{2 \times 100 \times 3 \times 10^{19}}} \simeq 2.0 \times 10^5\,\mathrm{m\,s^{-1}} \sim 200\,\mathrm{km\,s^{-1}}.$$

is a typical random velocity of a dark particle.

1.3 Thermal equilibrium?

It's natural to imagine that the Galaxy comprises a gravitationally confined system of dark particles in thermal equilibrium. It's a monatomic gas so its temperature is given by $\frac{3}{2} N k_\mathrm{B} T = K$ and its internal energy is $U = K + W = -K$ is negative. The heat capacity of the system

$$C = \frac{\partial U}{\partial T} = -\frac{\partial K}{\partial T} = -\tfrac{3}{2} N k_\mathrm{B} \tag{1.8}$$

is also negative. A negative specific heat is highly problematic because it makes it impossible for the system to come into thermal equilibrium with a conventional heat bath: suppose the system and heat bath were in thermal equilibrium. Then a fluctuation could shift δU of energy from the system to the heat bath. The system would heat up by $\delta T = |\delta U/C|$ and the heat bath, if it was large, would heat by a smaller amount. Since the system would now be hotter than the bath, more heat would flow from the system to the bath and the system would get hotter and hotter, apparently without limit.

1.4 Escape

There's another conceptual problem associated with the system attaining thermal equilibrium: gravity confines particles only up to a finite escape speed $v_{\rm esc}(\mathbf{x}) = \sqrt{-2\Phi(\mathbf{x})}$, where Φ is the gravitational potential. Hence in thermal equilibrium the DF $f(\mathbf{x},\mathbf{v})$ would have to vanish for $v > v_{\rm esc}$, so could not be a Maxwellian since the latter is non-zero all the way to ∞. Yet surely the processes that maintain thermal equilibrium will scatter stars to $v > v_{\rm escape}$ and such stars will then escape ('evaporate') from the system. We can assess the scale of this issue by computing the mean-square value of $v_{\rm esc}$:

$$V^2_{\rm esacape} = \frac{1}{M}\int {\rm d}^3\mathbf{x}\,\rho v^2_{\rm esc} = -\frac{2}{M}\int {\rm d}^3\mathbf{x}\,\rho\Phi = -4\frac{W}{M} = 8\frac{K}{M} = 4\sigma^2. \tag{1.9}$$

So $V_{\rm esc} = 2\sigma$, i.e. twice the rms speed, which isn't far into the high-velocity tail. The fraction of a Maxwellian distribution with one-dimensional dispersion $\sigma/\sqrt{3}$ that lies above 2σ is

$$f_{\rm esc} = \frac{\int_{2\sigma}^\infty {\rm d}v\, v^2\exp(-3v^2/2\sigma^2)}{\int_0^\infty {\rm d}v\, v^2\exp(-3v^2/2\sigma^2)} = \frac{\int_{\sqrt{6}}^\infty {\rm d}x\, x^2{\rm e}^{-x^2}}{\int_0^\infty {\rm d}x\, x^2{\rm e}^{-x^2}} \sim \frac{1}{140}. \tag{1.10}$$

Since the velocities of stars will be reshuffled into a Maxwellian once every relaxation time, each such time we expect $\sim M/140$ of the mass to evaporate.

1.5 Fluctuations

So how long is the relaxation time? Consider a system of mass M and characteristic scale R, in which the characteristic internal speed is $\sigma = \sqrt{GM/R}$. Consider now a subregion of size $r = xR$, which contains mass $M_r \simeq x^3M$. If there are N stars in the entire system, then $n \simeq x^3N$ is the typical number of stars in the subregion, and on account of Poisson noise M_r fluctuates by $\delta M_r = M_r/\sqrt{n} = x^3M/\sqrt{x^3N}$ during times $\delta t = r/\sigma$. Consider a point that is distance yR from our subregion. At this point a single fluctuation in the subregion's gravitational attraction will change the velocity of a test star by

$$\delta v = \frac{G\delta M_r}{(yR)^2}\delta t = \frac{GMx^{3/2}}{(yR)^2\sqrt{N}}\frac{xR}{\sigma} = \frac{\sigma x^{5/2}}{y^2\sqrt{N}}. \tag{1.11}$$

This formula states that for given y, large volumes $x \simeq 1$ perturb v very much more strongly than small volumes $x \ll 1$. Against this trend we must bear in mind that (a) $y \geq x$, (b) the number of subregions perturbing increases as x^{-3} as x decreases, and (c) the time within which the contribution (1.11) comes about decreases with x, so in a given time each small subregion makes many more contributions to v than does a large subregion.

We assume that the contributions to v from different subregions are statistically independent, so it's appropriate to add the δv in quadrature. There are $\sim 4\pi(y/x)^2$ subregions of scale x that are distance yR from our point, and in a crossing time $t_{\rm cross} = R/\sigma$ each such subregion contributes x^{-1} times. So in a crossing time all these subregions change v^2 by

$$(\Delta v)^2 = 4\pi\frac{y^2}{x^3}(\delta v)^2 = 4\pi\frac{\sigma^2x^2}{y^2N}. \tag{1.12}$$

Now we have to sum over $y = x, 2x, 3x, \dots, 1$. We convert the sum to an integral using ${\rm d}y = x$ and have

$$\sum\frac{1}{y^2} \simeq \frac{1}{x}\int_x^1\frac{{\rm d}y}{y^2} = \frac{1}{x}\left(\frac{1}{x}-1\right) \simeq \frac{1}{x^2}. \tag{1.13}$$

Hence in a crossing time the subregions of scale x change v^2 by

$$(\Delta v)^2 \simeq 4\pi\sigma^2/N. \tag{1.14}$$

Remarkably, this is independent of x, so regions of each scale xR contribute equally to changing v^2. We obtain the total change by summing these contributions over all relevant values of x. We do this by multiplying equation (1.14) by $-\ln x_{\rm min}$, where $x_{\rm min}R$ is the smallest subregion it's sensible to consider. This clearly shouldn't be smaller than a decent multiple of the inter-particle distance $\lambda \sim R/N^{1/3}$.

$$(\Delta v)^2_{t_{\rm cross}} \simeq \frac{4\pi\sigma^2 \ln N}{3N} \tag{1.15}$$

The **relaxation time** is the time required for fluctuations to change any velocity by order of itself, thus for $(\Delta v)^2$ to accumulate to σ^2. From (1.15) it follows that

$$t_{\rm relax} \simeq \frac{N}{4\ln N} t_{\rm cross}. \tag{1.16}$$

In an ideal gas the number of molecules in a given volume experiences Poisson fluctuations as was assumed above, and these fluctuations can be considered to arise from thermally excited sound waves. The self-gravity of a stellar system makes the system more compressible on large scales than on small scales, where self gravity is unimportant and an ideal gas provides a valid model. Hence, large-scale fluctuations have a larger amplitude than simple Poisson fluctuations, with the consequence that contrary to our finding above of equal contributions from all scales, fluctuations on the size of the system are dominant. In Chapter 4 we will develop the apparatus required to include the amplifying effect of self gravity, and in Chapter 5 we will see that self-gravity accelerates the relaxation of stellar discs by orders of magnitude. Its effect is much smaller in star clusters.

The conclusions we've reached in §§1.3 and 1.4 make it clear that the statistical mechanics of self-gravitating systems must be very different from anything we have previously encountered.

2
Mean-field model

In this section we assemble the tools needed to figure out the long-term evolution of self-gravitating systems. The key step is to recognise that the evolution can be described as a sequence of steady states of a 'mean-field' model. Since the dominant forces come from remote particles (Figure 1.1), an excellent approximation to $\mathbf{F}$ can be obtained by smearing the masses of each particle over distances somewhat larger than the inter-particle distance. The gravitational potential Φ_0 of this **mean-field model** is the time-average of the system's real fluctuating Φ. The latter may be computed by smearing the passes of particles through volumes that extend just a bit further than the local inter-particle distance. This system has a pretty smooth density distribution $\rho(\mathbf{x})$ and consequently a very smooth gravitational potential

$$\Phi(\mathbf{x}) = -G\int \mathrm{d}^3\mathbf{x}'\, \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}. \tag{2.1}$$

Conservation of particles as they flow through phase space requires that the one-particle DF $f(\mathbf{x},\mathbf{v})$ of the mean-field model satisfies

$$0 = \frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\cdot(f\dot{\mathbf{x}}) + \frac{\partial}{\partial \mathbf{v}}\cdot(f\dot{\mathbf{v}}) = 0, \tag{2.2}$$

and since by Hamilton's equations

$$\frac{\partial}{\partial \mathbf{x}}\cdot\dot{\mathbf{x}} = \frac{\partial^2 H}{\partial \mathbf{x}\cdot\partial \mathbf{v}} = -\frac{\partial}{\partial \mathbf{v}}\cdot\dot{\mathbf{v}}, \tag{2.3}$$

where $H = \frac{1}{2}v^2 + \Phi$ is the Hamiltonian, we have that f satisfies the collisionless Boltzmann (Vlasov) equation

$$0 = \frac{\partial f}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{x}} + \dot{\mathbf{v}} \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial f}{\partial t} + [f, H], \tag{2.4}$$

where $[.,.]$ is the Poisson bracket.

Consider now a steady-state mean-field model. In this case $[f, H] = 0$, so the DF is a constant of the equations of particle motion in the mean-field potential. **Jeans' theorem** states (trivially because f *is* a constant of motion!) that the DF of a stationary mean-field model can be assumed to depend on $(\mathbf{x}, \mathbf{v})$ only through constants of motion.

Inhomogeneity is a major setback: when a system is translationally invariant, group theory guarantees that the linear equations governing small disturbances have a complete set of solutions of the form $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ and to understand the dynamics of disturbances we only have to determine the dispersion relation. When a system is inhomogeneous, we don't know the structure of the eigenfunctions up front and have to work hard to find them.

Next, we study orbits in the mean-field potential.

2.1 Angle-action variables

If you numerically integrate orbits in potentials similar to those of star clusters and galaxies and Fourier decompose the resulting time series $x(t)$, $y(t)$, etc, you generally find that these series are **quasiperiodic**.[1] That is, their Fourier decompositions are of the form

$$x(t) = \sum_{\mathbf{n}} X_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{\Omega}t}, \tag{2.5}$$

where the $2d$ or $3d$ vectors $\mathbf{n}$ have integer components and the vector $\mathbf{\Omega}$ is made up of 2 or 3 frequencies that are characteristic of the orbit.[2] From the quasiperiodic nature of $x(t)$ it can be shown (see V.I. Arnold *Mathematical Methods of Classical Mechanics* Springer) that the orbit admits at least as many independent **integrals of motion** $I(\mathbf{x}, \mathbf{v})$ as it has degrees of freedom. That is, there are at least 2 or 3 (depending on whether or not the orbit is confined to a plane) independent functions on phase space such that

$$\frac{d}{dt} I[\mathbf{x}(t), \mathbf{v}(t)] = 0. \tag{2.6}$$

In a time-independent potential, H is always an integral of motion, and in an axisymmetric potential the appropriate component of angular moment is always another integral. The non-trivial numerical result is that there is almost always a third integral of motion of unknown functional form.

Given a set of integrals of motion I_i, any function $J_i(I_1, I_2, I_3)$ of three variables provides another integral. Given this choice, it's natural to ask whether a set of integrals can be found that can be complemented by canonically-conjugate variables, θ_i. For if we had a system of canonical coordinates $(\boldsymbol{\theta}, \mathbf{J})$ such that the momenta were constant, half of Hamilton's equations would read

$$0 = \dot{J}_i = -\frac{\partial H}{\partial \theta_i}. \tag{2.7}$$

That is, these equations of motion establish that the Hamiltonian, and its derivatives, are functions of the J_i only and are therefore constant on each orbit. The other equations of motion are now trivially solved:

$$\dot{\theta}_i = \frac{\partial H}{\partial J_i} = \Omega_i(\mathbf{J}) \quad \text{a constant} \quad \Rightarrow \quad \theta_i(t) = \theta_i(0) + \Omega_i t. \tag{2.8}$$

So in the $(\boldsymbol{\theta}, \mathbf{J})$ coordinate system dynamics becomes trivial. The magic integrals J_i are called **actions** and their conjugate variables θ_i are called **angles** because one usually scales the actions

[1] Binney & Spergel, ApJ, 252, 308 (1982)

[2] Whereas the Fourier decomposition of a periodic function contains only integer multiples of a single fundamental frequency, a quasiperiodic function contains only integer linear combinations of 2 or more fundamental frequencies.

so ordinary phase-space coordinates such as x are 2π periodic in the angles. That is the function on phase space x can be expanded as

$$x(\boldsymbol{\theta},\mathbf{J})=\sum_{\mathbf{n}}X_{\mathbf{n}}(\mathbf{J})\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}. \tag{2.9}$$

The Fourier expansion (2.5) from which we started arises by eliminating $\boldsymbol{\theta}$ between equations (2.8) and (2.9).

Whenever the frequencies Ω_i are incommensurable (that is, no relation of the form $\mathbf{n}\cdot\boldsymbol{\Omega}=0$ exists) the actions constitute a complete set of integrals of motion in the sense that any integral of motion can be obtained as a function of them. Since almost all real numbers are irrational, the frequencies of most orbits are incommensurable and the actions are generically a complete set of integrals.

We have seen that by Jeans' theorem the DF of an equilibrium mean-field model is an integral of motion, so it is a function $f(\mathbf{J})$ of the actions. In a plasma we assume $f(\mathbf{v})$ because in a homogeneous system, $\mathbf{v}=$ constant. Many formulae derived for a plasma will go over to a stellar system with the substitutions $\mathbf{x}\to\boldsymbol{\theta}$, $\mathbf{v}\to\mathbf{J}$.

2.1.1 Adiabatic invariance

Action integrals are **adiabatic invariants**: if H evolves on a timescale that is longer than the dynamical time, an orbit of H evolves in such a way that $\mathbf{J}=$ constant. Consequently, when H evolves slowly, the number of particles with $\mathbf{J}$ in each element $\mathrm{d}^3\mathbf{J}$ of action space is unchanged, so the function $f(\mathbf{J})$ is constant.

2.1.2 Hamilton-Jacobi equation

Let $S(\mathbf{x},\mathbf{J})$ be the generating function of the canonical transformation $(\mathbf{x},\mathbf{p})\leftrightarrow(\boldsymbol{\theta},\mathbf{J})$. Then $\mathbf{p}=\partial S/\partial\mathbf{x}$ and we use this relation to eliminate $\mathbf{p}$ from the statement that the Hamiltonian is constant along the orbit:

$$H\left(\mathbf{x},\frac{\partial S}{\partial\mathbf{x}}\right)=E. \tag{2.10}$$

This **Hamilton-Jacobi equation**, which holds at all points that can be reached by the orbit, is a p.d.e. for S. In practice it can only be solved when the substitution $S(\mathbf{x},\mathbf{J})=\sum_i S_i(x_i,\mathbf{J})$ leads to a clean separation of variables. For example, for planar motion in an axisymmetric $\Phi(r)$ we have

$$H(r,\phi,p_r,p_\phi)=\tfrac12 p_r^2+\frac{p_\phi^2}{2r^2}+\Phi(r), \tag{2.11}$$

so $2r^2$ times the H-J eqn yields

$$r^2\left(\frac{\partial S_r}{\partial r}\right)^2+2r^2(\Phi-E)=-\left(\frac{\partial S_\phi}{\partial\phi}\right)^2=-L^2, \tag{2.12}$$

where $-L^2$ is a constant of separation. $S_\phi=L\phi$ follows trivially, and almost as easily we get

$$S_r=\int^r \mathrm{d}r\,\sqrt{2(E-\Phi)-\frac{L^2}{r^2}}. \tag{2.13}$$

These operations yield a function $S(\mathbf{x},E,L)$, which is not of the required form: the integrals of motion E,L are (unknown) functions of the required action integrals; the pair (E,L) cannot be complemented by variables to form a set of canonical coordinates. The actions J_i are defined by

$$J_i=\frac{1}{2\pi}\oint_{\gamma_i}\mathrm{d}\mathbf{x}\cdot\mathbf{p}, \tag{2.14}$$

where each path γ_i around the part of phase space accessible to the orbit cannot be deformed into another of the γ_i without leaving the accessible region.[3]

[3] The integrals (2.14) are unchanged by sliding γ_i over the accessible region because the latter has vanishing Poincaré invariant $\sum_i \mathrm{d}x_i\mathrm{d}p_i$.

Once we've separated the H-J eqn, we can evaluate the integral (2.14) that defines an action associated with each spatial coordinate because a separated equation such as (2.12) makes p_i a function of only its coordinate x_i. In the case of 2d motion, we hold r constant along one path, and ϕ constant along the other path. Then the first path trivially yields $J_\phi = L$ and the second path yields

$$J_r(E,L) = \frac{1}{2\pi}\oint \mathrm{d}r\, p_r = \frac{1}{\pi}\int_{r_{\rm min}}^{r_{\rm max}} \mathrm{d}r\,\sqrt{2(E-\Phi)-\frac{L^2}{r^2}}. \tag{2.15}$$

To find the angle variables we have to use the chain rule

$$\theta_i = \frac{\partial S_r}{\partial J_i} + \frac{\partial S_\phi}{\partial J_i} = \frac{\partial S_r}{\partial E}\frac{\partial E}{\partial J_i} + \frac{\partial S_r}{\partial L}\frac{\partial L}{\partial J_i} + \frac{\partial S_\phi}{\partial L}\frac{\partial L}{\partial J_i}. \tag{2.16}$$

2.1.3 Choice of actions

The **action integrals** J_i are defined up to a set of discrete canonical transformations (generating function $S(\boldsymbol{\theta},\mathbf{J}') = \boldsymbol{\theta}\cdot\mathbf{M}\cdot\mathbf{J}'$ where the matrix $\mathbf{M}$ has integer elements). For an axisymmetric system the actions are uniquely defined by requiring that

$$\begin{aligned} &J_r \text{ quantifies radial excursions}\\ &J_\phi = L_z \text{ is angular momentum about the symmetry axis}\\ &J_z \text{ quantifies oscilations perpendicular to the equatorial plane.}\end{aligned} \tag{2.17}$$

In the spherical limit $J_z = L - |L_z|$ is the angular momentum in the (x,y) plane.

2.2 Self-consistent, mean-field model

A stellar system's DF $f(\mathbf{x},\mathbf{v})$ specifies the mass $\mathrm{d}m = f\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{v}$ in each infinitesimal volume of phase space. If the system is in a statistically steady state, Jeans' theorem tells us that f can depend on $(\mathbf{x},\mathbf{v})$ only through $\mathbf{J}(\mathbf{x},\mathbf{v})$, so it can be expressed as a function $f(\mathbf{J})$. In fact any non-negative function of three variables $0 \le J_r < \infty$, $-\infty < J_\phi < \infty$ and $0 \le J_z < \infty$ specifies an axisymmetric stellar system, and a really powerful way of generating models that can be fitted to observational data is simply to write down a likely function.[4]

Given some function $f(\mathbf{J})$ how do we discover what the system looks like in real space?

1) Make a guess $\rho_0(\mathbf{x})$ at its density distribution. The guess doesn't have to be a good one, but you should ensure that its mass satisfies

$$M \equiv (2\pi)^3\int \mathrm{d}^3\mathbf{J}\,f(\mathbf{J}) = \int \mathrm{d}^3\mathbf{x}\,\rho_0. \tag{2.18}$$

2) Solve Poisson's equation for the potential $\Phi_0(\mathbf{x})$ generated by ρ_0.
3) Obtain the angle-action coordinates for $\rho_0(\mathbf{x})$ and use them to determine a new density distribution

$$\rho_1(\mathbf{x}) = \int \mathrm{d}^3\mathbf{v}\,f[J(\mathbf{x},\mathbf{v})]. \tag{2.19}$$

4) Return to step (2) with ρ_0 replaced by ρ_1 and iterate until $\rho_n(\mathbf{x})$ differs negligibly from ρ_{n-1}. This typically requires ~ 5 iterations.[5]

The only tricky part of this procedure is obtaining the angle-action coordinates of Φ_n. In practice approximations to the true $(\boldsymbol{\theta},\mathbf{J})$ coordinates are used.[6]

[4] In the 20th c. composers appeared who argued that any series of notes constitutes music. We disagree: writing music involves observing rules regarding scales, chords, etc. Similarly, creating plausible stellar systems requires adherence to rules regarding how $f(\mathbf{J})$ behaves in certain parts of action space. But these rules are a matter of good taste.

[5] Binney, MNRAS, **440**, 787 (2014)

[6] Sanders & Binney, MNRAS, **457**, 2107 (2016)

2.3 Biorthogonal potential-density pairs

Unfortunately, while Φ is a function of only $\mathbf{x}$, it becomes a function of both $\boldsymbol{\theta}$ and $\mathbf{J}$. So while angle-action variables make dynamics trivial (advance $\boldsymbol{\theta}$ linearly in t), they seriously complicate the solution of Poisson's eqn.

We finesse this difficulty by introducing a basis of **biorthogonal potential-density** pairs. That is, a set of pairs $(\rho^{(\alpha)}, \Phi^{(\alpha)})$ such that

$$4\pi G\rho^{(\alpha)} = \nabla^2\Phi^{(\alpha)} \quad \text{and} \quad \int \mathrm{d}^3\mathbf{x}\,\Phi^{(\alpha)*}\rho^{(\alpha')} = -\mathcal{E}\delta_{\alpha\alpha'}, \tag{2.20}$$

where $\mathcal{E}$ is an arbitrary constant with the dimensions of energy. Given a density distribution $\rho(\mathbf{x})$, we expand it in the basis

$$\rho(\mathbf{x}) = \sum_\alpha A_\alpha\rho^{(\alpha)}(\mathbf{x}) \quad \Rightarrow \quad \begin{cases} \Phi(\mathbf{x}) = \sum_\alpha A_\alpha\Phi^{(\alpha)}(\mathbf{x}), \\ A_\alpha = -\dfrac{1}{\mathcal{E}}\displaystyle\int \mathrm{d}^3\mathbf{x}\,\Phi^{(\alpha)*}(\mathbf{x})\rho(\mathbf{x}). \end{cases} \tag{2.21}$$

If ρ and Φ are time-dependent, the A_α become time-dependent.

In practice potential-density pairs are complex because they are based on the spherical harmonics Y_l^m – see §2.8 of BT08 for more information. However, we could regard the real and imaginary parts of $Y_l^m(\theta,\phi) = p_l^m(\cos\theta)(\cos\phi + \mathrm{i}\sin\phi)$ as (real) basis functions in their own right. Below, we will find it useful to assume that we are in fact working with real basis functions.

3

Perturbing the DF

The full DF $f(\mathbf{x},\mathbf{v})$ satisfies

$$0 = \frac{\partial f}{\partial t} + [f, H]. \tag{3.1}$$

Breaking f and H into their mean-field and fluctuating parts, we obtain

$$0 = \frac{\partial f_0}{\partial t} + \frac{\partial f_1}{\partial t} + [f_0, H_0] + [f_0, H_1] + [f_1, H_0] + [f_1, H_1]. \tag{3.2}$$

By Jeans' theorem, $[f_0, H_0] = 0$. When we ensemble-average the equation, the parts linear in f_1 or H_1 vanish, so we are left with

$$0 = \frac{\partial f_0}{\partial t} + \langle [f_1, H_1] \rangle. \tag{3.3a}$$

The second term in this equation is clearly $\mathrm{O}(f_1^2)$ or smaller, so the time derivative of f_0 is small, as expected.

Since we are not formally expanding in some small parameter (e.g., $1/N$) we haven't yet defined f_1 exactly: our only requirement is that its ensemble average vanishes, so $\langle f\rangle = f_0$. Hence we are free to *define* f_1 such that the part of eqn (3.2) that is $\mathrm{O}(f_1)$ is identically zero, which is a stronger statement that f_1 has vanishing ensemble average. That is, we now require

$$0 = \frac{\partial f_1}{\partial t} + [f_0, H_1] + [f_1, H_0], \tag{3.3b}$$

where

$$H_1 = \Phi_1(\mathbf{x}) = -G\int \mathrm{d}^3\mathbf{x}'\,\mathrm{d}^3\mathbf{v}\,\frac{f_1(\mathbf{x}',\mathbf{v})}{|\mathbf{x}-\mathbf{x}'|}$$

because only the potential term fluctuates and it is related to the perturbation to the density, $\rho_1(\mathbf{x}) = \int \mathrm{d}^3\mathbf{v}\, f_1(\mathbf{x},\mathbf{v})$, by the Poisson integral.

Since the $(\boldsymbol{\theta},\mathbf{J})$ system, like the $(\mathbf{x},\mathbf{v})$ one, is canonical and Poisson brackets are invariant under changes of canonical coordinates, we can substitute $\mathbf{x}\to\boldsymbol{\theta}$, $\mathbf{v}\to\mathbf{J}$ in all these formulae if we wish. Then we have

$$f(\boldsymbol{\theta},\mathbf{J},t) = f_0(\mathbf{J}) + f_1(\boldsymbol{\theta},\mathbf{J},t) \tag{3.4}$$

and

$$H = H_0(\mathbf{J}) + \Phi_1(\boldsymbol{\theta},\mathbf{J},t) \tag{3.5}$$

and eqn (3.4) becomes

$$0 = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial \boldsymbol{\theta}}\cdot\frac{\partial H_0}{\partial \mathbf{J}} - \frac{\partial f_0}{\partial \mathbf{J}}\cdot\frac{\partial \Phi_1}{\partial \boldsymbol{\theta}} + \mathrm{O}(f_1^2). \tag{3.6}$$

From (2.8) we identify $\partial H_0/\partial\mathbf{J} = \boldsymbol{\Omega}(\mathbf{J})$ as the frequency vector of the unperturbed orbit $\mathbf{J}$ Moreover, incrementing any angle coordinate by 2π brings us back to the same point in phase space (eq. 2.9), so all functions of $\boldsymbol{\theta}$ can be expressed as Fourier series:

$$\begin{aligned} f_1(\boldsymbol{\theta},\mathbf{J},t) &= \sum_{\mathbf{n}} \hat{f}_1(\mathbf{n},\mathbf{J},t)\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}} & & & \hat{f}_1(\mathbf{n},\mathbf{J},t) &= \int \frac{\mathrm{d}^3\boldsymbol{\theta}}{(2\pi)^3} f_1(\boldsymbol{\theta},\mathbf{J},t)\mathrm{e}^{-\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}} \\ & & \leftrightarrow & & & \\ \Phi_1(\boldsymbol{\theta},\mathbf{J},t) &= \sum_{\mathbf{n}} \hat{\Phi}_1(\mathbf{n},\mathbf{J},t)\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}} & & & \hat{\Phi}_1(\mathbf{n},\mathbf{J},t) &= \int \frac{\mathrm{d}^3\boldsymbol{\theta}}{(2\pi)^3} \Phi_1(\boldsymbol{\theta},\mathbf{J},t)\mathrm{e}^{-\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}, \end{aligned} \tag{3.7}$$

Using these results, we can rewrite the linearised Vlasov equation (3.6) as

$$0 = \sum_{\mathbf{n}} \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}\left(\frac{\partial \hat{f}_1}{\partial t} + \mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}\hat{f}_1 - \mathrm{i}\mathbf{n}\cdot\frac{\partial \hat{f}_0}{\partial \mathbf{J}}\hat{\Phi}_1\right). \tag{3.8}$$

Since $\boldsymbol{\theta}$ is arbitrary, for this equation to hold, every coefficient of $\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}$ must separately vanish, so we obtain an infinite set of equations

$$\frac{\partial \hat{f}_1}{\partial t} = \mathrm{i}\mathbf{n}\cdot\frac{\partial \hat{f}_0}{\partial \mathbf{J}}\hat{\Phi}_1 - \mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}\hat{f}_1 \quad \text{for } \mathbf{n} \text{ with integer components.} \tag{3.9}$$

We use Laplace transforms to solve (3.9): multiplying by e^{-pt} (with $\Re(p) > 0$) and integrating over t, we get[1]

$$p\widetilde{f}_1(\mathbf{n},\mathbf{J},p) - \hat{f}_1(\mathbf{n},\mathbf{J},0) + \mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}\widetilde{f}_1(\mathbf{n},\mathbf{J},p) - \mathrm{i}\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}\widetilde{\Phi}_1(\mathbf{n},\mathbf{J},p) = 0, \tag{3.10}$$

where the tildes denote Laplace transforms:

$$\widetilde{f}_1(\mathbf{n},\mathbf{J},p) \equiv \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-pt}\hat{f}_1(\mathbf{n},\mathbf{J},t). \tag{3.11}$$

Solving for $\widetilde{f}_1$ we have (cf. Schekochihin eqn. 3.8)

$$\widetilde{f}_1(\mathbf{n},\mathbf{J},p) = \frac{\mathrm{i}\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}\widetilde{\Phi}_1(\mathbf{n},\mathbf{J},p) + \hat{f}_1(\mathbf{n},\mathbf{J},0)}{p + \mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}}. \tag{3.12}$$

This equation provides one connection between a perturbation to the potential Φ_1 and the response f_1 it induces dynamically.

We now need to put into maths the principle that Φ_1 is the potential generated by the perturbation to the density that's associated with f_1. To obtain the coefficients A_α of the

[1] While the dimensions of a quantity are unchanged by a hat, a tilde raises the dimensions by a factor T.

potential-density expansion (2.21) of the perturbed density (or at any rate their temporal Laplace transforms), we multiply the left side of (3.12) by $\Phi^{(\alpha)*}\sum_{\mathbf{n}} \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}$ and integrate over phase space:

$$\begin{aligned}\int \mathrm{d}^3\boldsymbol{\theta}\mathrm{d}^3\mathbf{J}\,\Phi^{(\alpha)*}(\mathbf{x})\sum_{\mathbf{n}} \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}\widetilde{f}_1(\mathbf{n},\mathbf{J},p) &= \int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{v}\,\Phi^{(\alpha)*}(\mathbf{x})\widetilde{f}_1(\mathbf{x},\mathbf{v},p)\\ &= \int \mathrm{d}^3\mathbf{x}\,\Phi^{(\alpha)*}(\mathbf{x})\widetilde{\rho}_1(\mathbf{x},p) = -\mathcal{E}\widetilde{A}_\alpha(p).\end{aligned} \tag{3.13}$$

Here we have exploited the fact the Jacobian between any two sets of canonical coordinates is unity, so $\mathrm{d}^3\boldsymbol{\theta}\mathrm{d}^3\mathbf{J} = \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{v}$. Now operating in the same way on the rhs of eqn (3.12) we have

$$\begin{aligned}&\int \mathrm{d}^3\boldsymbol{\theta}\mathrm{d}^3\mathbf{J}\sum_{\mathbf{n}} \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}\Phi^{(\alpha)*}(\mathbf{x})\frac{\mathrm{i}\mathbf{n}\cdot\frac{\partial f_0}{\partial\mathbf{J}}\widetilde{\Phi}_1(\mathbf{n},\mathbf{J},p)+\hat{f}_1(\mathbf{n},\mathbf{J},0)}{p+\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}}\\ &\qquad = (2\pi)^3\int \mathrm{d}^3\mathbf{J}\sum_{\mathbf{n}}[\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})]^*\frac{\mathrm{i}\mathbf{n}\cdot\frac{\partial f_0}{\partial\mathbf{J}}\sum_{\alpha'}\widetilde{A}_{\alpha'}(p)\hat{\Phi}^{(\alpha')}(\mathbf{n},\mathbf{J})+\hat{f}_1(\mathbf{n},\mathbf{J},0)}{p+\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}}.\end{aligned} \tag{3.14}$$

Uniting the two sides (3.13) and (3.14) of equation (3.12) we obtain an equation for $\widetilde{A}_\alpha$:

$$\widetilde{A}_\alpha(p) = -\frac{(2\pi)^3}{\mathcal{E}}\int \mathrm{d}^3\mathbf{J}\sum_{\mathbf{n}}\frac{\mathrm{i}\mathbf{n}\cdot\frac{\partial f_0}{\partial\mathbf{J}}\sum_{\alpha'}\widetilde{A}_{\alpha'}(p)[\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})]^*\hat{\Phi}^{(\alpha')}(\mathbf{n},\mathbf{J})+\hat{f}_1(\mathbf{n},\mathbf{J},0)[\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})]^*}{p+\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}}. \tag{3.15}$$

We move the term on the right containing $A_{\alpha'}$ to the left side so we can write

$$\sum_{\alpha'}\epsilon_{\alpha\alpha'}(p)\widetilde{A}_{\alpha'}(p) = -\frac{(2\pi)^3}{\mathcal{E}}\int \mathrm{d}^3\mathbf{J}\sum_{\mathbf{n}}\frac{\hat{f}_1(\mathbf{n},\mathbf{J},0)[\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})]^*}{p+\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}}, \tag{3.16a}$$

where

$$\epsilon_{\alpha\alpha'}(p) \equiv \delta_{\alpha\alpha'} + \frac{(2\pi)^3}{\mathcal{E}}\mathrm{i}\int \mathrm{d}^3\mathbf{J}\sum_{\mathbf{n}}\frac{\mathbf{n}\cdot\frac{\partial f_0}{\partial\mathbf{J}}}{p+\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}}[\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})]^*\hat{\Phi}^{(\alpha')}(\mathbf{n},\mathbf{J}) \tag{3.16b}$$

is the analogue of the dielectric function (cf. Schekochihin eqn. 3.11). In both integrals over $\mathbf{J}$ in equations (3.16) we must use the Landau prescription. That is, we must ensure that $\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}$ passes to the left of p in the complex plane (Box 3.2).

After computing the inverse of the dimensionless matrix $\boldsymbol{\epsilon}$, we have an explicit expression for $\widetilde{A}_\alpha(p)$. Multiplying this by $\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})$ and summing over α we obtain the Laplace transform of the potential perturbation arising from the initial condition $f_1(\mathbf{n},\mathbf{J},0)$:

$$\begin{aligned}\widetilde{\Phi}_1(\mathbf{n}',\mathbf{J}',p) &= \sum_{\alpha'}\widetilde{A}_{\alpha'}(p)\hat{\Phi}^{(\alpha')}(\mathbf{n}',\mathbf{J}')\\ &= -\frac{(2\pi)^3}{\mathcal{E}}\int \mathrm{d}^3\mathbf{J}\sum_{\mathbf{n}}\frac{\hat{f}_1(\mathbf{n},\mathbf{J},0)}{p+\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}}\sum_{\alpha\alpha'}\hat{\Phi}^{(\alpha')}(\mathbf{n}',\mathbf{J}')\epsilon^{-1}_{\alpha'\alpha}(p)[\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})]^*\\ &= -(2\pi)^3\int \mathrm{d}^3\mathbf{J}\sum_{\mathbf{n}}E_{\mathbf{n}'\mathbf{n}}(\mathbf{J}',\mathbf{J},p)\frac{\hat{f}_1(\mathbf{n},\mathbf{J},0)}{p+\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}},\end{aligned} \tag{3.17a}$$

where

$$E_{\mathbf{n}'\mathbf{n}}(\mathbf{J}',\mathbf{J},p) \equiv \frac{1}{\mathcal{E}}\sum_{\alpha\alpha'}\hat{\Phi}^{(\alpha')}(\mathbf{n}',\mathbf{J}')\epsilon^{-1}_{\alpha'\alpha}(p)[\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})]^*, \tag{3.17b}$$

has dimensions $M^{-1}L^2T^{-2}$ and is (to within a factor $\mathcal{E}$) $\boldsymbol{\epsilon}^{-1}$ written in the $(\mathbf{n},\mathbf{J})$ basis rather than the $(\boldsymbol{\alpha},\mathbf{J})$ basis. Equation (3.17a) is analogous to Schekochihin eqn. (3.13) in giving the Laplace transform of the response potential set up by a specified initial condition. It's more complicated than Schekochihin eqn. (3.13) because: (a) in the latter Poisson's equation is solved by simply dividing by k^2 while here we do acrobatics with the potential basis functions; (b) we have $\mathbf{E}$ where Schekochihin eqn. 3.13 has $1/\epsilon$ and the case $\epsilon = 0$ becomes the case in which our matrix $\boldsymbol{\epsilon}$ has no inverse, so $\mathbf{E}$, which is basically this inverse, diverges; (c) Schekochihin eqn. (3.13) involves an integral over $\mathbf{v}$ with the denominator of the integrand linear in $\mathbf{v}$, while here we integrate over $\mathbf{J}$ and the denominator involves the non-linear function $\mathbf{n}\cdot\boldsymbol{\Omega}(\mathbf{J})$. The generalisation of the Landau prescription to this more complex context is given in Box 3.2.

Box 3.1: Stability of a collisionless system

If we recover the temporal dependence of Φ_1 by taking the inverse Laplace transform of equation (3.17a), we obtain a sum of terms with exponential time dependence $e^{p_i t}$ (Schekochihin eqn. 3.16), where p_i is the value of the Laplace transform variable at which the matrix $\mathbf{E}$ has a pole in the sense that it is the inverse of a singular matrix $\boldsymbol{\epsilon}$. Consequently, the stability of a system at the level of collisionless dynamics is determined by whether the dielectric matrix $\boldsymbol{\epsilon}$ is singular at any value p_0 of the Laplace transform variable with $\Re p_0 > 0$. We say that each p_i is associated with a **normal mode** of the system. In a stable system the normal modes are all neutral ($\Re p_i = 0$) or damped ($\Re p_i < 0$).

Box 3.2: The Landau prescription with actions

We often encounter, as in eqns (3.16), an integral over action space with a denominator that vanishes if $p = -\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}(\mathbf{J})$. In a plasma, analogous integrals occur with denominator $p + \mathrm{i}\mathbf{k}\cdot\mathbf{v}$ and we evaluate them using the Landau contour. To solve our more complex problem we make a coordinate change from $\mathbf{J} \to (x, y, z)$, where $z \equiv \mathbf{n}\cdot\boldsymbol{\Omega}$ and (x, y) is a coordinate system for the 2-surfaces $z = \text{const}$. Then

$$\int \mathrm{d}^3\mathbf{J}\,\frac{k(\mathbf{J})}{p + \mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}} = \int_{-\infty}^{\infty} \mathrm{d}z\,\frac{K(z)}{p + \mathrm{i}z}, \tag{1}$$

where

$$K(z) \equiv \int \mathrm{d}x\mathrm{d}y\,\frac{\partial(\mathbf{J})}{\partial(x, y, z)}k(\mathbf{J}).$$

The integral on the right of (1) is now in just the form considered by Landau. We write $p = \gamma - \mathrm{i}\omega$ with $\gamma > 0$, and have

$$\int_{-\infty}^{\infty} \mathrm{d}z\,\frac{K(z)}{p + \mathrm{i}z} = -\mathrm{i}\int \mathrm{d}z\,\frac{K(z)}{z - \mathrm{i}p} = -\mathrm{i}\int \mathrm{d}z\,\frac{K(z)}{z - (\omega + \mathrm{i}\gamma)}.$$

The z contour (real axis) passes under the pole, so in the limit $\gamma \to 0$ this becomes

$$\int_{-\infty}^{\infty} \mathrm{d}z\,\frac{K(z)}{p + \mathrm{i}z} = -\mathrm{i}\left(\mathcal{P}\int \mathrm{d}z\,\frac{K(z)}{z - \omega} + \mathrm{i}\pi K(\omega)\right) = -\mathrm{i}\mathcal{P}\int \mathrm{d}z\,\frac{K(z)}{z - \omega} + \pi K(\omega). \tag{2}$$

Now let's transform $\int \mathrm{d}^3\mathbf{J}\,k(\mathbf{J})\delta(\mathbf{n}\cdot\boldsymbol{\Omega} - \omega)$ into the (x, y, z) system:

$$\int \mathrm{d}^3\mathbf{J}\,k(\mathbf{J})\delta(\mathbf{n}\cdot\boldsymbol{\Omega} - \omega) = \int \mathrm{d}z\,K(z)\delta(z - \omega) = K(\omega).$$

When we use this equation in (2), we obtain the needed analogue of the Plemelj formula.

$$\int \mathrm{d}^3\mathbf{J}\,\frac{k(\mathbf{J})}{p + \mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}} = -\mathrm{i}\mathcal{P}\int \mathrm{d}z\,\frac{K(z)}{z - \omega} + \pi\int \mathrm{d}^3\mathbf{J}\,k(\mathbf{J})\delta(\mathbf{n}\cdot\boldsymbol{\Omega} - \omega) \qquad (p = -\mathrm{i}\omega + 0). \tag{3}$$

4

Evolution of the mean-field model

We have been studying the properties of mean-field equilibrium systems. Such systems are fully characterised by a non-negative DF of the form $f(\mathbf{J})$. We have shown how to compute the evolution of the DF when at $t = 0$ it differs very slightly from $f(\mathbf{J})$. In all the above we have been imagining that the system comprises an extremely large number of particles with extremely low masses, so statistical fluctuations of the density around its mean value, $\rho(\mathbf{x}) = \int \mathrm{d}^3\mathbf{v}\, f(\mathbf{x},\mathbf{v})$, vanish. In this section we explore how to compute the evolution of f that occurs because its constituent particles have non-zero masses, so ρ and Φ fluctuate around their mean values.

Recall from Paul Dellar's discussion of the BBGKY hierarchy that the 1-particle DF $f(\mathbf{x},\mathbf{v})$ satisfies a Boltzmann equation in which the 2-particle correlation function $g^{(2)}(\mathbf{x},\mathbf{v},\mathbf{x}',\mathbf{v}')$ appears (Problem 7):

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{\mathbf{w}} = (N-1)\int \mathrm{d}^3\mathbf{x}'\mathrm{d}^3\mathbf{v}'\,\frac{\partial u(\mathbf{x}-\mathbf{x}')}{\partial \mathbf{x}'}\cdot\frac{\partial g^{(2)}(\mathbf{w},\mathbf{w}')}{\partial \mathbf{v}}, \tag{4.1}$$

where $\mathbf{w} \equiv (\mathbf{x},\mathbf{v})$ denotes position in phase space and $u(\mathbf{x}-\mathbf{x}')$ is the interaction potential between two particles. The physical content of this equation is that evolution of the mean-field model, $f(\mathbf{x},\mathbf{v})$, is driven by the tendency, encoded in $g^{(2)}$ for particles to cluster together, so you are more likely to find a second particle near you if you stand on a particle than if you stand in a random location. Heyvaerts[1] obtains from equation (4.1) the equation for the evolution of f, which is what we seek in this section, but we'll proceed along a different path, similar to that laid out by Chavanis.[2]

4.1 Dynamics of fluctuations

We argue that the small-scale structure is unimportant so we should be able to compute everything in terms of a smooth potential $\Phi(\mathbf{x},t)$ providing we properly account for the fluctuations in Φ.

Equation (3.3a) shows that evolution of f_0 is driven by the 'collision integral' $-\langle[f_1,H_1]\rangle$, and the evolution of f_1 is given by equation (3.3b). Our the strategy is to use the solutions to (3.3b) that we obtained in Chapter 3 to compute $\langle[f_1,H_1]\rangle$. We replace f_1 and Φ_1 in $\langle[f_1,H_1]\rangle$ by their Fourier expansions in $\boldsymbol{\theta}$ (eq. 3.7). We also take advantage of the expectation operator $\langle.\rangle$ to integrate over all angles. Then we have

$$\begin{aligned}\langle[f_1,\Phi_1]\rangle = \Bigg\langle \int \frac{\mathrm{d}^3\boldsymbol{\theta}}{(2\pi)^3}\Bigg(&\sum_{\mathbf{n}} \hat{f}_1(\mathbf{n},\mathbf{J},t)\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}\mathrm{i}\mathbf{n}\cdot\sum_{\mathbf{n}'}\frac{\partial\hat{\Phi}_1(\mathbf{n}',\mathbf{J},t)}{\partial\mathbf{J}}\mathrm{e}^{\mathrm{i}\mathbf{n}'\cdot\boldsymbol{\theta}}\\ &-\sum_{\mathbf{n}}\frac{\partial\hat{f}_1(\mathbf{n},\mathbf{J},t)}{\partial\mathbf{J}}\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}\cdot\sum_{\mathbf{n}'}\mathrm{i}\mathbf{n}'\hat{\Phi}_1(\mathbf{n}',\mathbf{J},t)\mathrm{e}^{\mathrm{i}\mathbf{n}'\cdot\boldsymbol{\theta}}\Bigg)\Bigg\rangle \\ = \mathrm{i}\frac{\partial}{\partial\mathbf{J}}\cdot\Bigg\langle\sum_{\mathbf{n}}&\mathbf{n}\hat{f}_1(\mathbf{n},\mathbf{J},t)\hat{\Phi}_1(-\mathbf{n},\mathbf{J},t)\Bigg\rangle.\end{aligned} \tag{4.2}$$

[1] J. Heyvaerts, MNRAS, 407, 355 (2010)
[2] P.-H. Chavanis, Physica A, 391, 3680 (2012).

Hence the equation for the evolution of the mean-field model is

$$\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial \mathbf{J}} \cdot \mathbf{F}, \tag{4.3a}$$

where the flux of stars in action space[3] is

$$\mathbf{F} = \mathrm{i}\left\langle \sum_{\mathbf{n}} \mathbf{n} \hat{f}_1(\mathbf{n},\mathbf{J},t)\hat{\Phi}_1(-\mathbf{n},\mathbf{J},t) \right\rangle . \tag{4.3b}$$

The divergence on the right of (4.3a) guarantees conservation of stars.

Rewriting (4.3b) in terms of Laplace transforms, it becomes

$$\mathbf{F}(\mathbf{J}) = \mathrm{i}\left\langle \sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi \mathrm{i}} \mathrm{e}^{pt} \widetilde{f}_1(\mathbf{n},\mathbf{J},p) \int \frac{\mathrm{d}p'}{2\pi \mathrm{i}} \mathrm{e}^{p't} \widetilde{\Phi}_1(-\mathbf{n},\mathbf{J},p') \right\rangle . \tag{4.4}$$

Now we use equation (3.12) to eliminate $\widetilde{f}_1$

$$\mathbf{F}(\mathbf{J}) = \mathrm{i}\left\langle \sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi \mathrm{i}} \mathrm{e}^{pt} \left(\frac{\mathrm{i}\mathbf{n}\cdot \frac{\partial f_0}{\partial \mathbf{J}} \widetilde{\Phi}_1(\mathbf{n},\mathbf{J},p) + \hat{f}_1(\mathbf{n},\mathbf{J},0)}{p + \mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}} \right) \int \frac{\mathrm{d}p'}{2\pi \mathrm{i}} \mathrm{e}^{p't} \widetilde{\Phi}_1(-\mathbf{n},\mathbf{J},p') \right\rangle . \tag{4.5}$$

This expression for the diffusive flux is made up of a part that's proportional to $\left\langle \widetilde{\Phi}_1(\mathbf{n})\widetilde{\Phi}_1(-\mathbf{n}) \right\rangle$ that will be non-vanishing regardless of the physical cause of fluctuations in the potential, and a part $\left\langle \hat{f}_1(\mathbf{n})\widetilde{\Phi}_1(-\mathbf{n}) \right\rangle$ that will be non-vanishing only to the extent that the fluctuations in Φ are generated by the fluctuations in f. Moreover, the first term is proportional to the gradient of $f_0(\mathbf{J})$ while the second is not. These distinctions will prove important (e.g., Problem 10), so we explicitly break $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ into two parts,

$$\begin{aligned} \mathbf{F}_1 &= \mathrm{i}\sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi \mathrm{i}} \mathrm{e}^{pt} \int \frac{\mathrm{d}p'}{2\pi \mathrm{i}} \mathrm{e}^{p't} \frac{\left\langle \hat{f}_1(\mathbf{n},\mathbf{J},0)\widetilde{\Phi}_1(-\mathbf{n},\mathbf{J},p') \right\rangle}{p + \mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}} \\ \mathbf{F}_2 &= -\sum_{\mathbf{n}} \mathbf{n}\,\mathbf{n}\cdot \frac{\partial f_0}{\partial \mathbf{J}} \int \frac{\mathrm{d}p}{2\pi \mathrm{i}} \mathrm{e}^{pt} \int \frac{\mathrm{d}p'}{2\pi \mathrm{i}} \mathrm{e}^{p't} \frac{\left\langle \widetilde{\Phi}_1(\mathbf{n},\mathbf{J},p)\widetilde{\Phi}_1(-\mathbf{n},\mathbf{J},p') \right\rangle}{p + \mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}} . \end{aligned} \tag{4.6}$$

Using (3.17a) to eliminate $\widetilde{\Phi}_1$, these fluxes become

$$\begin{aligned} \mathbf{F}_1(\mathbf{J}) &\equiv -(2\pi)^3 \mathrm{i}\left\langle \sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi \mathrm{i}} \mathrm{e}^{pt} \frac{\hat{f}_1(\mathbf{n},\mathbf{J},0)}{p + \mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}} \int \frac{\mathrm{d}p'}{2\pi \mathrm{i}} \mathrm{e}^{p't} \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{n}'} E_{-\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',p') \frac{\hat{f}_1(\mathbf{n}',\mathbf{J}',0)}{p' + \mathrm{i}\mathbf{n}'\cdot\mathbf{\Omega}'} \right\rangle \\ \mathbf{F}_2(\mathbf{J}) &\equiv (2\pi)^6 \mathrm{i}\left\langle \sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi \mathrm{i}} \mathrm{e}^{pt} \frac{\mathrm{i}\mathbf{n}\cdot \frac{\partial f_0}{\partial \mathbf{J}}}{p + \mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}} \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{n}'} E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',p) \frac{\hat{f}_1(\mathbf{n}',\mathbf{J}',0)}{p + \mathrm{i}\mathbf{n}'\cdot\mathbf{\Omega}'} \right. \\ &\qquad \left. \times \int \frac{\mathrm{d}p'}{2\pi \mathrm{i}} \mathrm{e}^{p't} \int \mathrm{d}^3\mathbf{J}'' \sum_{\mathbf{n}''} E_{-\mathbf{n}\mathbf{n}''}(\mathbf{J},\mathbf{J}'',p') \frac{\hat{f}_1(\mathbf{n}'',\mathbf{J}'',0)}{p' + \mathrm{i}\mathbf{n}''\cdot\mathbf{\Omega}''} \right\rangle . \end{aligned} \tag{4.7}$$

The expectation-value brackets $\langle . \rangle$ imply that we require the expectation $\hat{f}_1(\mathbf{n},\mathbf{J},0)\hat{f}_1(\mathbf{n}',\mathbf{J}',0)$ of the initial conditions. In Box 4.1 we show that

$$\left\langle \hat{f}_1(\mathbf{n},\mathbf{J},0)\hat{f}_1(\mathbf{n}',\mathbf{J}',0) \right\rangle = \frac{1}{(2\pi)^3} \delta_{\mathbf{n},-\mathbf{n}'} \delta(\mathbf{J}-\mathbf{J}') m f_0(\mathbf{J}). \tag{4.8}$$

[3] Strictly, the density of stars in action space is $(2\pi)^3 f_0(\mathbf{J})$ and the action-space flux is $(2\pi)^3 \mathbf{F}(\mathbf{J})$ rather than $\mathbf{F}(\mathbf{J})$, but in heuristic discussions it's convenient to ignore the factor $(2\pi)^3$.

Box 4.1: Expectation value of the initial conditions

We require $\left\langle \hat{f}_1(\mathbf{n},\mathbf{J},0)\hat{f}_1(\mathbf{n}',\mathbf{J}',0)\right\rangle$. We drop the time slot for brevity and recall that f_1 is the difference between the actual DF and the mean-field model, which has DF $f_0(\mathbf{J})$. The actual DF is a sum of one delta-function for each particle:

$$f(\boldsymbol{\theta},\mathbf{J}) = m\sum_i \delta(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\delta(\mathbf{J}-\mathbf{J}_i).$$

Thus bearing in mind that $\langle f(\boldsymbol{\theta},\mathbf{J})\rangle = f_0(\mathbf{J})$,

$$\begin{aligned}\langle f_1(\boldsymbol{\theta},\mathbf{J})f_1(\boldsymbol{\theta}',\mathbf{J}')\rangle &= \big\langle\big(f(\boldsymbol{\theta},\mathbf{J})-f_0(\mathbf{J})\big)\big(f(\boldsymbol{\theta}',\mathbf{J}')-f_0(\mathbf{J}')\big)\big\rangle\\ &= \langle f(\boldsymbol{\theta},\mathbf{J})f(\boldsymbol{\theta}',\mathbf{J}')\rangle - f_0(\mathbf{J})f_0(\mathbf{J}')\\ &= m^2\sum_{ij}\langle\delta(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\delta(\mathbf{J}-\mathbf{J}_i)\delta(\boldsymbol{\theta}'-\boldsymbol{\theta}_j)\delta(\mathbf{J}'-\mathbf{J}_j)\rangle - f_0(\mathbf{J})f_0(\mathbf{J}').\end{aligned}$$

Now

$$\begin{aligned}\sum_{ij}\langle\delta(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\delta(\mathbf{J}-\mathbf{J}_i)\delta(\boldsymbol{\theta}'-\boldsymbol{\theta}_j)\delta(\mathbf{J}'-\mathbf{J}_j)\rangle &= \sum_{i\neq j}\langle\delta(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\delta(\mathbf{J}-\mathbf{J}_i)\delta(\boldsymbol{\theta}'-\boldsymbol{\theta}_j)\delta(\mathbf{J}'-\mathbf{J}_j)\rangle\\ &\quad+\sum_i\langle\delta(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\delta(\mathbf{J}-\mathbf{J}_i)\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\delta(\mathbf{J}'-\mathbf{J})\rangle\\ &= m^{-2}f_0(\mathbf{J})f_0(\mathbf{J}') + m^{-1}f_0(\mathbf{J})\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')\delta(\mathbf{J}-\mathbf{J}'),\end{aligned}$$

where we have assumed that the particles are uniformly distributed in $\boldsymbol{\theta}$ and uncorrelated (so the expectation value of products of delta-functions associated with different particles is the product of the expectation values of the individual terms). When the last equation is used in the previous equation, we obtain

$$\langle f_1(\boldsymbol{\theta},\mathbf{J})f_1(\boldsymbol{\theta}',\mathbf{J}')\rangle = mf_0(\mathbf{J})\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')\delta(\mathbf{J}-\mathbf{J}'),$$

which simply states that particles are only correlated with themselves. Finally Fourier transforming

$$\begin{aligned}\left\langle\hat{f}_1(\mathbf{n},\mathbf{J})\hat{f}_1(\mathbf{n}',\mathbf{J}')\right\rangle &= mf_0(\mathbf{J})\delta(\mathbf{J}-\mathbf{J}')\int\frac{\mathrm{d}^3\boldsymbol{\theta}}{(2\pi)^3}\int\frac{\mathrm{d}^3\boldsymbol{\theta}'}{(2\pi)^3}\mathrm{e}^{-\mathrm{i}(\mathbf{n}\cdot\boldsymbol{\theta}+\mathbf{n}'\cdot\boldsymbol{\theta}')}\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')\\ &= (2\pi)^{-3}mf_0(\mathbf{J})\delta(\mathbf{J}-\mathbf{J}')\delta_{\mathbf{n},-\mathbf{n}'}.\end{aligned}$$

Inserting this and using the δ-function to carry out the integral over $\mathbf{J}'$ in the equation for $\mathbf{F}_1$ and over $\mathbf{J}''$ in the equation for $\mathbf{F}_2$, we get

$$\begin{aligned}\mathbf{F}_1(\mathbf{J}) &= -\mathrm{i}m\sum_{\mathbf{n}}\mathbf{n}\int\frac{\mathrm{d}p}{2\pi\mathrm{i}}\mathrm{e}^{pt}\frac{1}{p+\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}}\int\frac{\mathrm{d}p'}{2\pi\mathrm{i}}\mathrm{e}^{p't}E_{-\mathbf{n}-\mathbf{n}}(\mathbf{J},\mathbf{J},p')\frac{f_0(\mathbf{J})}{p'-\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}}\\ \mathbf{F}_2(\mathbf{J}) &= -(2\pi)^3m\sum_{\mathbf{n}}\mathbf{n}\int\frac{\mathrm{d}p}{2\pi\mathrm{i}}\mathrm{e}^{pt}\frac{\mathbf{n}\cdot\frac{\partial f_0}{\partial\mathbf{J}}}{p+\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}}\int\mathrm{d}^3\mathbf{J}'\sum_{\mathbf{n}'}E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',p)\frac{1}{p+\mathrm{i}\mathbf{n}'\cdot\boldsymbol{\Omega}'}\\ &\quad\times\int\frac{\mathrm{d}p'}{2\pi\mathrm{i}}\mathrm{e}^{p't}E_{-\mathbf{n}-\mathbf{n}'}(\mathbf{J},\mathbf{J}',p')\frac{f_0(\mathbf{J}')}{p'-\mathrm{i}\mathbf{n}'\cdot\boldsymbol{\Omega}'}.\end{aligned}\qquad(4.9)$$

The expression for $\mathbf{F}_1$ is easy to simplify further because $E_{-\mathbf{n}-\mathbf{n}}$ won't contribute a pole at $\Re(p')\geq 0$: if it had such a pole, the underlying model would be unstable (Box 3.1), and we are interested in the case when it's stable. So the only singularity we need consider is the obvious one when $p' = \mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}$. Similarly, the integration over p follows immediately from the pole at $p = -\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega}$. So we have

$$\mathbf{F}_1(\mathbf{J}) = -\mathrm{i}m\sum_{\mathbf{n}}\mathbf{n}E_{-\mathbf{n}-\mathbf{n}}(\mathbf{J},\mathbf{J},\mathrm{i}\mathbf{n}\cdot\boldsymbol{\Omega})f_0(\mathbf{J}).\qquad(4.10)$$

Notice that the time dependencies introduced by the two inverse Laplace transforms have cancelled, so the flux $\mathbf{F}_1$ is constant.

Now we turn to $\mathbf{F}_2$. The integral over p' is straightforward because the integrand has only the obvious pole at $p' = \mathrm{i}\mathbf{n}' \cdot \mathbf{\Omega}'$. After doing the p' integral we have

$$\begin{aligned}\mathbf{F}_2(\mathbf{J}) = -(2\pi)^3 m \sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi\mathrm{i}} \mathrm{e}^{pt} \frac{\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}}{p + \mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}} \\ \times \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{n}'} \mathrm{e}^{\mathrm{i}\mathbf{n}'\cdot\mathbf{\Omega}'t} E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',p) E_{-\mathbf{n}-\mathbf{n}'}(\mathbf{J},\mathbf{J}',\mathrm{i}\mathbf{n}'\cdot\mathbf{\Omega}') \frac{f_0(\mathbf{J}')}{p + \mathrm{i}\mathbf{n}'\cdot\mathbf{\Omega}'}\end{aligned} \tag{4.11}$$

Now we perform the integral over $\mathbf{J}'$ using the Landau prescription (Box 3.2) to handle the pole at $\mathrm{i}\mathbf{n}'\cdot\mathbf{\Omega}' = -p$:

$$\begin{aligned}\mathbf{F}_2(\mathbf{J}) = -(2\pi)^3 m \sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi\mathrm{i}} \mathrm{e}^{pt} \frac{\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}}{p + \mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}} \Big(-\mathrm{i}P + \pi \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{n}'} \mathrm{e}^{-pt} \delta(\mathrm{i}\mathbf{n}'\cdot\mathbf{\Omega}' - \mathrm{i}p) \\ \times E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',p) E_{-\mathbf{n}-\mathbf{n}'}(\mathbf{J},\mathbf{J}',-p) f_0(\mathbf{J}') \Big),\end{aligned} \tag{4.12}$$

where P is the (real) principal part of the integral. It is now straightforward to execute the integral over p because the integrand has just the simple pole at $p = -\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}$. After integration over p we have

$$\begin{aligned}\mathbf{F}_2(\mathbf{J}) = -(2\pi)^3 m \sum_{\mathbf{n}} \mathbf{n}\mathrm{e}^{-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}t} \mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}} \Big(-\mathrm{i}P + \pi \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{n}'} \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}t} \delta(\mathbf{n}'\cdot\mathbf{\Omega}' - \mathbf{n}\cdot\mathbf{\Omega}) \\ \times E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}) E_{-\mathbf{n}-\mathbf{n}'}(\mathbf{J},\mathbf{J}',\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}) f_0(\mathbf{J}') \Big),\end{aligned} \tag{4.13}$$

We now argue that since $\mathbf{F}_2$ is real, the contribution from the principal part, P, must vanish, and we have finally

$$\begin{aligned}\mathbf{F}_2(\mathbf{J}) = -\tfrac{1}{2}(2\pi)^4 m \sum_{\mathbf{n}} \mathbf{n}\Big(\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}\Big) \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{n}'} \delta(\mathbf{n}'\cdot\mathbf{\Omega}' - \mathbf{n}\cdot\mathbf{\Omega}) \\ \times E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}) E_{-\mathbf{n}-\mathbf{n}'}(\mathbf{J},\mathbf{J}',\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}) f_0(\mathbf{J}').\end{aligned} \tag{4.14}$$

Notice that the time dependence has disappeared from $\mathbf{F}_2$ as it did from $\mathbf{F}_1$.

At this point we assume that we are working with real basis functions $\Phi^{(\alpha)}$ for then by the bottom-right equation of (3.7), $[\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})]^* = \hat{\Phi}^{(\alpha)}(-\mathbf{n},\mathbf{J})$. Also $[\boldsymbol{\epsilon}(p)]^* = \boldsymbol{\epsilon}(p^*)$ (Problem 8). Consequently, from (3.17b)

$$\begin{aligned}[E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega})]^* &= \frac{1}{\mathcal{E}} \sum_{\alpha\alpha'} [\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J})]^* [\boldsymbol{\epsilon}^{-1}_{\alpha\alpha'}(-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega})]^* \hat{\Phi}^{(\alpha')}(\mathbf{n}',\mathbf{J}') \\ &= \frac{1}{\mathcal{E}} \sum_{\alpha\alpha'} \hat{\Phi}^{(\alpha)}(-\mathbf{n},\mathbf{J}) \boldsymbol{\epsilon}^{-1}_{\alpha\alpha'}(\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}) [\hat{\Phi}^{(\alpha')}(-\mathbf{n}',\mathbf{J}')]^* \\ &= E_{-\mathbf{n}-\mathbf{n}'}(\mathbf{J},\mathbf{J}',\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega}).\end{aligned} \tag{4.15}$$

Consequently, our expression (4.14) can be simplified to

$$\mathbf{F}_2(\mathbf{J}) = -\tfrac{1}{2}(2\pi)^4 m \sum_{\mathbf{n}\mathbf{n}'} \mathbf{n}\big(\mathbf{n}\cdot\frac{\partial f_0}{\partial \mathbf{J}}\big) \int \mathrm{d}^3\mathbf{J}' \big|E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega})\big|^2 f_0(\mathbf{J}') \delta(\mathbf{n}'\cdot\mathbf{\Omega}' - \mathbf{n}\cdot\mathbf{\Omega}). \tag{4.16}$$

This completes our computation of the diffusive flux in action space that's engendered by Poisson fluctuations in the density:

$$\begin{aligned}\mathbf{F}(\mathbf{J}) &= \mathbf{F}_1(\mathbf{J}) + \mathbf{F}_2(\mathbf{J}) \\ &= -\mathbf{D}_1(\mathbf{J}) f_0 - \mathbf{D}_2(\mathbf{J}) \cdot \frac{\partial f_0}{\partial \mathbf{J}},\end{aligned} \tag{4.17}$$

where $\mathbf{D}_1$ is the (vector) **drag coefficient** and $\mathbf{D}_2$ is the (tensor) **diffusion coefficient**:

$$\begin{aligned}\mathbf{D}_1(\mathbf{J}) &= \mathrm{i}m\sum_{\mathbf{n}} E_{-\mathbf{n}-\mathbf{n}}(\mathbf{J},\mathbf{J},\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega})\,\mathbf{n}\\ \mathbf{D}_2(\mathbf{J}) &= \tfrac{1}{2}(2\pi)^4 m\sum_{\mathbf{n}\mathbf{n}'}\int \mathrm{d}^3\mathbf{J}'\big|E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega})\big|^2 f_0(\mathbf{J}')\delta(\mathbf{n}'\cdot\mathbf{\Omega}'-\mathbf{n}\cdot\mathbf{\Omega})\,\mathbf{n}\otimes\mathbf{n}.\end{aligned} \tag{4.18}$$

Notice that the sign of $\mathbf{D}_2$ is positive, so the flux that it generates is in the opposite direction to the gradient of f_0: stars diffuse away from regions of high phase-space density. Whereas the flux of heat in a metal bar, $\mathbf{q} = -\kappa\nabla T$ is simply proportional to the gradient of the heat-density T, our diffusive flux has, in addition to a term that's proportional to the gradient of the star density, a term that's proportional to the density itself. To understand the necessity of this additional term, consider how the system would evolve if it were absent. Then stars would diffuse from modest initial actions to ever higher actions, so eventually the density of stars would become uniform throughout phase space, just as heat diffusion will eventually make the temperature uniform throughout a bar. However, energy conservation, which is encoded in the dynamics we have been using, excludes a uniform distribution of stars in action space, since larger actions are associated with more energy. Consequently, the tendency of the term in $\mathbf{F}$ proportional to $\partial f_0/\partial\mathbf{J}$ to drive the system to uniformity in action space has to be counteracted by the term proportional to f_0, which generates a net drift towards the origin of action space.

In thermal equilibrium, $\mathbf{F}$ must vanish by detailed balance. Then the DF $f_0 = \exp(-\beta H)$, where H is the Hamiltonian and $\beta = (k_\mathrm{B}T)^{-1}$ is the inverse temperature. Since $\partial H/\partial\mathbf{J} = \mathbf{\Omega}$, for $\mathbf{F}$ to vanish the diffusion coefficients (which depend on f_0) must satisfy

$$\mathbf{D}_1(\mathbf{J}) - \beta\mathbf{D}_2(\mathbf{J})\cdot\mathbf{\Omega}(\mathbf{J}) = 0 \tag{4.19}$$

everywhere in action space. This relation provides a useful check on any formulae for the diffusion coefficients (Problem 9). It also suggests that whatever the origin of the fluctuations that drive diffusion (here Poisson fluctuations), $\mathbf{D}_1$ and $\mathbf{D}_2$ will be closely related to one another. In fact, from our expression for $\mathbf{D}_1$ one can derive (Appendix A)

$$\mathbf{D}_1(\mathbf{J}) = -\tfrac{1}{2}(2\pi)^4 m\sum_{\mathbf{n}\mathbf{n}'}\int \mathrm{d}^3\mathbf{J}'\big|E_{\mathbf{n}\mathbf{n}'}(\mathbf{J},\mathbf{J}',-\mathrm{i}\mathbf{n}\cdot\mathbf{\Omega})\big|^2\mathbf{n}'\cdot\frac{\partial f_0}{\partial\mathbf{J}'}\delta(\mathbf{n}'\cdot\mathbf{\Omega}'-\mathbf{n}\cdot\mathbf{\Omega})\,\mathbf{n}, \tag{4.20}$$

which is extremely similar to our expression for $\mathbf{D}_2$.

Equation (4.20) for $\mathbf{D}_1$ and our equation (4.18) for $\mathbf{D}_2$ give the $\mathbf{D}_i(\mathbf{J})$ as sums of contributions from stars at any point $\mathbf{J}'$ at which stars "resonate" with stars at $\mathbf{J}$ – two stars resonate in the sense that the $\mathbf{n}'$ harmonic of one star coincides with the $\mathbf{n}$ harmonic of the other. $\mathbf{D}_1$ and $\mathbf{D}_2$ are proportional to the values taken by $\partial f_0/\partial\mathbf{J}'$ and $f_0(\mathbf{J}')$, respectively, because the strength of the oscillating field that's created by the stars at $\mathbf{J}'$ is proportional to the number of stars at $\mathbf{J}'$. On account of the vector $\mathbf{n}$ that occurs in $\mathbf{D}_1$ and the diadic $\mathbf{n}\otimes\mathbf{n}$ in $\mathbf{D}_2$, the diffusion tensor is highly anisotropic in the sense that stars diffuse anomalously fast in the direction $\mathbf{n}$ that yields the largest number of resonant stars.

5

Diffusion in a galactic disc

The formalism developed in the last section gives fascinating insight into the dynamics of galactic discs similar to that in which we reside. These systems were among the first to be studied by N-body simulation when electronic computers became widely available, but it is only recently that we have achieved a reasonable understanding of their dynamics.

Fouvry et al. (arXiv150706887) have applied the formalism of Chapter 4 to razor-thin discs: restricting motion to the xy plane significantly simplifies the computations. First, angle-action coordinates are readily constructed for an axisymmetric disc (Problem 3). Second, Kalnajs (1976) has defined a convenient set of orthonormal potential-density pairs

$$\Phi^\alpha(r,\phi) = \mathrm{e}^{\mathrm{i}l\phi}\Phi^l_n(r) \qquad \rho^\alpha(r,\phi) = \mathrm{e}^{\mathrm{i}l\phi}\rho^l_n(r), \tag{5.1}$$

where $\alpha = (l,n)$. Φ^l_n is a specified polynomial and ρ^l_n is a polynomial in r times a half power of $1 - r^2/r_0^2$, where r_0 is the edge of the disc.

Next they compute the AA representation of their basis potentials:

$$\hat{\Phi}^{(\alpha)}(\mathbf{n},\mathbf{J}) = \delta_{\alpha_2,n_2}\frac{1}{\pi}\int_{r_\mathrm{p}}^{r_\mathrm{a}} \mathrm{d}r\, \Phi^l_n(r)\cos[n_1\theta_1 + n_2(\theta_2 - \phi)].$$

They considered a disc that is confined by a potential that generates a circular speed $v_\mathrm{c} = (R\partial\Phi/\partial R)^{1/2}$ that is everywhere constant. If the disc generated this potential on its own, its surface density $\Sigma(R)$ would be proportional to R^{-1}. It is more realistic (and numerically more convenient) to assume that Φ is generated by three components: (i) a bulge that dominates the mass density near the origin, (ii) a dark halo that dominates the mass density far from the centre, and (iii) the disc, which contributes ~ 0.5 of the radial force at intermediate radii. One says that a "Mestel" disc with $\Sigma(R) \propto R^{-1}$ has been "tapered" at small and large radii to accommodate the bulge and the dark halo. The unperturbed DF is

$$f_0(E,J_\phi) = \xi C J_\phi^q \mathrm{e}^{-E/\sigma_r^2} T_\mathrm{in}(J_\phi) T_\mathrm{out}(J_\phi), \tag{5.2a}$$

where $E = \frac{1}{2}(v_R^2 + v_\phi^2) + \Phi$, C normalises the DF such that with $\xi = T_\mathrm{in} = T_\mathrm{out} = 1$ the disc generates the entire potential, σ_r is a parameter that controls the magnitude of stars' random motions, and

$$q = (v_\mathrm{c}/\sigma_r)^2 - 1 \tag{5.2b}$$

was taken to have the value 11.4. Finally the taper functions are

$$T_\mathrm{in}(J_\phi) = \frac{J_\phi^4}{(R_\mathrm{in}v_\mathrm{c})^4 + J_\phi^4} \qquad T_\mathrm{out}(J_\phi) = \frac{(R_\mathrm{out}v_\mathrm{c})^5}{(R_\mathrm{out}v_\mathrm{c})^5 + |J_\phi|^5}, \tag{5.2c}$$

where $R_\mathrm{out} = 11.5R_\mathrm{in}$. By increasing ξ between zero and unity, the dynamical importance of the disc's self-gravity can be increased from unimportant to dominant. For $\xi \simeq 0.5$ this disc is known to be stable in the sense (Box 3.1) that all its normal modes are damped (Toomre 1981).

In Figure 5.1 arrows show the diffusive flux $\mathbf{F}$ computed from equation (A.7). We see that $\mathbf{F}$ is small except along a ridge that slopes leftwards up from the J_ϕ axis (which is where the