Classical Fields
Part I:
Relativistic Covariance

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Books: (i) *Introduction to Einstein's Relativity*, Ray d'Inverno, OUP; (ii) *The Classical Theory of Fields*, L.D. Landau & E.M. Lifschitz (Pergamon); (iii) *Gravitation and Cosmology*, S., Weinberg (Wiley)

Vacation work: Study §1 Relativistic Covariance and work the eight embedded Exercises

1 Relativistic Covariance

Observers who move relative to one another do not always agree about the values of quantities, such as speed, mass, energy etc, associated with the same physical system. The special theory of relativity tells us how we may predict the values measured by any observer once we know the values assigned by one particular observer, for example ourselves.

Special relativity teaches us to think of experience as being made up of 'events', each with a definite location in the four-dimensional continuum of spacetime. Any given observer assigns to each event a unique 4-tuple of numbers (t, x, y, z). Of course he can do this in many, many ways. But special relativity claims that there are certain specially favoured systems for assigning coordinates to events, the so-called inertial coordinate systems. O chooses one inertial system and another observer, O$'$, sets up a different one. But according to special relativity the coordinates (t', x', y', z') O$'$ assigns to any event can be related to O's coordinates (t, x, y, z) of the same event by

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} ct_0 \\ x_0 \\ y_0 \\ z_0 \end{pmatrix} + \mathbf{L} \cdot \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \tag{1.1}$$

where c is the speed of light and (t_0, x_0, y_0, z_0) is a set of numbers characteristic of the two observers, as is the 4×4 matrix $\mathbf{L}$.

Clearly, (t_0, x_0, y_0, z_0) are the coordinates O$'$ assigns to the event that marks the origin of O's coordinates. For simplicity we shall assume that $(t_0, x_0, y_0, z_0) = \mathbf{0}$. In general $\mathbf{L}$ can be represented as the product of matrices generating a rotation, a boost parallel to a coordinate direction and a second rotation: $\mathbf{L} = \mathbf{R}' \cdot \mathbf{L}_0 \cdot \mathbf{R}$, where $\mathbf{R}$ rotates the coordinate axes so as to align the boost direction with a coordinate direction, $\mathbf{L}_0$ effects the boost along the given axis and $\mathbf{R}'$ rotates the coordinates to any desired final orientation. If $\mathbf{R}$ is chosen such that the x-axis becomes the boost direction, $\mathbf{L}_0$ has the form

$$\mathbf{L}_0 = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{where} \quad \begin{array}{c} \beta \equiv v/c \\ \gamma \equiv 1/\sqrt{1-\beta^2} \end{array}. \tag{1.2}$$

For simplicity we confine ourselves to observers whose spatial coordinate systems are aligned, and whose relative motion lies along their (mutually parallel) x-axes. Then in (1.1) $\mathbf{L} = \mathbf{L}_0$ and we get the familiar equations of a Lorentz transformation:

$$\begin{aligned} t' &= \gamma t - \gamma v x/c^2 \\ x' &= \gamma x - \gamma v t \\ y' &= y \\ z' &= z \end{aligned} \tag{1.3}$$

4-vectors Lorentz transformations mix up space and time, so it is useful to define new coordinates which all have dimensions of length. We write $x^0 \equiv ct$, $x^1 \equiv x$, $x^2 \equiv y$, $x^3 \equiv z$, and refer to a general component of the 4-vector (x^0, x^1, x^2, x^3) as x^μ. (The reason for labelling the components with superscripts rather than subscripts will emerge shortly.) Then we write a Lorentz transformation as

$$x^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{1.4a}$$

where

$$\boldsymbol{\Lambda} \equiv \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.4b}$$

In (1.4a) the **Einstein summation convention** is being used in that the summation sign $\sum_{\nu=0}^{1}$ has been omitted for brevity. You know it's really there because ν appears twice on the right-hand side of the equation, once up and once down.

Why do we write the row index of $\boldsymbol{\Lambda}$ as a superscript and the column index as a subscript?

A key property of a Lorentz transformation is that $-(ct')^2 + x'^2 + y'^2 + z'^2 = -(ct)^2 + x^2 + y^2 + z^2$. This is analogous to the fact that if two vectors $\boldsymbol{a}$ and $\boldsymbol{a}'$ are related by a rotation matrix, then $a_x'^2 + a_y'^2 + a_z'^2 = a_x^2 + a_y^2 + a_z^2$. So a Lorentz transformation is a sort of modified, four-dimensional rotation. When we rotate a vector $\boldsymbol{a}$ we like to say that the length $|\boldsymbol{a}|$ is invariant (i.e., stays constant). Analogously we define the length of the 4-vector $\mathbf{x}$ to be

$$|\mathbf{x}| \equiv -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2. \tag{1.5}$$

Notes:

(i) We don't extract a square root because we have no guarantee that $|\mathbf{x}| \geq 0$.

(ii) 4-vectors that have negative lengths are called **time-like**, while those with positive lengths are **space-like**. Vectors with zero length are said to be **null**.

(iii) Every book on relativity uses a different convention. The sign of the lengths of space-like vectors is called the "signature of the metric".

The lengths of 4-vectors are sufficiently important for it to be useful to have a way of writing them that does not involve writing out all the components explicitly. To achieve this we introduce this matrix, called the **Minkowski metric**:

$$\boldsymbol{\eta} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.6}$$

Then we have

$$|\mathbf{x}| = \mathbf{x} \cdot \boldsymbol{\eta} \cdot \mathbf{x}, \tag{1.7a}$$

or in component form

$$|\mathbf{x}| = x^\mu \eta_{\mu\nu} x^\nu. \tag{1.7b}$$

The Einstein convention is here being used to drop two summation signs. We write both of $\boldsymbol{\eta}$'s indices as subscripts so that each sum is over one up and one down index.

Covariant and contravariant vectors We write the result of matrix multiplication of $\mathbf{x}$ by $\boldsymbol{\eta}$ as

$$x_\mu \equiv \eta_{\mu\nu} x^\nu .$$

We have $x_0 = -x^0 = -ct$, $x_1 = x^1$, $x_2 = x^2$ and $x_3 = x^3$. Thus the length of $\mathbf{x}$ is

$$x^\mu x_\mu = -c^2t^2 + x^2 + y^2 + z^2 .$$

Notice that here as everywhere else, we are summing over one up and one down index. In order to stick rigidly to this rule, we define

$$\eta^{\mu\nu} \equiv \eta_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{1.8}$$

Note:

We have $\eta^{\mu\gamma}\eta_{\gamma\nu} = \delta^\mu_\nu$, or in matrix form $\boldsymbol{\eta}\cdot\boldsymbol{\eta} = \mathbf{I}$, where $\mathbf{I}$ and δ^μ_ν are two ways of writing the 4×4 identity matrix. Also $\eta^{\mu\nu} = \eta^{\mu\gamma}\delta^\nu_\gamma$, so in a sense $\boldsymbol{\eta}$ is merely the up-up and down-down forms of the identity matrix.

From x_μ we can recover x^μ;

$$x^\mu = \eta^{\mu\nu} x_\nu . \tag{1.9}$$

x_μ is a 4-vector, but of a slightly different type than x^μ, because under a Lorentz transformation we have

$$\begin{aligned} x'_\mu &= \eta_{\mu\nu} x'^\nu = \eta_{\mu\nu}\Lambda^\nu{}_\kappa x^\kappa = \eta_{\mu\nu}\Lambda^\nu{}_\kappa \eta^{\kappa\lambda} x_\lambda \\ &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \equiv \Lambda_\mu{}^\nu x_\nu , \end{aligned} \tag{1.10}$$

where we have defined a new matrix

$$\Lambda_\mu{}^\lambda \equiv \eta_{\mu\nu}\Lambda^\nu{}_\kappa \eta^{\kappa\lambda} . \tag{1.11}$$

Notice that the transpose of $\Lambda_\mu{}^\nu$ is the inverse of $\Lambda^\mu{}_\nu$:

$$\Lambda^\mu{}_\kappa \Lambda_\mu{}^\nu = \delta^\nu_\kappa , \tag{1.12}$$

where we have again written the 4×4 identity matrix as δ^ν_κ.

Exercise (1):

Obtain (1.12) from the requirement that for any two vectors $\mathbf{x}$, $\mathbf{y}$, we have $x'_\mu y'^\mu = x_\mu y^\mu$.

Vectors with their indices below are called **covariant** (x_μ). Vectors with indices above are called **contravariant** (x^μ). I shall call them down and up vectors. The operation of setting two indices equal and summing from 0 to 3 is called **contraction**. In a contraction one index must be up and one down. Quantities like $\sum_\mu x_\mu x_\mu$ have nothing to do with physics. An important motivation for writing x^μ rather than $\mathbf{x}$ is to distinguish the up from the down form of $\mathbf{x}$. Often an expression is equally valid for up or down vectors provided the basic rules are obeyed, and then it is neater to use conventional vector notation than to stick in indices. For example, if $\mathbf{a}$ and $\mathbf{b}$ are vectors and $\mathbf{M}$ is a matrix, we can interpret $\mathbf{a} = \mathbf{M} \cdot \mathbf{b}$ as $a^\mu = M^{\mu\nu} b_\nu$, as $a_\mu = M_{\mu\nu} b^\nu$, or in yet other ways. But if you ever express a 4-vector in component form, you *must* come clean and say whether you're giving the up or the down vector, as in $x^\mu = (ct, x, y, z)$.

According to special relativity, all quantities of physical interest can be grouped into n-tuples.

1.1 1-tuples (4-scalars)

On some things all observers agree, for example the charge and total spin of the an electron. These quantities are called **4-scalars** or relativistic invariants. The length of a 4-vector is a 4-scalar.

1.2 4-tuples (4-vectors)

If O measures the wave-vector and frequency of a photon to be $\boldsymbol{k}$ and ω, then an observer O$'$ who moves at speed v along O's x-axis measures wave-vector $\boldsymbol{k}'$ and frequency ω' given by

$$\begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix}. \tag{1.13a}$$

The matrix form of this equation is

$$\begin{pmatrix} \omega'/c \\ \boldsymbol{k}' \end{pmatrix} = \boldsymbol{\Lambda} \cdot \begin{pmatrix} \omega/c \\ \boldsymbol{k} \end{pmatrix} \quad \text{where} \quad \boldsymbol{\Lambda} \equiv \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.13b}$$

Notes:

(i) The Lorentz transformation matrix $\boldsymbol{\Lambda}$ is dimensionless, so ω has to be divided by c to give the same dimensions as $\boldsymbol{k}$ before being put into the last place of a 4-vector with $\boldsymbol{k}$.

(ii) Vectors written in italic boldface ($\boldsymbol{k}$) are 3-vectors, while those written in Roman boldface ($\mathbf{k}$) are 4-vectors.

If we define $k^0 \equiv \omega/c$, then

$$\mathbf{k}' = \boldsymbol{\Lambda} \cdot \mathbf{k} \quad \text{i.e.,} \quad k'^{\mu} = \Lambda^{\mu}{}_{\nu} k^{\nu}. \tag{1.14}$$

Exercise (2):

Determine whether the photon is blue or red shifted between its emission by O and its detection by O′. Relate this to the question of whether O′ is approaching or receding from O.

The length of a photon's 4-vector is the scalar

$$|\mathbf{k}| \equiv -(k^0)^2 + (k^1)^2 + (k^2)^2 + (k^3)^2 = -\frac{\omega^2}{c^2} + |\boldsymbol{k}|^2 = 0.$$

One can prove that this really is a scalar by brute force:

$$\begin{aligned} |\mathbf{k}'| &= -(k'^0)^2 + (k'^1)^2 + (k'^2)^2 + (k'^3)^2 \\ &= -\Big(\gamma\frac{\omega}{c} - \beta\gamma k^1\Big)^2 + \Big(-\beta\gamma\frac{\omega}{c} + \gamma k^1\Big)^2 + (k^2)^2 + (k^3)^2 \\ &= -\gamma^2\Big(1-\beta^2\Big)\frac{\omega^2}{c^2} + \gamma^2\Big(1-\beta^2\Big)(k^1)^2 + (k^2)^2 + (k^3)^2 \\ &= -(k^0)^2 + (k^1)^2 + (k^2)^2 + (k^3)^2. \end{aligned}$$

Another familiar 4-tuple: if observer O measures energy E and momentum $\mathbf{p}$ for some particle, then O′ will measure E' and $\boldsymbol{p}'$ given by

$$\begin{pmatrix} E'/c \\ \boldsymbol{p}' \end{pmatrix} = \boldsymbol{\Lambda} \cdot \begin{pmatrix} E/c \\ \boldsymbol{p} \end{pmatrix}, \tag{1.15}$$

or setting $p^0 \equiv E/c$, we have $p'^{\mu} = \Lambda^{\mu}{}_{\nu} p^{\nu}$.

The length of the momentum-energy 4-vector of a particle of rest mass $m_0 \neq 0$ is just $-c^2$ times the square of its rest mass m_0. We show this by arguing that it doesn't matter in whose frame we evaluate a scalar. We choose the particle's rest frame. Then $\boldsymbol{p} = 0$ and $E = cp^0 = m_0 c^2$, so

$$-(p^0)^2 + (p^1)^2 + (p^2)^2 + (p^3)^2 = -m_0^2 c^2.$$

1.3 6-tuples (antisymmetric 2nd rank tensors)

If the electric and magnetic fields measured by O are arranged into the antisymmetric matrix $\mathbf{F}$,

$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \quad \text{(SI units)}, \tag{1.16}$$

then O$'$ will measure E' and B' as

$$\begin{pmatrix} 0 & E'_x/c & E'_y/c & E'_z/c \\ -E'_x/c & 0 & B'_z & -B'_y \\ -E'_y/c & -B'_z & 0 & B'_x \\ -E'_z/c & B'_y & -B'_x & 0 \end{pmatrix} \equiv F'^{\mu\nu} = \Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda F^{\kappa\lambda}. \tag{1.17}$$

Note that $F^{\mu\nu}$ transforms *as if* it were the product $p^\mu p^\nu$ of two down-vectors (which it isn't). Objects that transform in this way are called second-rank tensors.

F is called the **Maxwell field tensor**.

Exercise (3):

Transform $F^{\kappa\lambda}$ with the matrix $\Lambda^\mu{}_\nu$ defined by (1.13b) to show that an observer who moves at speed v down the x-axis of an observer who sees fields $\boldsymbol{E} = (E_x, E_y, 0)$ and $\boldsymbol{B} = 0$, perceives fields $\boldsymbol{E}' = (E_x, \gamma E_y, 0)$ and $\boldsymbol{B}' = (0, 0, \gamma v E_y/c)$. [Hint: since $\boldsymbol{\Lambda}$ is symmetric, we can write $\mathbf{F}' = \boldsymbol{\Lambda}\cdot\mathbf{F}\cdot\boldsymbol{\Lambda}$.] Hence deduce the general rules $\boldsymbol{E}'_\| = \boldsymbol{E}_\|$, $\boldsymbol{E}'_\perp = \gamma(\boldsymbol{E}_\perp + \boldsymbol{v}\times\boldsymbol{B})$, $\boldsymbol{B}'_\| = \boldsymbol{B}_\|$, $\boldsymbol{B}_\perp = \gamma(\boldsymbol{B}_\perp - \boldsymbol{v}\times\boldsymbol{E}/c^2)$. Verify that $(B^2 - E^2/c^2) = (B'^2 - E'^2/c^2)$.

Some 6-tuples correspond to elements of area. This correspondence works as follows. With any two displacements, say $\mathbf{u}$ and $\mathbf{v}$, we associate the parallelogram bounded by $\mathbf{u}$ and $\mathbf{v}$. Information about the size and orientation of this parallelogram is conveyed by the antisymmetric tensor $S^{\alpha\beta} \equiv u^\alpha v^\beta - u^\beta v^\alpha$; in particular, if $\mathbf{u} = \mathbf{v}$, then $\mathbf{S} = 0$. $\mathbf{S}$ has fewer degrees of freedom than the eight numbers involved in $\mathbf{u}$ and $\mathbf{v}$ because we can add to $\mathbf{u}$ any multiple of $\mathbf{v}$ without affecting $\mathbf{S}$, and vice versa for $\mathbf{v}$ and $\mathbf{u}$.

Exercise (4):

Consider transformation $\mathbf{u} \to \mathbf{u}' = a\mathbf{u} + b\mathbf{v}$, $\mathbf{v} \to \mathbf{v}' = c\mathbf{u} + d\mathbf{v}$ with the corresponding mapping $\mathbf{S} \to \mathbf{S}'$. Show that the equation $\mathbf{S}' = \mathbf{S}$ imposes one constraint on the numbers a, b, c, d. Hence only $8 - 3 = 5$ numbers are needed to specify $\mathbf{S}$. Give a geometrical interpretation of this result.

In three-space the size and orientation of a parallelogram may be specified by giving the magnitude and direction of the normal. Hence in three-space full information about an antisymmetric 2^{nd} rank tensor can be packed into the three components of the 3-vector which we call the cross-product of the parallelogram's sides. In four-dimensional spacetime each parallelogram has a magnitude and two mutually perpendicular normals, requiring five numbers for its full specification. Consequently there is no direct analogue of the cross product and we must represent areas directly with antisymmetric tensors.

Exercise (5):

Compare the above statements to the number of independent components of an antisymmetric $n \times n$ matrix for $n = 2, 3, 4$.

A physically interesting 6-tuple that describes an area is the tensor $(x^\mu p^\nu - x^\nu p^\mu)$ formed from the space-time coordinate vector $x^\mu = (ct, x, y, z)$ and the 4-momentum of a particle. If the angular momentum about the origin is $\boldsymbol{L}$, we have

$$H^{\mu\nu} \equiv (x^\mu p^\nu - x^\nu p^\mu) = \begin{pmatrix} 0 & \ddots & \ddots & \\ c(xE/c^2 - tp_x) & 0 & \ddots & \ddots \\ c(yE/c^2 - tp_y) & -L_z & 0 & \ddots \\ c(zE/c^2 - tp_z) & L_y & -L_x & 0 \end{pmatrix}, \tag{1.18}$$

where the diagonal dots stand for minus the quantities in the lower left triangle of the matrix. The numbers in the first column of this matrix give mc times the particle's initial position vector.

With every 6-tuple we get two free scalars. If the 6-tuple is of the form $(u^\alpha v^\beta - u^\beta v^\alpha)$, then one of these is twice the squared magnitude of the corresponding parallelogram:

$$\begin{aligned} S^{\mu\nu}(\eta_{\mu\kappa}\eta_{\nu\lambda}S^{\kappa\lambda}) &\equiv S^{\mu\nu}S_{\mu\nu} = -\operatorname{Tr}\mathbf{S}\cdot\mathbf{S} \\ &= (u^\mu v^\nu - u^\nu v^\mu)(u_\mu v_\nu - u_\nu v_\mu) = 2[|\mathbf{u}||\mathbf{v}| - (\mathbf{u}\cdot\mathbf{v})^2]. \end{aligned}$$

Note:

Here by $\operatorname{Tr}\mathbf{M}$ we mean $M_\alpha{}^\alpha = M^\alpha{}_\alpha$. That is, the sum implied by Tr must always be over one up and one down index.

Evaluation in the particle's rest frame shows that the scalar $\frac{1}{2}H_{\mu\nu}H^{\mu\nu} = [|\mathbf{x}||\mathbf{p}| - (\mathbf{x}\cdot\mathbf{p})^2] = -(m_0 c r_0)^2$, where r_0 is the distance (in the rest frame) between the particle and the origin at $t = 0$.

It is interesting to evaluate this same scalar for the Maxwell field tensor. Straightforward matrix multiplication shows that the down-down shadow of $F^{\mu\nu}$ is[1]

$$F_{\mu\nu} \equiv \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} \quad \text{(SI units)}, \tag{1.19}$$

Multiplying each element of $F_{\mu\nu}$ by the corresponding element of $F^{\mu\nu}$ we find

$$\begin{aligned} m &\equiv \tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} = -\tfrac{1}{2}\operatorname{Tr}\mathbf{F}\cdot\mathbf{F} \\ &= \tfrac{1}{2}(\text{each element of } F_{\mu\nu}) \times (\text{corresponding element of } F^{\mu\nu}) \\ &= (B^2 - E^2/c^2). \end{aligned} \tag{1.20}$$

To extract another scalar from a 6-tuple we need to introduce the **Levi-Civita symbol**:

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{if } \alpha\beta\gamma\delta \text{ is an even permutation of 0123} \\ -1 & \text{if } \alpha\beta\gamma\delta \text{ is an odd permutation of 0123} \\ 0 & \text{otherwise.} \end{cases} \tag{1.21}$$

[1] It is worth remembering that in special relativity the lowering operation only *changes the sign of the mixed space-time components.*

Note:

Whereas when n is odd, the cyclic interchange $i_1 \to i_2 \to \ldots \to i_{n-1} \to i_n \to i_1$ is an even permutation of the i_k, when n is even, this permutation is odd. (To prove this exchange i_1 and i_n and then make $n-2$ exchanges to work i_1 back to the second place.) So whereas for 3-dimensional tensors $\epsilon_{jki} = \epsilon_{ijk}$, we now have $\epsilon^{\beta\gamma\delta\alpha} = -\epsilon^{\alpha\beta\gamma\delta}$.

$\epsilon^{\alpha\beta\gamma\delta}$ allows us to form the **dual** $\overline{\mathbf{F}}$ of $\mathbf{F}$:

$$\begin{aligned} \overline{F}^{\alpha\beta} &\equiv \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} \\ &= \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}. \end{aligned} \tag{1.22}$$

$\overline{\mathbf{F}}$ can be obtained from $\mathbf{F}$ by the transformation $\boldsymbol{E} \to \boldsymbol{B}$, $\boldsymbol{B} \to -\boldsymbol{E}$. The other scalar is the trace of the product of $\mathbf{F}$ with its dual:

$$\begin{aligned} f &\equiv \mathrm{Tr}\,\mathbf{F}\cdot\overline{\mathbf{F}} \\ &= -(\text{each element of } F_{\alpha\beta}) \times (\text{corresponding element of } \overline{F}^{\alpha\beta}) \\ &= \frac{4}{c}\boldsymbol{E}\cdot\boldsymbol{B}. \end{aligned} \tag{1.23}$$

Exercise (6):

Show that with $S_{\mu\nu} = u_\mu v_\nu - u_\nu v_\mu$, $\mathrm{Tr}\,\mathbf{S}\cdot\overline{\mathbf{S}} = 0$. This result explains why $\mathbf{S}$ has only 5 degrees of freedom (Exercise 4).

1.4 10-tuples (symmetric 2^{nd} rank tensors)

Imagine that we move some charges around. Then the rate at which we do work *on* the e.m. field is

$$\begin{aligned} \dot{\mathcal{E}} &= -\int \boldsymbol{E}\cdot\boldsymbol{j}\,\mathrm{d}^3\boldsymbol{x} \\ &= -\frac{1}{\mu_0}\int \boldsymbol{E}\cdot\Big(\nabla\times\boldsymbol{B} - \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}\Big)\,\mathrm{d}^3\boldsymbol{x} \end{aligned} \tag{1.24}$$

But $\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{E}) - \boldsymbol{E}\cdot(\nabla\times\boldsymbol{B})$, so (1.24) can be rewritten

$$\begin{aligned} \dot{\mathcal{E}} &= \frac{1}{\mu_0}\int \nabla\cdot(\boldsymbol{E}\times\boldsymbol{B})\,\mathrm{d}^3\boldsymbol{x} + \frac{1}{\mu_0}\int\Big(-\boldsymbol{B}\cdot(\nabla\times\boldsymbol{E}) + \frac{1}{c^2}\boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t}\Big)\,\mathrm{d}^3\boldsymbol{x} \\ &= \frac{1}{\mu_0}\oint(\boldsymbol{E}\times\boldsymbol{B})\cdot\mathrm{d}^2\boldsymbol{S} + \frac{1}{2\mu_0}\int\frac{\partial}{\partial t}\big(B^2 + E^2/c^2\big)\,\mathrm{d}^3\boldsymbol{x}. \end{aligned} \tag{1.25}$$

If energy is to be conserved, the energy we deploy moving the charges has to go somewhere. According to (1.25) energy will be conserved if we interpret the **Poynting vector**

$$\boldsymbol{N} \equiv \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} \tag{1.26}$$

as the flux of e.m. energy, and

$$\frac{1}{2\mu_0}\left(B^2 + E^2/c^2\right) \tag{1.27}$$

as the density of e.m. energy.

How do the Poynting vector and the e.m. energy-density fit into the scheme of n-tuples? From $\mathbf{F}$ we can construct the following important tensor:

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{\mu_0}[-\tfrac{1}{4}(F_{\delta\gamma}F^{\delta\gamma})\eta^{\mu\nu} - F^{\mu}{}_{\gamma}F^{\gamma\nu}]; \\ \mathbf{T} &= \frac{1}{\mu_0}\left[\tfrac{1}{4}\operatorname{Tr}(\mathbf{F}\cdot\mathbf{F})\boldsymbol{\eta} - \mathbf{F}\cdot\mathbf{F}\right], \end{aligned} \tag{1.28}$$

where $\mathbf{F}$ is, as usual, the Maxwell field tensor (1.16). It's easy to see that $\operatorname{Tr}\mathbf{T} = 0$. A little slog shows that in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$ the tensor $\mathbf{T}$ is

$$T^{\mu\nu} = \begin{pmatrix} \frac{1}{2\mu_0}(B^2 + E^2/c^2) & N_x/c & N_y/c & N_z/c \\ N_x/c & & & \\ N_y/c & & P_{ij} & \\ N_z/c & & & \end{pmatrix}, \tag{1.29}$$

where

$$P_{ij} \equiv \frac{1}{\mu_0}\left[\tfrac{1}{2}\delta_{ij}\left(B^2 + \frac{E^2}{c^2}\right) - \left(B_iB_j + \frac{E_iE_j}{c^2}\right)\right] \quad (i,j = 1,2,3). \tag{1.30}$$

Thus the energy density in the e.m. field is the 00 component of $\mathbf{T}$ and the Poynting vector occupies the mixed space-time components of $\mathbf{T}$. It turns out that the 3×3 matrix P_{ij} describes the flux of the three kinds of momentum: P_{ix} = flux of x-momentum etc.

Exercise (7):

Show that a uniform magnetic field parallel to the z-axis is associated with tension (negative pressure) along the axis, and pressure in the perpendicular directions.

As an example of $\mathbf{T}$ consider a plane e.m. wave running along $\hat{\mathbf{i}}$ polarized parallel to $\hat{\mathbf{j}}$. Then

$$\begin{aligned} \boldsymbol{E} &= (0, E, 0)\cos(\omega t - kx) \\ \boldsymbol{B} &= (0, 0, B)\cos(\omega t - kx). \end{aligned}$$

E and B are related by $-\partial\boldsymbol{B}/\partial t = \nabla \times \boldsymbol{E} \quad\Rightarrow\quad B = kE/\omega = E/c$. Hence

$$\boldsymbol{N} = (E^2/\mu_0 c, 0, 0)\cos^2(\omega t - kx).$$

The first term in our expression (1.30) is non-zero only on the diagonal. The second term is non-zero only in the yy and zz slots and there cancels the first term. So $\boldsymbol{P}$ is

$$P_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{E^2}{\mu_0 c^2}\cos^2(\omega t - kx),$$

and finally

$$T^{\mu\nu} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{E^2}{\mu_0 c^2} \cos^2(\omega t - kx). \tag{1.31}$$

The stress tensor $\boldsymbol{P}$ has only an entry in the xx slot because our wave is engaged in the business of carrying x-type momentum in the x-direction; the wave would push back a mirror placed in a plane $x =$ constant. Clearly the Poynting vector is also directed along the x axis, which accounts for the off-diagonal units in $\mathbf{T}$. In proper relativistic units the wave employs unit energy density ("capital employed") to carry unit fluxes of energy and momentum ("turnover"). Notice that the wave's phase is the scalar $-\mathbf{k}\cdot\mathbf{x}$.

When we do cosmology we'll need $T^{\mu\nu}$ for a fluid. At each event a fluid has a streaming motion that's characterized by the 4-velocity u^α and an associated rest frame. In this rest frame there's an energy density ρc^2 and a pressure P. If the fluid is "perfect" there are no other stresses (such as viscous shear) and we'll only consider perfect fluids. $T^{\mu\nu}$ has to be a symmetric second-rank tensor made from the scalars ρ and P, the vector u^μ and the tensor $\eta^{\mu\nu}$. A candidate is

$$T^{\mu\nu} = (\rho + P/c^2)u^\mu u^\nu + P\eta^{\mu\nu}. \tag{1.32}$$

It's the tensor we want because in the fluid's rest frame it becomes

$$\begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}.$$

1.5 Derivatives of tensors

Derivatives with respect to any system of coordinates can be expressed in terms of derivatives w.r.t. any other system by use of the chain rule:

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}. \tag{1.33}$$

If the primed and unprimed systems are linked by a Lorentz transformation,

$$x'^\nu = \Lambda^\nu{}_\mu x^\mu, \tag{1.34}$$

we have on multiplying by $\Lambda_\nu{}^\kappa$ and summing over ν,

$$\Lambda_\nu{}^\kappa x'^\nu = \Lambda_\nu{}^\kappa \Lambda^\nu{}_\mu x^\mu = x^\kappa,$$

where the last step follows by (1.12). Differentiating we get

$$\frac{\partial x^\kappa}{\partial x'^\nu} = \Lambda_\nu{}^\kappa. \tag{1.35}$$

Thus

$$\frac{\partial}{\partial x'^\mu} = \Lambda_\mu{}^\nu \frac{\partial}{\partial x^\nu}, \tag{1.36}$$

and we see that

$$\partial_\mu \equiv \partial/\partial x^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{1.37}$$

transforms like a down vector.

Notes:

(i)

$$\frac{\partial}{\partial x^\mu} \text{ operates on scalars to produce vectors:} \quad G_\mu \equiv \frac{\partial \phi}{\partial x^\mu} \equiv \partial_\mu \phi \equiv \phi,_\mu$$

$\frac{\partial}{\partial x^\mu}$ operates on vectors to produce 2^{nd} rank tensors:

$$G_{\mu\nu} \equiv \frac{\partial A_\nu}{\partial x^\mu} \equiv \partial_\mu A_\nu \equiv A_\nu,_\mu$$

$\frac{\partial}{\partial x^\mu}$ operates on tensors to produce higher-rank tensors:

$$G_{\mu\lambda\nu} \equiv \frac{\partial B_{\lambda\nu}}{\partial x^\mu} \equiv \partial_\mu B_{\lambda\nu} \equiv B_{\lambda\nu},_\mu$$

The operand's indices can be either up or down: ${G_\mu}^\nu = \partial_\mu A^\nu$.

(ii) If we contract the tensor produced by operating on a vector, we get a scalar, the 4-divergence $\psi = \partial_\mu A^\mu$.

(iii) We can reduce the number of indices on a higher-rank tensor by contraction: $A^\nu = \partial_\mu G^{\mu\nu}$.

(iv) The 4-analogue of taking the curl of a vector is to antisymmetrize the tensor formed by operating on a vector: $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$. If $A_\nu = \partial_\nu \phi$, then $F_{\mu\nu} = 0$ because partial derivatives commute.

(v) A natural generalization of the divergence theorem reads

$$\int_V \mathrm{d}^4\mathbf{x}\, \frac{\partial T_{\alpha\ldots}}{\partial x^\mu} = \oint_S (\mathrm{d}^3\mathbf{x})_\mu T_{\alpha\ldots}, \tag{1.38}$$

where S is the boundary of the 4-d region V. Notice that $\mathbf{T}$ may have as many indices as it pleases and that one of them may be contracted with μ if you wish.

Example:

In e.m. the usual vector potential $\boldsymbol{A}$ and the electrostatic potential ϕ form the four components of an up vector

$$A^\mu = (\phi/c, A_x, A_y, A_z) \quad [\Rightarrow \quad A_\mu = (-\phi/c, A_x, A_y, A_z)]. \tag{1.39}$$

Our old friend the Maxwell field tensor $\mathbf{F}$ is then

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.40}$$

Thus $F_{12} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_z$ and $F_{01} = \frac{\dot{A}_x}{c} + \frac{1}{c}\frac{\partial \phi}{\partial x} = -E_x/c$.

Derivatives with respect to proper time The history of a particle defines a curve in space-time. Let λ be a parameter which labels points on the curve in a continuous way. Then the coordinates x^μ of points on the curve are continuous functions $x^\mu(\lambda)$. For $\delta\lambda \ll 1$ the small vector

$$\delta\mathbf{x} \equiv \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\lambda}\,\delta\lambda$$

almost joins two points on the curve. Hence it is time-like and $|\delta\mathbf{x}| < 0$. For any two points A and B on the curve, we define

$$\tau \equiv \frac{1}{c}\int_{\mathrm{A}}^{\mathrm{B}} \sqrt{-\left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\lambda}\right|}\,\mathrm{d}\lambda \tag{1.41}$$

to be the **proper time** difference between A and B along the curve. If the curve is a straight line, we may transform to the coordinate system in which $x^\mu = (ct, 0, 0, 0)$ at all points on the curve, and then

$$\tau = \frac{1}{c}\int_{\mathrm{A}}^{\mathrm{B}} \sqrt{-\frac{\mathrm{d}ct}{\mathrm{d}\lambda}\frac{\mathrm{d}(-ct)}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda = [t_{\mathrm{B}} - t_{\mathrm{A}}]. \tag{1.42}$$

Hence the name. We regard the coordinates x^μ of events along the trajectory as functions $x^\mu(\tau)$ of the proper time. Differentiating w.r.t. τ and multiplying through by the rest mass m_0 we obtain a 4-vector, the momentum

$$\mathbf{p} \equiv m_0\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau}. \tag{1.43}$$

From the zeroth component of the up version of this equation we have $\mathrm{d}t = \gamma\mathrm{d}\tau$; the hearts of passengers on a fast train (they mark off units of τ) appear to beat slowly to a medic on the station platform (whose watch keeps t).

1.6 Laws of e.m. and mechanics in tensor form

The relativistic generalization of Newton's second law is

$$m_0\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}\tau^2} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(m_0\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = \mathbf{f}, \tag{1.44}$$

where $\mathbf{f}$ is the **4-force**. The last three components of f^μ are just the Newtonian force components f_i. With $\mu = 0$ equation (1.44) states that the zeroth component of f^μ is to $1/c$ times the rate of change of the particle's energy cp^0; hence physically f^0 is $1/c$ times the rate of working of the force w. In summary

$$f^\mu = (w/c, f_x, f_y, f_z). \tag{1.45}$$

The divergence of (1.16) consists of these four equations:

$$F^{\mu\nu}{}_{,\nu} = \begin{pmatrix} \dfrac{1}{c}\dfrac{\partial E_x}{\partial x} + \dfrac{1}{c}\dfrac{\partial E_y}{\partial y} + \dfrac{1}{c}\dfrac{\partial E_z}{\partial z} \\ \partial B_z/\partial y - \partial B_y/\partial z - \frac{1}{c^2}\partial E_x/\partial t \\ -\partial B_z/\partial x + \partial B_x/\partial z - \frac{1}{c^2}\partial E_y/\partial t \\ \partial B_y/\partial x - \partial B_x/\partial y - \frac{1}{c^2}\partial E_z/\partial t \end{pmatrix} = \begin{pmatrix} \frac{1}{c}\nabla\cdot\boldsymbol{E} \\ \nabla\times\boldsymbol{B} - \dfrac{1}{c^2}\dfrac{\partial \boldsymbol{E}}{\partial t} \end{pmatrix}. \tag{1.46}$$

The zeroth component is by Poisson's equation equal to $\rho/(c\epsilon_0) = c\mu_0\rho$, where ρ is the charge density. By Ampere's law, the last three of these equations are equal to $\mu_0\boldsymbol{j}$, where $\boldsymbol{j}$ is the current density. Hence if we form a 4-vector

$$j^\mu = (c\rho, j_x, j_y, j_z), \tag{1.47}$$

we may write four of Maxwell's equations as

$$F^{\mu\nu}{}_{,\nu} = \mu_0 j^\mu. \tag{1.48}$$

It is straightforward to verify that Maxwell's other four equations can be written

$$F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0 \quad (\mu \neq \nu \neq \lambda). \tag{1.49}$$

Exercises (8):

(i) Show that when λ, μ and ν equal 1, 2 and 3 respectively, (1.49) becomes $\nabla\cdot\boldsymbol{B} = 0$.

(ii) Show that with equation (1.22) equation (1.49) may also be written $\overline{F}^{\mu\nu}{}_{,\nu} = 0$.

Charge conservation is expressed as

$$\mu_0 \partial \cdot \mathbf{j} = \mu_0 j^\mu{}_{,\mu} = F^{\mu\nu}{}_{,\nu\mu} = 0, \tag{1.50}$$

where the last step follows by the antisymmetry of $\mathbf{F}$.

The natural definition of the 4-current associated with a particle of charge q is

$$\mathbf{J} = q\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau}. \tag{1.51}$$

Since the force exerted on a charged particle by an e.m. field has to be linear in q, the fields represented by $\mathbf{F}$, and the particle's velocity vector, a suitable 4-vector to try as the force is

$$\mathbf{f} = \mathbf{F}\cdot\mathbf{J}. \tag{1.52}$$

Tentatively inserting this into (1.44) and multiplying through by $\mathrm{d}\tau/\mathrm{d}t = 1/\gamma$ to obtain the acceleration as measured in the laboratory frame, we get

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\mathbf{F}\cdot\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}. \tag{1.53}$$

It is straightforward to check that the last three components of the up form of this vector are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m_0\gamma\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right) = q(\boldsymbol{v}\times\boldsymbol{B} + \boldsymbol{E}),$$

while the zeroth component is

$$\frac{\mathrm{d}(m_0 c\gamma)}{\mathrm{d}t} = \frac{q}{c}\boldsymbol{E}\cdot\boldsymbol{v},$$

or, in words, "the rate of change of the particle's energy mc^2 is equal to the rate of working of the Lorentz force."

Gauge invariance At a classical (i.e. non-quantum level) only $\boldsymbol{E}$ and $\boldsymbol{B}$ are physically meaningful—$\boldsymbol{A}$ is just an abstraction from which $\boldsymbol{E}$ and $\boldsymbol{B}$ can be calculated via $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$. So nothing physical changes if we replace $\mathbf{A}$ by

$$\mathbf{A}' \equiv \mathbf{A} + \partial\Lambda, \tag{1.54}$$

where $\Lambda(\mathbf{x})$ is any scalar-valued function of space-time coordinates. The change (1.54) in $\mathbf{A}$ is called a **gauge transformation**.

Gauge transformations can be used to ensure that $\mathbf{A}$ satisfies an additional equation. In particular, given $\mathbf{A}$ we can choose Λ s.t. $\mathbf{A}'$ satisfies one of these gauge conditions:

(i) **Lorentz gauge**:[2]

$$\partial \cdot \mathbf{A}' = 0 \quad \Rightarrow \quad \Box\Lambda = \partial \cdot \mathbf{A} \tag{1.55}$$

The Lorentz condition (1.55) does not uniquely specify $\mathbf{A}'$ since many non-trivial functions satisfy $\Box\phi = 0$ and so given one Λ satisfying the 2$^{\rm nd}$ of eqs (1.55), we can construct many others $\Lambda' = \Lambda + \phi$.

(ii) **Coulomb** or **radiation** or transverse gauge

$$\nabla \cdot \boldsymbol{A}' = 0 \quad \Rightarrow \quad \nabla^2\Lambda = \nabla \cdot \boldsymbol{A} \tag{1.56}$$

In this gauge the 0$^{\rm th}$ eqn of the set $\partial^\nu F_{\mu\nu} = \mu_0 j_\mu$ reads

$$\begin{aligned} \frac{\rho}{c\epsilon_0} &= -\mu_0 j_0 = -\partial^\nu(\partial_0 A_\nu - \partial_\nu A_0) \\ &= -\partial_0\partial^\nu A_\nu + \partial^\nu\partial_\nu A_0 \\ &= -\partial_0\partial^0 A_0 + \partial^\nu\partial_\nu A_0 \\ &= \partial^i\partial_i A_0 \\ &= -\nabla^2\phi/c \end{aligned} \tag{1.57}$$

i.e., in this gauge the electrostatic potential satisfies Poisson's eqn, which explains the gauge's name.

1.7 Summary

The special theory of relativity requires that any physical quantity must fit into an n-tuple of numbers, where $n = 1, 4, 6, 10, \ldots$. Physical laws must be expressed as equations connecting the n-tuples associated with different physical quantities. These equations must be constructed in accordance with the rules of tensor calculus, which permit only:

(i) the multiplication of n-tuples to form either higher-rank n-tuples (as in $H_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$) or lower-rank n-tuples (as in $f_\mu = F_\mu{}^\nu J_\nu$), or

[2] We denote the d'Alembertian opertor by $\Box \equiv \partial_\mu\partial^\mu$ by analogy with the notation $\triangle \equiv \nabla^2 = \partial_i\partial^i$ for the Laplacian operator.

(ii) the addition of n-tuples *of the same rank.*

In particular, both sides of every acceptable equation always form valid n-tuples of the same kind.

Rest-mass, electric charge and total spin are scalars (1-tuples). The most important 4-vectors (4-tuples) include any particle's energy-momentum $\mathbf{p}$, e.m. current $\mathbf{J}$ or acceleration $\mathrm{d}\mathbf{p}/\mathrm{d}\tau$, and the potential $\mathbf{A}$ of the e.m. field. Important 6-tuples include any particle's angular momentum $\mathbf{H}$ and the Maxwell field tensor $\mathbf{F}$. An important 10-tuple is the density $\mathbf{T}$ of the energy-momentum due to the e.m. field.

In 4-vector notation the key equation of mechanics and e.m. are

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} \quad ; \quad \mathbf{p} = m_0\mathbf{v} \quad ; \quad \mathbf{J} = q\mathbf{v}$$

$$\mathbf{f} = \mathbf{F}\cdot\mathbf{J} \quad ; \quad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = \mathbf{f}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad ; \quad F^{\mu\nu}{}_{,\nu} = \mu_0 j_\mu \quad ; \quad \overline{F}^{\mu\nu}{}_{,\nu} = 0,$$

where $F^{\mu\nu} \equiv \eta^{\mu\gamma}\eta^{\nu\delta}F_{\gamma\delta}$ and $\overline{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\gamma\delta}F_{\gamma\delta}$. The energy-monentum tensor of the e.m. field is

$$T^{\mu\nu} = \frac{1}{\mu_0}\left[\tfrac{1}{4}\,\mathrm{Tr}(\mathbf{F}\cdot\mathbf{F})\eta^{\mu\nu} - F^{\mu}{}_{\gamma}F^{\gamma\nu}\right].$$

2 Groups & their representations

Rotations (and Lorentz transformations) form what mathematicians call a group because:

i) If you follow one rotation by another, the result could be achieved by a single rotation; in mathematical language, the product of two group members is itself a member of the group.

ii) Doing nothing can be considered to be a rotation about zero angle; in mathematical language there is an identity element I such that $IR = R$ for all group members R.

iii) Any rotation can be reversed, that is each rotation R has an inverse R^{-1} such that $R^{-1}R = I$.

The group of rotations is called the three-dimensional **special orthogonal group** or **SO(3)**.

If we are concerned with the effect of rotations on vectors, we associate each rotation with an orthogonal matrix such as

$$\mathbf{M}(\hat{\mathbf{k}}, \psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.1}$$

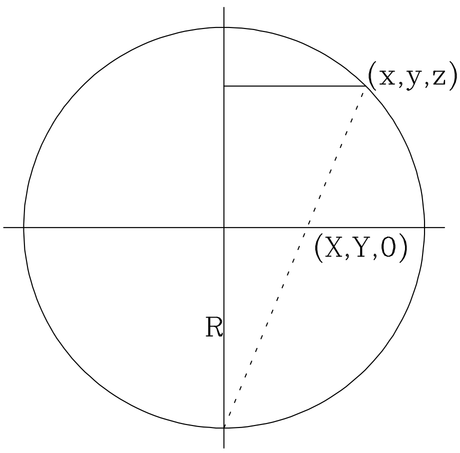
When these matrices are multiplied, we get the matrix associated with the product of the two rotations:

$$R_3 = R_2 R_1 \quad \leftrightarrow \quad \mathbf{M}_3 = \mathbf{M}_2 \mathbf{M}_1 \tag{2.2}$$

Matrices that are associated with all group members such that this relation holds, are said to form a **representation** of the group.

Arbitrarily many different representations of a group like SO(3) are possible. To widen our horizons away from 3×3 rotation matrices, consider the following scheme.

Each rotation moves points around spheres. Consider the sphere of radius R. Positions on this sphere are elegantly described by stereographically projecting points (x, y, z) on the sphere to points $(X, Y, 0)$ in the plane $z = 0$ as shown in the figure. (Stereographic projections are much used by crystallographers.)



The upper hemisphere is mapped to $X^2+Y^2 < R^2$, while the lower hemisphere is mapped to the rest of the XY plane. Suppose $y = Y = 0$. Then from the triangles $x = X(R + z)/R$. Using this to eliminate x from the equation of the circle we get a quadratic in z with solution $z = R(R^2 - X^2)/(R^2 + X^2)$. Back-substituting we then get $x = 2XR^2/(R^2 + X^2)$.

We define $\zeta \equiv (X + \mathrm{i}Y)/R$. It's clear that the phase of ζ will be the same as the phase of $x + \mathrm{i}y$. So from $X^2 + Y^2 = R^2\zeta\zeta^*$ and the results we already have, it follows that

$$x + \mathrm{i}y = 2R\frac{\zeta}{1+\zeta\zeta^*} \quad ; \quad z = R\frac{1-\zeta\zeta^*}{1+\zeta\zeta^*}. \tag{2.3}$$

Writing $\zeta = \eta_2/\eta_1$, we have

$$x + \mathrm{i}y = 2R\frac{\eta_2\eta_1^*}{|\eta_1|^2 + |\eta_2|^2} \quad ; \quad z = R\frac{|\eta_1|^2 - |\eta_2|^2}{|\eta_1|^2 + |\eta_2|^2}. \tag{2.4}$$

We fix the length of the complex 2-vector (**Pauli spinor**) $\boldsymbol{\eta} \equiv (\eta_1, \eta_2)$ by setting $R = |\eta_1|^2 + |\eta_2|^2$ so we have simply

$$x + \mathrm{i}y = 2\eta_2\eta_1^* \quad ; \quad z = |\eta_1|^2 - |\eta_2|^2. \tag{2.5}$$

A unitary transformation $\boldsymbol{\eta} \to \boldsymbol{\eta}' \equiv \mathbf{U}\cdot\boldsymbol{\eta}$ leaves the normalization invariant and through equations (2.5) generates a new point on the sphere. We can show (see Problems) that a given unitary transformation leaves invariant the distance between different points on the sphere, so the transformation is a rotation, potentially plus an inversion. Conversely, any rotation of the sphere transforms $\boldsymbol{\eta}$ into some other spinor $\boldsymbol{\eta}'$ in a unitary way. Thus a rotation is associated with each 2×2 unitary matrix $\mathbf{U}$, and *any* rotation is generated by some $\mathbf{U}$.

Exercise (9):

Show that

$$x = \boldsymbol{\eta}^\dagger\sigma_x\boldsymbol{\eta} \qquad y = \boldsymbol{\eta}^\dagger\sigma_y\boldsymbol{\eta} \qquad z = \boldsymbol{\eta}^\dagger\sigma_z\boldsymbol{\eta}, \tag{2.6a}$$

where $\boldsymbol{\eta}^\dagger$ is the complex-conjugate-transpose of $\boldsymbol{\eta}$ and

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma_y \equiv \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad ; \quad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.6b}$$

are the **Pauli spin matrices**. Notice that they are Hermitian and that $[\sigma_i, \sigma_j] = 2\mathrm{i}\epsilon_{ijk}\sigma_k$.

Bearing in mind that $|\eta_2|^2 = R - |\eta_1|^2$, let's arrange the orginal coordinates into a matrix:

$$\mathbf{X} \equiv \tfrac{1}{2}\begin{pmatrix} z & x - \mathrm{i}y \\ x + \mathrm{i}y & -z \end{pmatrix} = \begin{pmatrix} |\eta_1|^2 - \frac{1}{2}R & \eta_1\eta_2^* \\ \eta_2\eta_1^* & |\eta_2|^2 - \frac{1}{2}R \end{pmatrix}, \tag{2.7}$$

which can also be written

$$X_{ij} = \eta_i\eta_j^* - \tfrac{1}{2}R\delta_{ij}. \tag{2.8}$$

The transformation $\boldsymbol{\eta} \to \widetilde{\boldsymbol{\eta}} \equiv \mathbf{U}\cdot\boldsymbol{\eta}$ maps $\mathbf{X} \to \widetilde{\mathbf{X}}$ where

$$\widetilde{X}_{ij} = U_{ik}\eta_k(U_{jl}\eta_l)^* - \tfrac{1}{2}R\delta_{ij} = U_{ik}(\eta_k\eta_l^* - \tfrac{1}{2}R\delta_{kl})U^\dagger_{lj} \quad \text{i.e.} \quad \widetilde{\mathbf{X}} = \mathbf{U}\mathbf{X}\mathbf{U}^\dagger. \tag{2.9}$$

To this point we have confined ourselves to unitary matrices in order to preserve the normalization $|\boldsymbol{\eta}|^2 = R$. However, a general linear transformation $\boldsymbol{\eta} \to \widetilde{\boldsymbol{\eta}} = \mathbf{M}\boldsymbol{\eta}$ induces the transformation

$$\eta_i\eta_j^* = X_{ij} + \tfrac{1}{2}R\delta_{ij} \to \widetilde{\eta}_i\widetilde{\eta}_j^* = \tfrac{1}{2}\mathbf{M}\begin{pmatrix} R+z & x-\mathrm{i}y \\ x+\mathrm{i}y & R-z \end{pmatrix}\mathbf{M}^\dagger = \tfrac{1}{2}\begin{pmatrix} R'+z' & x'-\mathrm{i}y' \\ x'+\mathrm{i}y' & R'-z' \end{pmatrix}. \tag{2.10}$$

If we impose the restriction $\det(\mathbf{M}) = \pm 1$, we will be making a transformation such that $R'^2 - x'^2 - y'^2 - z'^2 = R^2 - x^2 - y^2 - z^2$. Hence, if we set $R = ct$, we will be performing a Lorentz transformation. The 2×2 complex matrices with unit determinant are considered to form the group SL(2,C) (SL = special linear).

Exercise (10):

Show that with $R = ct$ we can complement equations (2.6a) with

$$ct = \boldsymbol{\eta}^\dagger I \boldsymbol{\eta}. \tag{2.11}$$

The rotations are the sub-group of the Lorentz group that are obtained by requiring $\mathbf{M}$ to be not merely of unit determinant, but unitary. The 2×2 unitary matrices with unit determinant form the group SU(2). Thus we have shown that SL(2,C) can be mapped into the Lorentz group, and SU(2) can be mapped onto SO(3).

Notice that these transformations cannot change the sign of of $R = ct$, so they do not include reversals of time. It turns out that they do not include inversions of space either. The mappings are not 1-1 because $-\mathbf{M}$ induces the same transformation of space-time as does $\mathbf{M}$. So we have found a representation of the subgroup of **proper orthochronous Lorentz transformations** or **proper Lorentz group** for short.

In classical physics spinors are no more than mathematical devices. But the amplitudes $a_\pm$ for a spin-half particle to have its spin up or down along any chosen axis transform under Lorentz transformations like the components of a spinor.

2.1 Generators

It's easy to show that the (Hermitian) Pauli matrices [eq. (2.6b)] all square up to the identity matrix: $\sigma_i^2 = I$. Let $\mathbf{n}$ be a unit vector, then this property applies equally to the matrix

$$\sigma_{\mathbf{n}} \equiv \mathbf{n}\cdot\boldsymbol{\sigma} = \begin{pmatrix} n_z & n_x - \mathrm{i}n_y \\ n_x + \mathrm{i}n_y & -n_z \end{pmatrix}. \tag{2.12}$$

We define the exponential of $\mathrm{i}\theta\sigma_{\mathbf{n}}$ through the power series

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\theta\sigma_{\mathbf{n}}} &= I + \mathrm{i}\theta\sigma_{\mathbf{n}} - \frac{\theta^2}{2!}\sigma_{\mathbf{n}}^2 - \mathrm{i}\frac{\theta^3}{3!}\sigma_{\mathbf{n}}^3 + \cdots \\ &= \Big(1 - \frac{\theta^2}{2!} + \cdots\Big)I + \mathrm{i}\Big(\theta - \frac{\theta^3}{3!} + \cdots\Big)\sigma_{\mathbf{n}} \\ &= \cos\theta\, I + \mathrm{i}\sin\theta\,\sigma_{\mathbf{n}}. \end{aligned} \tag{2.13}$$

Now for any θ, $e^{i\theta\sigma_{\mathbf{n}}}$ is a unitary matrix:

$$\left(e^{i\theta\sigma_{\mathbf{n}}}\right)^{\dagger} e^{i\theta\sigma_{\mathbf{n}}} = \left(\cos\theta\, I - i\sin\theta\,\sigma_{\mathbf{n}}\right)\left(\cos\theta\, I + i\sin\theta\,\sigma_{\mathbf{n}}\right) = \left(\cos^2\theta + \sin^2\theta\right) I. \quad (2.14)$$

Moreover, $e^{i\theta\sigma_{\mathbf{n}}}$ contains three free parameters (θ and the two angles required to specify the direction $\mathbf{n}$). Given that any rotation can be specified by three parameters (for example the Euler angles), we might suspect that the unitary matrix required to generate any rotation can be obtained as $e^{i\theta\sigma_{\mathbf{n}}}$ for appropriate θ and $\mathbf{n}$. In fact, $e^{i\theta\sigma_{\mathbf{n}}}$ is the matrix that rotates the coordinates by angle -2θ about the axis $\mathbf{n}$ – as one may easily verify when $\mathbf{n}$ is one of the coordinate vectors $\mathbf{i}$, $\mathbf{j}$ or $\mathbf{k}$.

Exercise (11):

Show that "rotating" $\boldsymbol{\eta}$ with the matrix

$$s_z(\phi) \equiv \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix} \quad (2.15)$$

has the effect of rotating the (x, y, z) coordinates through ϕ about the z axis. What happens to $\boldsymbol{\eta}$ when the (x, y, z) axes are rotated through 2π?

Since the Pauli matrices enable us to generate any member of SU(2) through this mechanism, we refer to them as the **generators** of SU(2). (To be pedantic, the generators are $\frac{1}{2}\sigma_i$.)

Exponentiating $\theta\sigma_{\mathbf{n}}$ we obtain

$$\begin{aligned} e^{\theta\sigma_{\mathbf{n}}} &= \left(1 + \frac{\theta^2}{2!} + \cdots\right) I + \left(\theta + \frac{\theta^3}{3!} + \cdots\right)\sigma_{\mathbf{n}} \\ &= \cosh\theta\, I + \sinh\theta\,\sigma_{\mathbf{n}}. \end{aligned} \quad (2.16)$$

The determinant of this matrix is 1:

$$\begin{aligned} |\cosh\theta\, I + \sinh\theta\,\sigma_{\mathbf{n}}| &= \begin{vmatrix} \cosh\theta + n_z\sinh\theta & (n_x - in_y)\sinh\theta \\ (n_x + in_y)\sinh\theta & \cosh\theta - n_z\sinh\theta \end{vmatrix} \\ &= \cosh^2\theta - n_z^2\sinh^2\theta - (n_x^2 + n_y^2)\sinh^2\theta = 1. \end{aligned} \quad (2.17)$$

Hence through (2.10) $e^{\theta\sigma_{\mathbf{n}}}$ generates a Lorentz transformation. To see which transformation we align the z axis with $\mathbf{n}$. Then $e^{\theta\sigma_{\mathbf{n}}}$ is a diagonal matrix and

$$\begin{aligned} \begin{pmatrix} ct' + z' & x' - iy' \\ x' + iy' & ct' - z' \end{pmatrix} &= e^{\theta\sigma_{\mathbf{n}}} \begin{pmatrix} ct + z & x - iy \\ x + iy & ct - z \end{pmatrix} \begin{pmatrix} \cosh\theta + \sinh\theta & 0 \\ 0 & \cosh\theta - \sinh\theta \end{pmatrix} \\ &= \begin{pmatrix} \cosh\theta + \sinh\theta & 0 \\ 0 & \cosh\theta - \sinh\theta \end{pmatrix} \begin{pmatrix} (ct + z)(\cosh\theta + \sinh\theta) & (x - iy)(\cosh\theta - \sinh\theta) \\ (x + iy)(\cosh\theta + \sinh\theta) & (ct - z)(\cosh\theta - \sinh\theta) \end{pmatrix} \\ &= \begin{pmatrix} (ct + z)(\cosh\theta + \sinh\theta)^2 & (x - iy) \\ (x + iy) & (ct - z)(\cosh\theta - \sinh\theta)^2 \end{pmatrix} \end{aligned} \quad (2.18)$$

From the off-diagonal components of this equation, $x' = x$, $y' = y$. Adding and subtracting the diagonal components we learn that

$$\begin{aligned} ct' &= ct(\cosh^2\theta + \sinh^2\theta) + z2\sinh\theta\cosh\theta \\ z' &= ct2\sinh\theta\cosh\theta + z(\cosh^2\theta + \sinh^2\theta) \end{aligned} = \begin{pmatrix} \cosh 2\theta & \sinh 2\theta \\ \sinh 2\theta & \cosh 2\theta \end{pmatrix} \begin{pmatrix} ct \\ z \end{pmatrix}. \tag{2.19}$$

Thus $e^{\theta\sigma_{\mathbf{n}}}$ generates the boost along $\mathbf{n}$ with Lorentz factor $\gamma = \cosh 2\theta$ and speed $\beta = \tanh 2\theta$. We say that $\mathrm{i}\frac{1}{2}\sigma_{\mathbf{n}}$ is the generator of this Lorentz transformation.

The boosts taken on their own do not form a group because the product of boosts along two non-parallel axes cannot always be expressed as a boost along a third axis: in general a rotation is required in addition to a boost.[3] As a specific example, consider the product

$$A \equiv e^{-\theta\sigma_y} e^{-\phi\sigma_x} e^{\theta\sigma_y} e^{\phi\sigma_x}, \tag{2.20}$$

which effects a boost along the x axis, followed by one along the y axis, followed by inverse bosts along the x and then the y axes. For infinitesimal θ, ϕ we have

$$\begin{aligned} B_\pm \equiv e^{\pm\theta\sigma_y} e^{\pm\phi\sigma_x} &= (I \pm \theta\sigma_y + \tfrac{1}{2}\theta^2 I + \cdots)(I \pm \phi\sigma_x + \tfrac{1}{2}\phi^2 I + \cdots) \\ &= [1 + \tfrac{1}{2}(\theta^2 + \phi^2)]I \pm [\theta\sigma_y + \phi\sigma_x] + \theta\phi\sigma_y\sigma_x + \cdots \end{aligned} \tag{2.21}$$

Hence

$$\begin{aligned} A = B_- B_+ &\simeq \{[I + \tfrac{1}{2}(\theta^2 + \phi^2)I + \theta\phi\sigma_y\sigma_x] - [\theta\sigma_y + \phi\sigma_x]\} \\ &\qquad \times \{[I + \tfrac{1}{2}(\theta^2 + \phi^2)I + \theta\phi\sigma_y\sigma_x] + [\theta\sigma_y + \phi\sigma_x]\} + \cdots \\ &= [I + \tfrac{1}{2}(\theta^2 + \phi^2)I + \theta\phi\sigma_y\sigma_x]^2 - [\theta\sigma_y + \phi\sigma_x]^2 + \mathrm{O}(\theta^3) \\ &= I + \theta\phi[\sigma_y, \sigma_x] + \mathrm{O}(\theta^3) = I - 2\mathrm{i}\theta\phi\sigma_z + \mathrm{O}(\theta^3) \end{aligned} \tag{2.22}$$

Thus this sequence of boosts effects a rotation by angle $4\theta\phi$ around z. Consequently, boosts are inextricably intertwined with rotations, and we must consider the form taken by a general Lorentz transformation, that is, a transformation that combines a boost with a rotation. The natural object to consider is

$$M \equiv e^{(\mathrm{i}\theta\mathbf{n} + \phi\mathbf{m})\cdot\boldsymbol{\sigma}}, \tag{2.23}$$

which combines a boost along $\mathbf{m}$ with a rotation around $\mathbf{n}$. A 2×2 complex matrix is defined by eight real numbers, and when we require the matrix to have unit determinant, we impose two restrictions on these numbers, leaving six degrees of freedom. Equation (2.23) for M has six parameters, so *any* matrix with unit determinant should be of this form. Consequently, the product of two objects of this type will be a third object of the same type, so these objects provide a representation of the proper Lorentz group.

Remarkably, (2.23) combines the pseudo-vector $\mathbf{n}$ with the polar vector $\mathbf{m}$. If we transform to axes that are mirror images of our original axes, $\mathbf{n}$ won't change sign, but $\mathbf{m}$ will, and M will change into

$$M' \equiv e^{(\mathrm{i}\theta\mathbf{n} - \phi\mathbf{m})\cdot\boldsymbol{\sigma}}. \tag{2.24}$$

[3] This phenomenon is the origin of Thomas precession in the theory of spin-orbit coupling.

It follows that the objects M' must also provide a representation of the proper Lorentz group. The representations provided by M and M' are inequivalent in the sense that there is no matrix S such that $M' = SMS^{-1}$ for all M.

There are two types of Pauli spinors. A right-handed Pauli spinor $\boldsymbol{\eta}_R$ is transformed by M under a Lorentz transformation, while a left-handed one $\boldsymbol{\eta}_L$ is transformed by M':

$$\boldsymbol{\eta}_R \to \widetilde{\boldsymbol{\eta}}_R = \mathrm{e}^{(\mathrm{i}\theta\mathbf{n}+\phi\mathbf{m})\cdot\boldsymbol{\sigma}}\boldsymbol{\eta}_R \quad ; \quad \boldsymbol{\eta}_L \to \widetilde{\boldsymbol{\eta}}_L = \mathrm{e}^{(\mathrm{i}\theta\mathbf{n}-\phi\mathbf{m})\cdot\boldsymbol{\sigma}}\boldsymbol{\eta}_L. \tag{2.25}$$

Under a coordinate inversion a right-handed spinor transforms into a left-handed one, and vice versa. Consequently, the Pauli spinors of one type do not support a representation of the full Lorentz group (the group you get by adding inversion through the origin and time reversal to the proper Lorentz group). A **Dirac spinor** is a pair of spinors, one of each type:

$$\boldsymbol{\psi} = (\boldsymbol{\eta}_R, \boldsymbol{\eta}_L). \tag{2.26}$$

It has four components, the first two being the components of $\boldsymbol{\eta}_R$, etc. We represent a coordinate inversion by the operation of swapping $\boldsymbol{\eta}_R$ with $\boldsymbol{\eta}_L$. This convention makes sense because after a coordinate inversion $\boldsymbol{\eta}_R$ remains a right-handed Pauli spinor, but because we are now using a left-handed coordinate system, its transformation rule is the one we previously associated with a left-handed spinor. By moving $\boldsymbol{\eta}_R$ to the lower slot in $\boldsymbol{\psi}$ we arrange that we don't have to change the transformation rules we apply to the top & bottom slots. In summary, a coordinate inversion reprsented by

$$\boldsymbol{\psi} \to \widetilde{\boldsymbol{\psi}} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\eta}_R \\ \boldsymbol{\eta}_L \end{pmatrix} \quad \text{or} \quad \widetilde{\boldsymbol{\psi}} = \gamma^0 \boldsymbol{\psi} \quad \text{where} \quad \gamma^0 \equiv \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{2.27}$$

with I the 2×2 identity matrix. In this way Dirac spinors support a representation of the full Lorentz group.

2.2 Spinor invariants

When we do Lagrangian field theory, we'll be interested in Lorentz invariants. So now we ask what invariants we can make out of spinors. If the Lorentz transformation matrices M and M' were unitary (as they are for a pure rotation; $\phi = 0$), $\boldsymbol{\eta}^\dagger \cdot \boldsymbol{\eta}$ would be a Lorentz invariant. But in the presence of a non-zero boost, M and M' are not unitary. Taking the Hermitian adjoints of equations (2.25) we find

$$\boldsymbol{\eta}_R^\dagger \to \widetilde{\boldsymbol{\eta}}_R^\dagger = \boldsymbol{\eta}_R^\dagger \mathrm{e}^{(-\mathrm{i}\theta\mathbf{n}+\phi\mathbf{m})\cdot\boldsymbol{\sigma}} \quad ; \quad \boldsymbol{\eta}_L^\dagger \to \widetilde{\boldsymbol{\eta}}_L^\dagger = \boldsymbol{\eta}_L^\dagger \mathrm{e}^{(-\mathrm{i}\theta\mathbf{n}-\phi\mathbf{m})\cdot\boldsymbol{\sigma}} \tag{2.28}$$

From equations (2.25) and (2.28) we see that under proper Lorentz transformations both $\boldsymbol{\eta}_L^\dagger \cdot \boldsymbol{\eta}_R$ and $\boldsymbol{\eta}_R^\dagger \cdot \boldsymbol{\eta}_L$ are invariant. To obtain a quantity that's still invariant when inversions are included, we add these two invariants. In terms of the **adjoint spinor**

$$\overline{\boldsymbol{\psi}} \equiv \boldsymbol{\psi}^\dagger \gamma^0 = (\boldsymbol{\eta}_L^\dagger, \boldsymbol{\eta}_R^\dagger) \tag{2.29}$$

our invariant is

$$\overline{\boldsymbol{\psi}} \cdot \boldsymbol{\psi} = \boldsymbol{\eta}_L^\dagger \cdot \boldsymbol{\eta}_R + \boldsymbol{\eta}_R^\dagger \cdot \boldsymbol{\eta}_L. \tag{2.30}$$

We'll also find it useful to know how to construct a 4-vector from a Dirac spinor. Equations (2.6a) and (2.11) imply that under rotations $\boldsymbol{\eta}^\dagger \boldsymbol{I} \boldsymbol{\eta}$, $\boldsymbol{\eta}^\dagger \sigma_x \boldsymbol{\eta}$, $\boldsymbol{\eta}^\dagger \sigma_y \boldsymbol{\eta}$,and $\boldsymbol{\eta}^\dagger \sigma_z \boldsymbol{\eta}$ transform like the components of a four vector. How should we generalize these expressions to the case in which right- and left-handed spinors are distinguishable because boosts occur? A component of a vector should not be invariant, so contrary to what happens in equation (2.30), the left and right spinors should be of the same handedness. But both halves of the Dirac spinor must be used. Moreover, under interchange of $\boldsymbol{\eta}_L$ and $\boldsymbol{\eta}_R$ the time component should stay the same, while the space components should change sign. This suggests that the time component is $\boldsymbol{\eta}_R^\dagger \cdot \boldsymbol{\eta}_R + \boldsymbol{\eta}_L^\dagger \cdot \boldsymbol{\eta}_L$ while the space components are $\boldsymbol{\eta}_R^\dagger \sigma_i \boldsymbol{\eta}_R - \boldsymbol{\eta}_L^\dagger \sigma_i \boldsymbol{\eta}_L$. To achieve this result in an elegant notation we define three new matrices

$$\gamma^1 \equiv \begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix} \quad ; \quad \gamma^2 \equiv \begin{pmatrix} 0 & -\sigma_y \\ \sigma_y & 0 \end{pmatrix} \quad ; \quad \gamma^3 \equiv \begin{pmatrix} 0 & -\sigma_z \\ \sigma_z & 0 \end{pmatrix}. \tag{2.31}$$

Then bearing in mind the definitions (2.27) and (2.29) of γ^0 and $\overline{\boldsymbol{\psi}}$, we have that

$$\overline{\boldsymbol{\psi}}\gamma^0\boldsymbol{\psi} = (\boldsymbol{\eta}_L^\dagger, \boldsymbol{\eta}_R^\dagger)(\boldsymbol{\eta}_L, \boldsymbol{\eta}_R) \quad \overline{\boldsymbol{\psi}}\gamma^i\boldsymbol{\psi} = (\boldsymbol{\eta}_L^\dagger, \boldsymbol{\eta}_R^\dagger)(-\sigma_i\boldsymbol{\eta}_L, \sigma_i\boldsymbol{\eta}_R) \tag{2.32}$$

as required – so $\overline{\boldsymbol{\psi}}\gamma^\mu\boldsymbol{\psi}$ is a 4-vector.

Exercise (12):
Show that $\gamma^0\gamma^i = -\gamma^i\gamma^0$ and that $\gamma^i\gamma^j = -\gamma^j\gamma^i$. (This anticommutation property is often written $\{\gamma^\mu, \gamma^\nu\} = 0$.)

The spinor representation of the Lorentz group is fundamental in the sense that every other representation can be constructed from it. We started by studying an example of this phenomenon: the components of a second-rank tensor in spinor space transform like the combinations $ct + z$, $x - \mathrm{i}y$, etc, of the components of a 4-vector. From the rule for transforming third-rank tensors on spinor space, we could extract the spin-$\frac{3}{2}$ representation of the Lorentz group, and so on. This corner of group theory is taught in quantum-mechanics courses under the heading of 'addition of angular momenta'. The total spin angular momentum of two spin-half particles can be zero (spin-0 representation of the LG) or one (spin-1 rep.). With three spin-half particles the possible spin angular momenta are $\frac{3}{2}$, and $\frac{1}{2}$.

The spin-n representations of the Lorentz group have a special property: they are **irreducible** (or an **irrep** for short) in the sense that no linear subspace of the representing space is invariant under the action of the matrices of the representation.

3 Lagrangian Dynamics

The sharp predictions that are characteristic of classical physics arise because destructive quantum interference excludes practically every future configuration of a system:

Box 1: Functionals and the Euler–Lagrange Equations

Let $y(t)$ be a function of the scalar parameter t. Then a functional $F[y(t)]$ is some rule that assigns to each function y a single number. For example F might be $F_1 \equiv \int_{t_1}^{t_2} \mathrm{d}t\, y(t)$ or $F_2 \equiv \int_{t_1}^{t_2} \mathrm{d}t\, y(t)\dot{y}(t)$ or $F_K \equiv \int_{t_1}^{t_2} \mathrm{d}t\, K(t)y(t)$, where $K(t)$ is any given function, or $F_{ab} \equiv y(a) - y(b)$, where a and b are any two given values of t. The function $y(t)$ may be scalar-, vector- or even tensor-valued. Vector-valued functions $\mathbf{y}(t)$ can be thought of as **paths**.

Physicists are particularly interested in extremizing functions of the type

$$F[y(t)] = \int \mathrm{d}t\, f(y, \dot{y}), \tag{B1.1}$$

where f is a known function of two variables. That is, they wish to find the function $y(t)$ such that $F[y(t)]$ takes a larger/smaller value than all nearby functions. The **calculus of variations** shows that the extremizing function is the one that satisfies the **Euler–Lagrange (EL)** equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{y}}\right) - \frac{\partial f}{\partial y} = 0. \tag{B1.2}$$

For given f this is an o.d.e. for $y(t)$.

a shell will blast through one spot on the roof of a dugout because it is at this spot alone that the quantum amplitudes for the shell's presence interfere constructively. Even in classical physics the most elegant way to do dynamics is to write down an expression for the phase of this amplitude for each path by which the system might travel between initial and final configuration, and find for what path it is stationary and constructive interference is possible.

This phase times $\hbar$ is called the **action** S. It is a scalar and is obtained by integrating along the prospective path the rate of change of phase with proper time, s:

$$S = \int \mathrm{d}\tau s. \tag{3.1}$$

Since S and τ are scalars, s must be too.

3.1 Single charged particle with given e.m. field

To determine s we have only to ask what scalars can be constructed from the world-line $\mathbf{x}(\tau)$ and quantities such as $\mathbf{A}$, $\mathbf{F}$ associated with the e.m. field.

First we note that S shouldn't depend on our choice of origin, so only derivatives $\dot{\mathbf{x}}$, $\ddot{\mathbf{x}}$ etc should occur in s, not $\mathbf{x}$ itself. Furthermore, the EL eqn (Box 1) involves differentiation with respect to the variable that parameterizes position along the extremal path, in this case τ. So we will get as 2^{nd}-order eqn of motion, if s depends on $\dot{\mathbf{x}}$, but not on higher derivatives of $\mathbf{x}(\tau)$. Similarly, the EL eqn involves differentiation w.r.t. the general position vector $\mathbf{x}$, so if the eqn of motion is to depend on $\mathbf{F}$ and not its derivatives, s should depend on $\mathbf{A}$ but not $\mathbf{F}$. So the invariants to consider are (i) $|\dot{\mathbf{x}}|^2 = -c^2$, (ii) $\dot{\mathbf{x}} \cdot \mathbf{A}$ and (iii) $|\mathbf{A}|^2$. We further require that any gauge-dependent contribution to S should be path-independent. $\dot{\mathbf{x}} \cdot \mathbf{A}$ satisfies this requirement, while $|\mathbf{A}|^2$ does not.

Exercise (13):

Show that the gauge-dependent contribution to S from $\dot{\mathbf{x}} \cdot \mathbf{A}$ is path-independent, while the gauge-dependent contribution from a term proportional to $|\mathbf{A}|^2$ would not be path-independent.

So the simplest thing to try is

$$S = \int \mathrm{d}\tau \, (-m_0 c^2 + q\dot{\mathbf{x}} \cdot \mathbf{A}), \tag{3.2}$$

where we've included the rest mass m_0 for future convenience and q is some constant.

Unfortunately we cannot apply the EL eqn (Box 1) to (3.2) as it stands because we want to hold constant the events of arrival and departure, $\mathbf{x}_1$ and $\mathbf{x}_2$, rather than the proper-time elapse between these events. So we have first to eliminate τ from (3.2) in favour of some parameter λ that always runs over the same range, say, 0 to 1. Using

$$\frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = \frac{1}{c}\sqrt{-\left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\lambda}\right|}, \tag{3.3}$$

we have

$$S = \int_0^1 \mathrm{d}\lambda \, \left(-m_0 c \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} + q\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}A_\mu \right), \tag{3.4}$$

which is now in a form that to which we can apply the EL eqn. Since

$$\frac{\partial}{\partial \dot{x}^\beta}\sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} = -\frac{\eta_{\beta\nu}\dot{x}^\nu + \eta_{\mu\beta}\dot{x}^\mu}{2\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}} = \frac{-\dot{x}_\beta}{\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}} = -\frac{\mathrm{d}x_\beta/\mathrm{d}\lambda}{c\mathrm{d}\tau/\mathrm{d}\lambda},$$

the EL eqn yields

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(m_0 \frac{\mathrm{d}x_\beta}{\mathrm{d}\tau} + qA_\beta \right) - q\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\partial A_\mu}{\partial x^\beta} = 0. \tag{3.5}$$

Multiplying through by $\mathrm{d}\lambda/\mathrm{d}\tau$ this becomes

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(m_0 \frac{\mathrm{d}x_\beta}{\mathrm{d}\tau} + qA_\beta \right) - q\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\partial A_\mu}{\partial x^\beta} \\ &= m_0\frac{\mathrm{d}v_\beta}{\mathrm{d}\tau} + q\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\left(\frac{\partial A_\beta}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\beta} \right) \\ &= m_0\frac{\mathrm{d}v_\beta}{\mathrm{d}\tau} + q\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}F_{\mu\beta}. \end{aligned} \tag{3.6}$$

Thus our action gives the required equation of motion.

Since $A_\mu = (-\phi/c, \boldsymbol{A})$, with $\lambda = t$, (3.4) can be written

$$S = \int \mathrm{d}t \left[-m_0^2 c^2 \sqrt{1 - v^2/c^2} + q(-\phi + \boldsymbol{v}\cdot\boldsymbol{A})\right]$$

If the field is electrostic ($\boldsymbol{A} = 0$) and the motion is non-relativistic, the action is

$$S \simeq \int \mathrm{d}t \left[- m_0 c^2 + L(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)\right], \quad \text{where} \quad L(\boldsymbol{x}, \dot{\boldsymbol{x}}) \equiv \tfrac{1}{2} m_0 \dot{\boldsymbol{x}}^2 - q\phi(\boldsymbol{x}, t). \tag{3.7}$$

Since $\int \mathrm{d}t\, m_0 c^2$ is the same for all paths that start and finish at the given events, it plays no role in picking out the true path. So it can be dropped, and we obtain the familiar **principle of least action**:

$$\delta S = 0 \quad \text{where} \quad S \equiv \int \mathrm{d}t\, L(\boldsymbol{x}, \dot{\boldsymbol{x}}, t). \tag{3.8}$$

The function L is called the **Lagrangian**. By (3.7) it is in this case the difference between the particle's kinetic and potential energies.

Starting with an action has many advantages:

- Since L is a scalar, transforming to new coordinates is easy;
- It's easy to ensure that the eqns of motion are Lorentz invariant (or Gallilean invariant as appropriate) by imposing the desired invariance on L;
- Given the required invariance and the basic form of the desired eqns (second-order, linear, say) only a few simple expressions are candidates for Lagrangians;
- Certain constants of motion can be readily derived from evident symmetries of L (Noether's theorem).

3.2 Principles of Lagrangian field theory

How do we obtain partial differential eqns such as the wave eqn or Maxwell's eqns from Lagrangians? Specimen problem: derive the wave eqn

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 0. \tag{3.9}$$

Regard $\phi(t, x)$ as a set of ∞-dimensional vectors $\phi_x(t)$, where x labels components of ϕ. The Lagrangian has to be a scalar, so ϕ's indices have to be 'soaked up' somehow. We make a scalar out of an ordinary vector by dotting it with another vector—this soaks up the indices of both vectors by introducing a sum over that index. Analogously, we soak up indices x with generalizations of dot products; that is, one sums over x by means of an integral:

$$s = \mathbf{a}\cdot\mathbf{b} = \sum_i a_i b_i \quad \leftrightarrow \quad s = (\psi, \phi) = \int \mathrm{d}x\, \psi(x)\phi(x). \tag{3.10}$$

This leads one to expect that many (but not all) actions for partial differential equations are evaluated by integrating a Lagrangian density $\mathcal{L}$ over space before performing the usual integral over time:

$$S[\phi] = \int \mathrm{d}t \int \mathrm{d}x\, \mathcal{L}(\phi, \dot{\phi}). \tag{3.11}$$

In Lagrangian mechanics, S is a functional of the particle's history $x(t)$. Now S is a functional of the field's history $\phi(t,x)$. So ϕ has stepped into x's place, and x has become an independent variable with a similar standing to that of t. Consequently, in (3.11) we're integrating over both space and time.

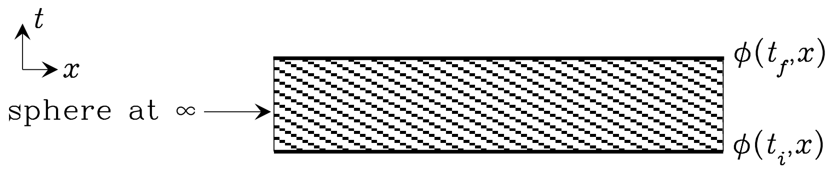
In order to make the symmetry between x and t complete we henceforth allow $\mathcal{L}$ to involve derivatives w.r.t. x as well as w.r.t. t; then $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$ and

$$S[\phi] = \int \mathrm{d}t\, \mathrm{d}x\, \mathcal{L}(\phi, \partial_\mu\phi). \tag{3.12}$$

Finally, it doesn't make things significantly more complicated to allow space to be fully three-dimensional. So x becomes the 3-vector $\boldsymbol{x}$ and $(ct, \boldsymbol{x})$ becomes the usual 4-vector $\mathbf{x}$. Since $\mathrm{d}^4\mathbf{x} = c\mathrm{d}t\mathrm{d}^3\boldsymbol{x}$, we write simply

$$S[\phi] = \frac{1}{c}\int \mathrm{d}^4\mathbf{x}\, \mathcal{L}(\phi, \partial_\mu\phi). \tag{3.13}$$

At each t between t_i and t_f the field's configuration $\phi(t, \boldsymbol{x})$ is chosen such that the integral (3.13) through the space-time volume bounded by $t = t_i$ and $t = t_f$ is extremized:



As in Lagrangian mechanics we are specifying a solution to the 2[nd] order equations of motion by giving values of the 'coordinates' at two times, t_i and t_f, rather than the coordinates and velocities at a single time. In this case specifying the 'coordinates' involves giving the functional dependence of ϕ on $\mathbf{x}$ at some fixed t.

Here's how we extremize S:

$$\begin{aligned} 0 = \delta S &= S[\phi + \psi] - S[\phi] \quad \text{where} \quad |\psi(t, \boldsymbol{x})| \ll |\phi(t, \boldsymbol{x})| \\ &\simeq \frac{1}{c}\int \mathrm{d}^4\mathbf{x} \left(\frac{\partial \mathcal{L}}{\partial \phi}\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\psi \right) \\ &= \frac{1}{c}\int \mathrm{d}^4\mathbf{x} \left(\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \right)\psi + \oint \mathrm{d}^3\mathbf{x}_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\psi. \end{aligned} \tag{3.14}$$

Here the final integral $\oint$ is over the closed 3-surface that bounds the 4-dimensional region of space-time through which $\mathcal{L}$ is integrated. The surface consists of the initial and final hypersurfaces, and the 3-surface swept out by a 2-surface at spatial ∞ as t

varies from t_i to t_f. This integral vanishes because ψ is zero throughout the domain integrated over: the variation ψ vanishes on the initial and final hypersurfaces by hypothesis, and we force it to vanish at spatial ∞ also in order to ensure that the varied field $\phi + \psi$ satisfies the same bdy condition as the unvaried field ϕ. Thus

$$\delta S = \frac{1}{c} \int \mathrm{d}^4\mathbf{x} \left(\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \psi \tag{3.15}$$

If this is to hold for any $\psi(t, \boldsymbol{x})$ that vanishes on the initial and final hypersurfaces, we clearly require that

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0. \tag{3.16}$$

This p.d.e. is the Euler-Lagrange equation for a field. It is the field equation that follows from the Lagrangian density $\mathcal{L}$.

3.3 Real scalar field

What p.d.e.s can we derive from a Langrangian density for a real scalar field ϕ? The scalars to consider are ϕ itself and powers of ϕ. The only way to make a scalar out of the gradient $\partial_\mu \phi$ is to contract it on itself. Consider therefore

$$\mathcal{L} = \tfrac{1}{2}(-|\partial\phi|^2 - K^2\phi^2) = \tfrac{1}{2}\left(-\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - K^2\phi^2\right), \tag{3.17}$$

where the sign of $|\partial\phi|^2$ has been chosen so that its contributions to $\mathcal{L}$ are k.e. – p.e. and the term with the constant K is the field's self-energy. Then $\partial\mathcal{L}/\partial\phi = -K^2\phi$ and $\partial\mathcal{L}/\partial(\partial_\mu\phi) = -\frac{1}{2}(\eta^{\mu\beta}\partial_\beta\phi + \eta^{\alpha\mu}\partial_\alpha\phi) = -\partial^\mu\phi$, so (3.16) yields

$$\begin{aligned} 0 = -\partial_\mu\partial^\mu\phi + K^2\phi &= \frac{\partial^2\phi}{\partial x^{0\,2}} - \nabla^2\phi + K^2\phi \\ &= \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + K^2\phi. \end{aligned}$$

Thus the wave equation emerges with $K = 0$ from the Lagrangian density which is the simplest possible function of $\partial_\mu\phi$ only. If $K \neq 0$ waves are evanescent (complex k) if $\omega < Kc$, just as electromagnetic waves are evanescent in a plasma below the plasma frequency.

3.4 Klein-Gordon equation

What p.d.e.s can we derive for a complex-valued scalar field ψ? Minor generalizaton of our work on the real scalar field leads us to

$$\mathcal{L}(\psi, \partial_\mu\psi) = -\tfrac{1}{2}\Big(|\partial\psi|^2 + K^2|\psi|^2\Big). \tag{3.18}$$

By $|\partial\psi|^2$ we mean

$$|\partial\psi|^2 = -\frac{1}{c^2}\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial t} + \nabla\psi^* \cdot \nabla\psi. \tag{3.19}$$

Differentiating w.r.t. ψ is slightly tricky because ψ^* is a function $\psi^*(\psi)$ of ψ. We handle this by writing $\psi = u + \mathrm{i}v$ and treating the real and imaginary parts of u and v as independent real fields:

$$\begin{aligned}\frac{\partial|\psi|^2}{\partial u} &= \frac{\partial}{\partial u}(u^2+v^2) = 2u,\\ \frac{\partial|\psi|^2}{\partial v} &= 2v.\end{aligned} \tag{3.20}$$

Further

$$|\partial\psi|^2 = \partial(u-\mathrm{i}v)\cdot\partial(u+\mathrm{i}v) = |\partial u|^2 + |\partial v|^2.$$

So

$$\frac{\partial|\partial\psi|^2}{\partial(\partial_\mu u)} = 2\partial^\mu u \quad ; \quad \frac{\partial|\partial\psi|^2}{\partial(\partial_\mu v)} = 2\partial^\mu v. \tag{3.21}$$

Hence the field eqns are

$$\left.\begin{aligned}\frac{\partial}{\partial x^\mu}\partial^\mu u - K^2 u = 0\\ \frac{\partial}{\partial x^\mu}\partial^\mu v - K^2 v = 0\end{aligned}\right\} \quad\Rightarrow\quad \partial_\mu\partial^\mu\psi - K^2\psi = 0. \tag{3.22}$$

Spin-0 particles of mass m_0 are excitations of a scalar field that satisfies $\hat{p}^2\psi = -m_0^2c^2\psi$. Substituting $\hat{E} = \mathrm{i}\hbar\partial_t$ and $\hat{p}_i = -\mathrm{i}\hbar\partial_i$ this becomes the Klein-Gordon eqn

$$\partial_\mu\partial^\mu\psi = \frac{m_0^2c^2}{\hbar^2}\psi. \tag{3.23}$$

The K–G eqn is obtained from equations (3.22) by setting $K = m_0c/\hbar$.

The following result simplifies the variation of an action that depends on a complex field ψ. Suppose $\delta f(\psi,\psi^*) = 0$. We have

$$0 = \delta f = \frac{\partial f}{\partial\psi}(\delta u + \mathrm{i}\delta v) + \frac{\partial f}{\partial\psi^*}(\delta u - \mathrm{i}\delta v).$$

Since δu and δv are arbitrary, we conclude

$$\left.\begin{aligned}0 = \frac{\partial f}{\partial\psi} + \frac{\partial f}{\partial\psi^*}\\ 0 = \frac{\partial f}{\partial\psi} - \frac{\partial f}{\partial\psi^*}\end{aligned}\right\} \quad\Leftrightarrow\quad \left\{\begin{aligned}0 &= \frac{\partial f}{\partial\psi}\\ 0 &= \frac{\partial f}{\partial\psi^*}.\end{aligned}\right.$$

Thus we can proceed as though $\delta\psi$ and $\delta\psi^*$ were independent, though they are not.

3.5 Dirac equation

In §2.2 we saw that from a Dirac spinor we can construct the scalar $\overline{\psi}\cdot\psi$ and the vector $\overline{\psi}\gamma^\mu\psi$. In light of our discussion of the Klein–Gordon equation it is natural to take the potential energy density of a Dirac field to be proportional to $\overline{\psi}\cdot\psi$. For the kinetic term we could choose $(\partial^\mu\overline{\psi})(\partial_\mu\psi)$ but a simpler choice is $\mathrm{i}\overline{\psi}\gamma^\mu\partial_\mu\psi$, where the factor i is inserted for later convenience. Consider therefore the field equation that follows from

$$\mathcal{L} = \overline{\psi}\mathrm{i}\gamma^\mu\partial_\mu\psi - \frac{m_0c}{\hbar}\overline{\psi}\cdot\psi. \tag{3.24}$$

A variation of ψ induces a corresponding variation in $\overline{\psi}$ and thus causes $\mathcal{L}$ to change by

$$\delta\mathcal{L} = \delta\overline{\psi}\Big(\mathrm{i}\gamma^\mu\partial_\mu - \frac{m_0c}{\hbar}\Big)\psi + \overline{\psi}\Big(\mathrm{i}\gamma^\mu\partial_\mu - \frac{m_0c}{\hbar}\Big)\delta\psi \tag{3.25}$$

Suppose we choose to vary only the first component of ψ, that is we choose $\delta\psi = (a+\mathrm{i}b, 0, 0, 0)$, where a and b are real functions on space-time. Then $\delta\overline{\psi} = (a-\mathrm{i}b, 0, 0, 0)\gamma^0$. We consider two variations, one with $b = 0$ and then one with $a = 0$ and b set equal to the function a that we used in the first case. From the stationarity of the action it follows that

$$\begin{aligned}
0 &= \int \mathrm{d}^4\mathbf{x}\ \Big\{(a,0,0,0)\gamma^0\Big(\mathrm{i}\gamma^\mu\partial_\mu - \frac{m_0c}{\hbar}\Big)\psi + \overline{\psi}\Big(\mathrm{i}\gamma^\mu\partial_\mu - \frac{m_0c}{\hbar}\Big)(a,0,0,0)\Big\}\\
0 &= \int \mathrm{d}^4\mathbf{x}\ \Big\{(-a,0,0,0)\gamma^0\Big(\mathrm{i}\gamma^\mu\partial_\mu - \frac{m_0c}{\hbar}\Big)\psi + \overline{\psi}\Big(\mathrm{i}\gamma^\mu\partial_\mu - \frac{m_0c}{\hbar}\Big)(a,0,0,0)\Big\}.
\end{aligned} \tag{3.26}$$

Subtracting the equations and exploiting the arbitrariness of $a(\mathbf{x})$, we obtain

$$0 = (a,0,0,0)\gamma^0\Big(\mathrm{i}\gamma^\mu\partial_\mu - \frac{m_0c}{\hbar}\Big)\psi. \tag{3.27}$$

Repeating this exercise for each component of ψ, we obtain the **Dirac equation**

$$0 = \Big(\mathrm{i}\gamma^\mu\partial_\mu - \frac{m_0c}{\hbar}\Big)\psi. \tag{3.28}$$

As in the case of the Klein–Gordon action, the equation we get at the end is the one we would have obtained if we had (incorrectly) argued that ψ and $\overline{\psi}$ are independent variables.

Exercise (14):

Show that when we add equations (3.26) we obtain

$$0 = -\partial_\mu\overline{\psi}\mathrm{i}\gamma^\mu - \frac{m_0c}{\hbar}\overline{\psi}$$

and show that this is just the adjoint of the Dirac equation. [Hint: $(\gamma^0\gamma^\mu)^\dagger = \gamma^0\gamma^\mu$.]

3.6 Maxwell's equations

What about Maxwell's equations? These are 2^{nd} order in $\mathbf{A}$, so we look for a Lagrangian density $\mathcal{L}$ that depends on $\mathbf{A}$ and its derivatives, $\partial_\mu \mathbf{A}$. Moreover, Maxwell's eqns are linear in the fields, and thus in $\mathbf{A}$. So $\mathcal{L}$ should be quadratic in $\mathbf{A}$ and $\partial_\mu \mathbf{A}$. Finally, $\mathcal{L}$ should be invariant under gauge transformations $\mathbf{A} \to \mathbf{A}' + \partial\Lambda$, and should involve $\partial_\mu \mathbf{A}$ only in the combination contained in $\mathbf{F}$. The shortlist of functions satisfying these criteria contains (up to an unimportant normalization) only one candidate:

$$\begin{aligned}\mathcal{L}_{\text{vac}}(\mathbf{A}, \partial_\mu \mathbf{A}) &= \frac{1}{4\mu_0} \operatorname{Tr} \mathbf{F} \cdot \mathbf{F} \\ &= -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} \\ &= \frac{1}{2\mu_0}(E^2/c^2 - B^2),\end{aligned} \tag{3.29}$$

where the last equality is from (1.20). (Notice that if we associate E with kinetic energy ($E = -\dot{\mathbf{A}}/c + \cdots$) and B with potential energy, $\mathcal{L}_{\text{vac}}$ is of the form k.e. – p.e..) The field equations associated with the Lagrangian (3.29) density are

$$\frac{\partial}{\partial x^\beta}\left(\frac{\partial \mathcal{L}_{\text{vac}}}{\partial(\partial_\beta A_\mu)}\right) = 0.$$

Now

$$\begin{aligned}\frac{\partial F_{\mu\nu}}{\partial(\partial_\beta A_\alpha)} &= \frac{\partial}{\partial(\partial_\beta A_\alpha)}(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &= \delta^\beta_\mu \delta^\alpha_\nu - \delta^\beta_\nu \delta^\alpha_\mu,\end{aligned} \tag{3.30}$$

so

$$\begin{aligned}\frac{\partial \mathcal{L}_{\text{vac}}}{\partial(\partial_\beta A_\alpha)} &= -\frac{1}{4\mu_0}\frac{\partial(F_{\mu\nu}F^{\mu\nu})}{\partial(\partial_\beta A_\alpha)} = -\frac{1}{4\mu_0}\frac{\partial(F_{\mu\nu}\eta^{\mu\kappa}\eta^{\nu\lambda}F_{\kappa\lambda})}{\partial(\partial_\beta A_\alpha)} \\ &= -\frac{1}{2\mu_0}(\delta^\beta_\mu \delta^\alpha_\nu - \delta^\beta_\nu \delta^\alpha_\mu)F^{\mu\nu} \\ &= -\frac{1}{2\mu_0}(F^{\beta\alpha} - F^{\alpha\beta}) \\ &= \frac{1}{\mu_0}F^{\alpha\beta}.\end{aligned} \tag{3.31}$$

The field equations are therefore

$$\frac{\partial F^{\alpha\beta}}{\partial x^\beta} = 0, \tag{3.32}$$

that is, 4 of Maxwell's 8 field eqns for an e.m. field in vacuo.

To get Maxwell's eqns in the presence of charges we need to add to the action S obtained by integrating (3.29) over spacetime, the action of particles in a given e.m. field. For a single charged particle the latter is given by (3.2). What does this suggest for the action associated with a swarm of particles of charge q, mass m_0 that are moving with 4-velocity $\mathbf{v}(\mathbf{x})$ and in their rest-frame have number density $n(\mathbf{x})$? Well, the form of (3.2) suggests that the part of $\mathcal{L}$ which depends on both the e.m. field and the particles (the 'interaction term'), is proportional to the dot product of $\mathbf{A}$ with the current density $\mathbf{j} = qn_0\mathbf{v}$ associated with the particles. So we speculate that the interaction term is $\mathbf{j}\cdot\mathbf{A}$. The current density contributed by a particle of charge q that moves on the world-line $\mathbf{X}(\tau)$, is

$$\mathbf{j}(\mathbf{x}) = qc\int \mathrm{d}\tau\,\dot{\mathbf{X}}\delta(\mathbf{x}-\mathbf{X}) = qc\int \mathrm{d}\mathbf{X}\,\delta(\mathbf{x}-\mathbf{X})\,. \tag{3.33}$$

Exercise (15):
Check the validity of (3.33) by (i) showing that it is dimensionally correct, (ii) showing that $\int \mathrm{d}^3\boldsymbol{x}\,\boldsymbol{j} = q(\mathrm{d}\boldsymbol{X}/\mathrm{d}t)$, i.e., the total current is just q times the Newtonian velocity, and (iii) showing similarly that the total charge in any spatial slice is always q.

Using this result, the contribution to the action from our conjectured term is

$$\begin{aligned} S_{\text{interaction}} &= \frac{1}{c}\int \mathrm{d}^4\mathbf{x}\,(\mathbf{j}\cdot\mathbf{A})\big|_{\mathbf{x}} \\ &= q\int \mathrm{d}^4\mathbf{x}\,\mathrm{d}\tau\,\dot{\mathbf{X}}\cdot\mathbf{A}(\mathbf{x})\delta(\mathbf{x}-\mathbf{X}) \\ &= q\int \mathrm{d}\tau\,\dot{\mathbf{X}}\cdot\mathbf{A}(\mathbf{X}) \end{aligned} \tag{3.34}$$

which agrees with (3.2).

So long as we are only interested in getting the field eqns, which are obtained by varying $\mathbf{A}$, we don't need to bother with the contribution to S from matter alone (which is independent of $\mathbf{A}$). So let's see whether this action begets Maxwell's eqns with sources:

$$S = \frac{1}{c}\int \mathrm{d}^4\mathbf{x}\,\big(\mathbf{j}\cdot\mathbf{A} + \frac{1}{4\mu_0}\,\mathrm{Tr}\,\mathbf{F}\cdot\mathbf{F}\big). \tag{3.35}$$

Varying $\mathbf{A}$ with the aid of previous results, the field eqns are found to be

$$j_\mu - \frac{1}{\mu_0}\frac{\partial F_{\mu\nu}}{\partial x^\nu} = 0 \tag{3.36}$$

in agreement with (1.48). The other four Maxwell's eqns don't come from minimizing the action but from the fact that $\mathbf{F}$ is the 4-curl of $\mathbf{A}$. So they are geometrical rather than dynamical in nature.

3.7 Noether's theorem for internal symmetries

Does Noether's theorem for the Lagrangians of particle motion extend to Lagrangian densities for fields? Actually it yields *two* closely related results: one for internal symmetries and one for external symmetries, such as Lorentz invariance. We deal with internal symmetries first.

Often $\mathcal{L}(\mathbf{A}, \partial_\mu \mathbf{A})$ is invariant under some transformation of the field $\mathbf{A}$. For example, in the case of e.m. $\mathcal{L}$ is invariant under $\mathbf{A} \to \mathbf{A} + \partial\Lambda$ where $\Lambda(\mathbf{x})$ is any scalar function.[4] Whenever there is a point-by-point invariance of this type, we can write

$$\begin{aligned} 0 = \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\mathbf{A}} \cdot \delta\mathbf{A} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\mathbf{A})} \cdot \delta(\partial_\mu\mathbf{A}) \\ &= \frac{\partial}{\partial x^\mu}\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\mathbf{A})}\Big) \cdot \delta\mathbf{A} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\mathbf{A})} \cdot \partial_\mu(\delta\mathbf{A}) \\ &= \frac{\partial}{\partial x^\mu}\Big(\delta\mathbf{A} \cdot \frac{\partial\mathcal{L}}{\partial(\partial_\mu\mathbf{A})}\Big), \end{aligned} \tag{3.37}$$

where the field eqns (3.16) have been used. The final line states that the current density j^μ has vanishing divergence, where

$$j^\mu \equiv \delta\mathbf{A} \cdot \frac{\partial\mathcal{L}}{\partial\partial_\mu\mathbf{A}}. \tag{3.38}$$

The vanishing of $\partial \cdot \mathbf{j}$ implies that the integral $J \equiv \int \mathrm{d}^3\mathbf{x}_\mu\, j^\mu \equiv \int \mathrm{d}x^\alpha \mathrm{d}x^\beta \mathrm{d}x^\gamma\, j^\mu \epsilon_{\mu\alpha\beta\gamma}$ is the same for any two large 3-dimensional spatial slices: Given two such slices we can extend these into the closed surface bounding a spacetime volume by adding the 3-surface formed by a very large spherical shell as it propagates in time from one spatial slice to the other [see fig. above (3.14)]. $\partial \cdot \mathbf{j} = 0$ implies that the flux into this volume has to equal that out of it, so provided $\mathbf{j}$ vanishes on the shell, the flux in through the earlier spatial slice has to equal the flux out through the later slice. Thus the internal symmetry of $\mathcal{L}$ has generated a conserved flux J.

E.m. charge conservation How does this idea work out in e.m? Setting $\delta\mathbf{A} = \partial\Lambda$, we have

$$\begin{aligned} j^\mu &= \big(\partial_\alpha\Lambda\big) \frac{\partial\mathcal{L}_{\rm vac}}{\partial(\partial_\mu A_\alpha)} \\ &= \frac{1}{\mu_0}\big(\partial_\alpha\Lambda\big) F^{\alpha\mu}, \end{aligned} \tag{3.39}$$

where use has been made of (3.31). Equating to zero the divergence of this we find that

$$\begin{aligned} 0 &= \frac{\partial^2\Lambda}{\partial x^\mu \partial x^\alpha} F^{\alpha\mu} + \frac{\partial\Lambda}{\partial x^\alpha}\frac{\partial F^{\alpha\mu}}{\partial x^\mu} \\ &= \frac{\partial\Lambda}{\partial x^\alpha}\partial_\mu F^{\alpha\mu}, \end{aligned}$$

where the first term on the right has been eliminated by virtue of $\mathbf{F}$'s antisymmetry. Since we can arrange for $\partial\Lambda$ to be any vector at a given point, (3.39) implies that $\partial_\mu F^{\alpha\mu} = 0$. This is just (3.32), the standard field eqn for e.m. in vacuo.

To obtain a more interesting Noether invariant one has to start from $\mathcal{L}$ for the e.m. field plus a matter field, say ψ.

[4] Notice the difference with the least-action principle, which states that $0 = c\delta S = \delta \int \mathrm{d}^4\mathbf{x}\, \mathcal{L}$ for *any* variation $\delta\mathbf{A}$; for most variations, $\mathcal{L}$ changes at each point, it is just its integral which is invariant.

Klein-Gordon current The Klein-Gordon $\mathcal{L}$ (3.18) is invariant under changes in the phase of ψ, i.e., $\psi \to e^{i\theta}\psi$. When θ is small we have $\delta u + i\delta v = \delta\psi \simeq i\theta\psi$, so the changes in the real and imaginary parts of ψ are

$$\delta u = -\theta v \quad ; \quad \delta v = \theta u. \tag{3.40}$$

Since we are considering $\mathcal{L}$ to be a function of (u, iv) and their derivatives, the dot in (3.38) has to be interpreted as a sum over u and iv. Using our results (3.21) we find that the conserved current is

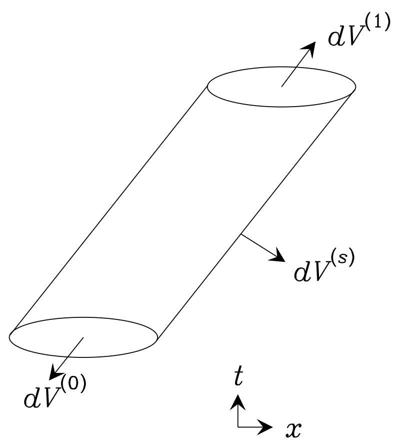
$$\begin{aligned} j_\mu &= \delta u \partial_\mu u + \delta iv \partial_\mu iv \\ &= \theta\Big(- v\frac{\partial u}{\partial x^\mu} + u\frac{\partial v}{\partial x^\mu}\Big) \\ &= \frac{\theta}{2i}\Big(\psi^* \frac{\partial \psi}{\partial x^\mu} - \psi\frac{\partial \psi^*}{\partial x^\mu}\Big). \end{aligned} \tag{3.41}$$

It is simple to verify $\partial \cdot \mathbf{j} = 0$ by taking the divergence and using the Klein-Gordon equation and its complex conjugate to eliminate $\Box$.

Consider the particle flux through some small region W of spacetime. To the past and future W is bounded by the 3-dimensional sets of events that occur in some physical container (an empty beer can?) at the times t_0 and $t_1 > t_0$ in the can's rest frame. In spacetime these sets are represented by 4-vectors $V^{(0)}_\mu$ and $V^{(1)}_\mu$. We orientate $V^{(0)}_\mu$ so that it points into the past, while $V^{(1)}_\mu$ looks to the future. Since the contents of the may not be uniform, we decompose both $\mathbf{V}^{(0)}$ and $\mathbf{V}^{(1)}$ into a large number of small pieces $d\mathbf{V}$, each centred on a different position within the can. The balance of W's boundary comprises the 3-dimensional set of events that occur on the can's surface at times between t_0 and t_1. We represent this part of W's boundary by elements $dV^{(s)}_\mu(\mathbf{x})$, each of which points out of the can.

The number of particles in the can at t_0 is $N(0) = -\int_{\mathrm{can},t_0} dV^{(0)}_\nu\, j^\nu$, while the number present at t_1 is $N(1) = \int_{\mathrm{can},t_1} dV^{(1)}_\nu\, j^\nu$. If particle number is to be conserved, the difference $N(1) - N(0)$ must represent the number of particles that that flow into the can between t_0 and t_1. Thus particle conservation requires that

$$\iiint_{\mathrm{can},t_1} dV^{(1)}_\nu\, j^\nu + \iiint_{\mathrm{can},t_0} dV^{(0)}_\nu\, j^\nu = - \iiint_{\substack{\mathrm{surface}\\ t_0<t<t_1}} dV^{(s)}_\nu\, j^\nu.$$



Thus in a natural notation we have

$$\oint \mathrm{d}V_\nu\, j^\nu = 0 \quad \Leftrightarrow \quad \partial_\nu j^\nu = 0. \tag{3.42}$$

This discussion and equation (3.41) show that

$$j_0 = \frac{\theta}{2\mathrm{i}c}\Big(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\Big) \tag{3.43}$$

is proportional to the particle density in the coordinate rest-frame, and because in that frame $\mathrm{d}V^{(s)} = \mathrm{d}^2\mathbf{x}_i c\mathrm{d}t$, the flux of particles in the coordinate rest frame is proportional to

$$j_i = \frac{\theta c}{2\mathrm{i}}\Big(\psi^*\frac{\partial\psi}{\partial x^i} - \psi\frac{\partial\psi^*}{\partial x^i}\Big). \tag{3.44}$$

By comparison, non-relativistic quantum mechanics yields for Hamiltonian $H = p^2/2m$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\boldsymbol{x}\,|\psi|^2 &= \int \mathrm{d}^3\boldsymbol{x}\,\Big(\frac{\partial\psi^*}{\partial t}\psi + \psi^*\frac{\partial\psi}{\partial t}\Big)\\ &= \int \mathrm{d}^3\boldsymbol{x}\,\Big(\frac{H\psi^*}{-\mathrm{i}\hbar}\psi + \psi^*\frac{H\psi}{\mathrm{i}\hbar}\Big)\\ &= \frac{\hbar}{2\mathrm{i}m}\int \mathrm{d}^3\boldsymbol{x}\,\Big((\nabla^2\psi^*)\psi - \psi^*\nabla^2\psi\Big)\\ &= \frac{\hbar}{2\mathrm{i}m}\oint \mathrm{d}^2\boldsymbol{x}_i\,\Big((\nabla_i\psi^*)\psi - \psi^*\nabla_i\psi\Big).\end{aligned} \tag{3.45}$$

Hence the Klein-Gordon expression for the particle flux is essentially identical with the non-relativistic one, but the expressions for the particle density are rather different in the two cases.

3.8 Noether's theorem and Lorentz invariance

The Lagrangian density $\mathcal{L}$ of a Lorentz-covariant theory depends on $\mathbf{x}$ only through the field $\mathbf{A}$ and its derivatives, i.e., it has no explicit space-time dependence. Consider an infinitesimal shift in the coordinate origin which changes the coordinates of the point $\mathbf{x}$ to $\mathbf{x}' \equiv \mathbf{x} + \mathbf{a}$, where $\mathbf{a}$ is very small. Then the difference in the value of $\mathcal{L}$ at $\mathbf{x}$ and at the point $\mathbf{x} + \mathbf{a}$ whose coordinates in the unprimed frame coincide with $\mathbf{x}$'s coordinates in the primed frame is

$$\begin{aligned}\delta\mathcal{L} &= \Big(\frac{\partial\mathcal{L}}{\partial\mathbf{A}}\cdot\frac{\partial\mathbf{A}}{\partial x^\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\mathbf{A})}\cdot\frac{\partial(\partial_\nu\mathbf{A})}{\partial x^\alpha}\Big)a^\alpha\\ &= \bigg(\frac{\partial}{\partial x^\nu}\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\mathbf{A})}\Big)\cdot\frac{\partial\mathbf{A}}{\partial x^\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\mathbf{A})}\cdot\frac{\partial^2\mathbf{A}}{\partial x^\alpha\partial x^\nu}\bigg)a^\alpha\\ &= \frac{\partial}{\partial x^\nu}\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\mathbf{A})}\cdot\frac{\partial\mathbf{A}}{\partial x^\alpha}\Big)a^\alpha.\end{aligned} \tag{3.46}$$

On the other hand, if we simply regard $\mathcal{L}$ as a function of $\mathbf{x}$ through the fields, we have

$$\delta\mathcal{L} = a^\alpha \frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{\partial}{\partial x^\nu}\big(\mathcal{L}\delta^\nu_\alpha a^\alpha\big). \tag{3.47}$$

Equating these two expressions for $\delta\mathcal{L}$ we have

$$0 = \frac{\partial}{\partial x^\nu}\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \mathbf{A})}\cdot\frac{\partial \mathbf{A}}{\partial x^\alpha} - \mathcal{L}\delta^\nu_\alpha\Big)a^\alpha. \tag{3.48}$$

Furthermore, $\mathbf{a}$ is an arbitrary small vector so its coefficient in (3.48) must vanish. Thus from the fact that $\mathcal{L}$ depends on $\mathbf{x}$ only through the fields we can conclude that the tensor

$$\hat{T}^\nu{}_\mu \equiv -\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \mathbf{A})}\cdot\frac{\partial \mathbf{A}}{\partial x^\mu} - \mathcal{L}\delta^\nu_\mu\right) \quad\Rightarrow\quad \hat{T}^{00} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{A}}}\cdot\dot{\mathbf{A}} - \mathcal{L} \tag{3.49}$$

has vanishing divergence: $\partial_\nu \hat{T}^\nu{}_\mu = 0$. $\mathbf{T}$ is the **canonical energy-momentum** tensor. The vanishing of its divergence expresses energy-momentum conservation in the same way that $\partial_\nu j^\nu = 0$ implies conservation of particles – the difference between the two cases is that $\partial_\nu \hat{T}^\nu{}_\mu = 0$ implies conservation of four quanties: energy and x, y and z momentum. Notice the similarity of (3.49) to the conventional definition of a system's Hamiltonian: $H = p_\mu \dot{q}^\mu - L$.

Again using (3.31), we find for the canonical energy-momentum tensor of the e.m. field

$$\hat{T}^\nu{}_\mu = -\frac{1}{\mu_0}\Big(F^{\alpha\nu}\frac{\partial A_\alpha}{\partial x^\mu} + \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta^\nu_\mu\Big). \tag{3.50}$$

Even when we lower $\mathbf{T}$'s first index by premultiplying by $\eta_{\kappa\nu}$, this isn't symmetric like the $\mathbf{T}$ of §1.4. We'd very much like $\hat{\mathbf{T}}$ to be symmetric, if only because Einstein's equations require it to be so. Also we'd like the energy-momentum tensor to depend on $\mathbf{A}$ only through $\mathbf{F}$. We can attain both goals by adding into $\hat{\mathbf{T}}$ what's necessary to upgrade the derivative of $\mathbf{A}$ in the first term into an $\mathbf{F}$. The required item is

$$\boldsymbol{\Delta}^\nu{}_\mu = \frac{1}{\mu_0}F^{\alpha\nu}\frac{\partial A_\mu}{\partial x^\alpha}. \tag{3.51}$$

In the absence of sources (which is when we would expect the energy-momentum tensor to be divergence-free) $\boldsymbol{\Delta}$ is itself divergence free:

$$\partial_\nu \boldsymbol{\Delta}^\nu{}_\mu = \frac{1}{\mu_0}\frac{\partial^2(F^{\alpha\nu}A_\mu)}{\partial x^\nu \partial x^\alpha} = 0. \tag{3.52}$$

So if we define $\mathbf{T} \equiv \hat{\mathbf{T}} + \boldsymbol{\Delta}$, $\mathbf{T}$ will be symmetric and divergence-free in vacuo. The energy-momentum tensor of the e.m. field is then

$$\begin{aligned} T^\nu_\mu &= -\frac{1}{\mu_0}\Big(F^{\alpha\nu}\frac{\partial A_\alpha}{\partial x^\mu} - F^{\alpha\nu}\frac{\partial A_\mu}{\partial x^\alpha} + \tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta}\delta^\nu_\mu\Big) \\ &= -\frac{1}{\mu_0}\Big(F_{\mu\alpha}F^{\alpha\nu} + \tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta}\delta^\nu_\mu\Big) \end{aligned} \tag{3.53}$$

in agreement with (1.28).

When charges are present, the field Lagrangian density includes a term $\mathbf{j}\cdot\mathbf{A}$ which breaks translational invariance if $\mathbf{j}$ is regarded as fixed. Consequently, the energy-momentum tensor made out of $\mathbf{j}\cdot\mathbf{A}+\mathcal{L}_{\rm vac}$ does *not* have vanishing divergvence. Physically, this is because the e.m. field is exchanging energy and momentum with the charges. If we add a term to the Lagrangian that describes the dynamics of the charges, the entire Langrangian – charges plus interaction plus vacuum field – *will* be translationally invariant and give rise to an energy-momentum tensor that has vanishing divergence. In fact, we cannot regard a system as isolated until it has been expanded to the point that its Lagrangian is translationally invariant, and gives rise to a conserved energy-momentum tensor.

We shall see below that one of the strange features of gravity is that a system that interacts with other systems only gravitationally has a conserved energy-momentum tensor even though, physically, it is exchanging energy and momentum with other systems – for example by emitting gravitational radiation. This singular feature of gravity makes it very hard to pin energy down in G.R.

4 Newton's Theory & the Principle of Equivalence

4.1 Newton's Theory

According to Newton, every body attracts every other body with a force that is proportional to the product of the masses of the two bodies and inversely proportional to the square of the distance between them. Hence the force on a unit mass at $\boldsymbol{x}$ that is generated by a distribution of matter of density $\rho(\boldsymbol{x}')$ is

$$\boldsymbol{f}(\boldsymbol{x}) = G\int \frac{\boldsymbol{x}'-\boldsymbol{x}}{|\boldsymbol{x}'-\boldsymbol{x}|^3}\rho(\boldsymbol{x}')\,\mathrm{d}^3\boldsymbol{x}', \tag{4.1}$$

where $G = 6.672(4)\times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{sec}^{-2}$ is Newton's constant. If we define the **gravitational potential** $\Phi(\boldsymbol{x})$ by

$$\Phi(\boldsymbol{x}) = -G\int \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}'-\boldsymbol{x}|}\,\mathrm{d}^3\boldsymbol{x}',$$

and notice that

$$\nabla_{\boldsymbol{x}}\left(\frac{1}{|\boldsymbol{x}'-\boldsymbol{x}|}\right) = \frac{\boldsymbol{x}'-\boldsymbol{x}}{|\boldsymbol{x}'-\boldsymbol{x}|^3},$$

we find that we may write $\boldsymbol{f}$ as

$$\begin{aligned}\boldsymbol{f}(\boldsymbol{x}) &= \nabla_{\boldsymbol{x}}\int \frac{G\rho(\boldsymbol{x}')}{|\boldsymbol{x}'-\boldsymbol{x}|}\,\mathrm{d}^3\boldsymbol{x}' \\ &= -\nabla\Phi.\end{aligned} \tag{4.2}$$

If we take the divergence of equation (4.1), we find

$$\nabla\cdot\boldsymbol{f}(\boldsymbol{x}) = G\int \nabla_{\boldsymbol{x}}\cdot\left(\frac{\boldsymbol{x}'-\boldsymbol{x}}{|\boldsymbol{x}'-\boldsymbol{x}|^3}\right)\rho(\boldsymbol{x}')\,\mathrm{d}^3\boldsymbol{x}'. \tag{4.3}$$

But

$$\nabla_{\boldsymbol{x}}\cdot\left(\frac{\boldsymbol{x}'-\boldsymbol{x}}{|\boldsymbol{x}'-\boldsymbol{x}|^3}\right) = -4\pi\delta(\boldsymbol{x}'-\boldsymbol{x}) \qquad \text{(where } \delta \text{ is the Dirac } \delta\text{-function)} \tag{4.4}$$

as one may show, on the one hand by evaluating the derivative at $\boldsymbol{x}\neq\boldsymbol{x}'$, and on the other hand by using the divergence theorem to integrate the left side through a small sphere centred on $\boldsymbol{x}=\boldsymbol{x}'$. Combining equations (4.2), (4.3) and (4.4) we obtain **Poisson's equation**

$$4\pi G\rho = \nabla^2\Phi \quad = -\nabla\cdot\boldsymbol{f}. \tag{4.5}$$

Elegant though it is, this equation cannot represent the whole truth about gravitational physics since it is not constructed according to the rules of tensor calculus summarized in §2.7; if the right side of equation (5) is to form an n-tuple, it must form a scalar since it has only one component. On the other hand, since mass is just a manifestation of energy, we expect the quantity ρ appearing on the left side of equation (5) to represent energy density, and this we know to form the 00-component of the 10-tuple $\mathbf{T}$. So we either have to think of some scalar thing to put on the left in the place of ρ, or we have to augment Φ with a whole bunch of extra potentials, its companions in some new 10-tuple $\mathbf{g}$, and somehow extend the single equation (4.5) to a set of ten equations from which the whole set of potentials can be determined.

Consideration of the predicament of a physicist who knows about relativity and electrostatics but not about magnetism will clarify this point. This person looks at the electrostatic form of Poisson's equation

$$\nabla^2\phi = -q/\epsilon_0, \quad \text{where } q \text{ is charge density,}$$

and thinks

“ q isn't a scalar because of the Lorentz-Fitzgerald contraction (in fact, q is the 0^{th} component of the current density $\mathbf{j}$),[5] so ϕ can't be a scalar either. Seems I'll have to augment ϕ with three other potentials, say A_x, A_y and A_z. Then that ∇^2 won't do either, because it's no kind of n-tuple. I'll replace it with the d'Alembertian, which is a scalar. Then I'll have

$$\Big(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)\phi = -\frac{q}{\epsilon_0} \quad \text{and} \quad \Big(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)A_i = -\frac{j_i}{\epsilon_0}. \text{”}$$

By this point our friend would be well on the way to a Nobel prize.

We shall see that the natural generalization of this argument to the case of gravity yields

$$\Big(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)\mathbf{g} = \text{constant} \times \mathbf{T}.$$

However, Einstein showed that the way forward is not to tinker thus with Newtonian gravity, but to assign to the gravitational force a unique position as the force generated by the very dynamics of spacetime itself. The stimulus for this remarkable intellectual leap was the modern form of Galileo's famous observation that all bodies fall at the same speed.

4.2 The Principle of Equivalence

Inertial & gravitational mass As conventionally stated Newton's laws of motion are part definition and part empirical law. The purely empirical content can be summed up by the statements:

(i) the more carefully one isolates a body from external influences, the more nearly does its velocity $\boldsymbol{v}$ remain constant;

[5] See equation (1.47).

(ii) when several otherwise isolated bodies $\alpha = 1, \ldots, N$ interact with one another, it is possible to assign a number m_α to each body such that the quantity $\boldsymbol{p} \equiv \sum_\alpha m_\alpha \boldsymbol{v}_\alpha$ remains constant.

We call m_α the **inertial mass** of body α. When bodies are interacting, and therefore have changing individual momenta $\boldsymbol{p}_\alpha \equiv m_\alpha \boldsymbol{v}_\alpha$, it is convenient to imagine that they are acting on one another with a quantity "force", $\boldsymbol{f}_\alpha \equiv \mathrm{d}p_\alpha/\mathrm{d}t$. By statement (ii), $\sum_\alpha \boldsymbol{f}_\alpha = 0$.

Again according to Newton, the gravitational force between bodies α and β is

$$\boldsymbol{f}_{\alpha\beta} = F \frac{\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta}{|\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta|^3},$$

where the constant $F = GM_\alpha M_\beta$ is proportional to the product of two numbers M_α and M_β characteristic of the bodies—we call these masses **gravitational masses** of the bodies. If we place two bodies β and γ at the same distance from α, their accelerations will be in the ratio

$$\frac{|\mathrm{d}\boldsymbol{v}_\beta/\mathrm{d}t|}{|\mathrm{d}\boldsymbol{v}_\gamma/\mathrm{d}t|} = \frac{M_\beta m_\gamma}{m_\beta M_\gamma} = \frac{\Gamma_\beta}{\Gamma_\gamma}, \quad \text{where} \quad \Gamma_\nu \equiv \frac{M_\nu}{m_\nu}.$$

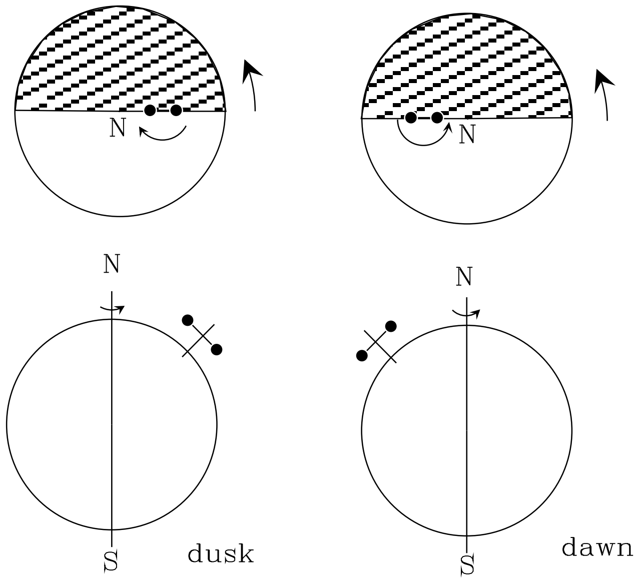
Thus β and γ will fall towards α at the same rate only if $\Gamma_\beta = \Gamma_\gamma$. Newton followed Galileo in thinking that all bodies fall at the same rate, and therefore assumed (with a suitable choice of G) that $\Gamma = 1$ for all particles. But in the 17^{th} century the experimental basis of this step was not strong.

4.3 Dicke–Eötvös Experiments

The most straightforward way to check whether Γ is the same for all masses is to compare the periods of pendulums made of different materials but having the same lengths. However, the impossibility of eliminating frictional resistance to the motion of a pendulum severely restricts the accuracy that can be attained in experiments of this kind.

In 1890 a Hungarian, Baron Roland v. Eötvös carried out a much more sensitive test of the proportionality of inertial and gravitational mass. A modified form of this experiment was repeated with greater accuracy by Robert Dicke and his students in Princeton in the 1960's.

Fig. 3 shows a schematic apparatus for the Dicke experiment. Two balls of approximately equal weight are attached to the ends of a short rod. This is attached to a wire so that it can execute torsional oscillations about a vertical axis. For simplicity we assume that a new moon is nearly eclipsing the Sun at the time of the experiment, which begins at dusk. Then in the lower panels the acceleration of the balls on account of the Earth's spin lies in the plane of the paper, while that due to the Earth's rotation about the Sun and Moon is perpendicular to the paper. Hence we may forget about the spin of the Earth as we balance the books as regards forces perpendicular to the paper. The bar is aligned North-South and released. If Γ is identical for both balls and equal to Γ for the Earth as a whole, the gravitational force towards the Sun and Moon exactly equals the acceleration due to their instantaneous motion transverse to the Earth-Sun line, and there is no tendency for the wire to twist. But if Γ is abnormally large for one of the balls, say that to the South, this ball will start to fall towards the Sun faster than the other ball, and the rod will start to twist in the direction indicated. Consequently, the bar (which has a period of about one hour) will oscillate about an equilibrium position that is skewed with respect to the N-S line.



Schematic of the Dicke experiment to determine Γ.

During the evening, the torque on the wire due to the extra gravitational force on the southern ball diminishes. After midnight the torque starts to grow again, but with reversed sign. By dawn its displacement of the centre of oscillation is exactly opposite to that operating at dusk. By looking for a component in the motion of the bar with period 24 hrs and the expected phase with respect to solar time, Dicke and his collaborators were able to establish the limit $|\Gamma - 1| < 10^{-11}$.

What material should be used for the balls? Various things were tried but it is most interesting to compare heavy with light atoms, for example aluminium with gold, because:

(i) the nuclei of such atoms have very different proton/neutron numbers (Al $= 13/14$, Au $= 79/118$).

(ii) such atoms have very different contributions to their mass from:

- (a) electrostatic energy $[\frac{3}{5}(Ze)^2 r^{-1}/mc^2 \simeq 0.003$ (Al) or 0.008 (Au)];
- (b) overall binding energy [Mass defect$/mc^2 = 0.0089$ (Al) or 0.0084 (Au)];
- (c) virtual positrons $[m_{e+}/mc^2 \simeq 3 \times 10^{-7}$ (Al) or 2×10^{-6} (Au); see p. 33 of *Gravitation & Relativity* by M. G. Bowler for details].

Hence from these experiments we may conclude that $|\Gamma - 1| \ll 1$ for all forms of mass-energy, with the exceptions of energy associated with weak and gravitational interactions.[6]

Extrapolating wildly from these experiments we hypothesize:

Strong Principle of Equivalence: *No experiment could distinguish between a homogeneous gravitational field and an accelerating frame of reference. In particular, in any frame which falls freely through such a field all the laws of physics are the same as if no field were present.*

Real gravitational fields are never homogeneous, so they *can* be distinguished from an accelerating frame of reference. For example, consider a star-warrior who regains consciousness in a closed cabin some time after being taken prisoner. He reaches for his watch and knocks it to the floor. Fortunately it falls only slowly, so it continues to tick. Is he in a (possibly elastic) accelerating spaceship, or is he on an asteroid? By now fully alert he determines that plumb bobs on either side of the cabin point towards a spot some ten miles away. He instantly concludes that he is either on an asteroid or that opposite sides of his cabin are accelerating away from one another. Moments later he verifies that his bobs have *not* moved apart. Hence he must be in the gravitational field of an asteroid.

Exercise (16):

What would he have concluded if he had found that his bobs pointed *away* from a spot thirty yards distant?

This example shows that a gravitational field is generally *not* equivalent to an accelerating frame of reference. From the Principle of Equivalence we merely conclude that physics in an accelerating frame of reference must look like physics in a particular type of gravitational field. However, this observation suggests a strategy for discovering how things behave in a strong gravitational field: we first work out the equations governing motion in the absence of a gravitational field (which we understand) when referred to a non-inertial frame of reference. This is a purely mathematical exercise. The equations we derive will contain terms associated with pseudo-forces generated by our accelerating frame of reference. Since there is really no gravitational field present, these pseudo-force terms will be restricted in form. The plan is to obtain equations for physics in the presence of a true gravitational field by lifting these restrictions.

5 Tensors in General Relativity

We start by discovering what the laws of e.m. and mechanics look like in a non-inertial frame. Let x'^μ be such a non-inertial frame and x^μ an inertial frame. Then each primed coordinate is a smooth function $x'^\mu(x^\nu)$ of the four inertial coordinates. Let $x^\mu(\tau)$ be an arbitrary trajectory through space-time and $\psi(x^\mu)$ an arbitrary scalar function of the inertial coordinates x^μ. Then the rate of change of ψ as perceived by an observer who moves along the trajectory $x^\mu(\tau)$ is

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\partial\psi}{\partial x^\mu} \equiv v^\mu \frac{\partial\psi}{\partial x^\mu},$$

[6] These contribute negligibly to the masses of atoms. However, since weak interactions are known to be intimately connected with electromagnetism, it is extremely unlikely that the value of Γ associated with weak-interaction energy differs from that associated with e.m. energy.

where we have defined the observer's 4-velocity $v^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}\tau$. Since by the chain rule

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} \tag{5.1}$$

we have

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = v^\mu \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial \psi}{\partial x'^\nu}.$$

If we define the observer's 4-velocity in the non-inertial primed frame to be

$$v'^\nu \equiv \frac{\partial x'^\nu}{\partial x^\mu} v^\mu, \tag{5.2}$$

then we may write

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = v'^\nu \frac{\partial \psi}{\partial x'^\nu}.$$

A natural extension of this argument leads us to define the primed components of any up vector A^μ as given in terms of the un-primed components by

$$A'^\nu \equiv \frac{\partial x'^\nu}{\partial x^\mu} A^\mu. \tag{5.3}$$

Note that if the primed frame were inertial, we would have $x'^\nu = x_0^\nu + \Lambda^\nu{}_\mu x^\mu$ (x_0^ν a constant 4-vector), so that $\partial x'^\nu/\partial x^\mu = \Lambda^\nu{}_\mu$ and the transformation (5.3) would reduce to a standard Lorentz transformation of an up vector.

If v^μ and u^μ are two up vectors, all inertial observers will agree on the value of the scalar

$$s \equiv \eta_{\mu\nu} u^\mu v^\nu. \tag{5.4}$$

How can we recover this number from the primed components v'^μ and u'^μ? First we express v^μ in terms of v'^μ. We use the chain rule to express $\mathrm{d}x'^\mu$ as

$$\mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\,\mathrm{d}x^\nu. \tag{5.5}$$

Dividing by $\mathrm{d}x'^\kappa$ and proceeding to the limit $\mathrm{d}x'^\kappa \to 0$ at fixed values of all the other coordinates, we get

$$\delta^\mu_\kappa = \frac{\partial x'^\mu}{\partial x'^\kappa} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\kappa}. \tag{5.6}$$

Thus the matrix $\partial x^\nu/\partial x'^\kappa$ is the inverse of the matrix $\partial x'^\mu/\partial x^\nu$. Premultiplying equation (2) by this matrix we solve for v^μ:

$$v^\mu = \frac{\partial x^\mu}{\partial x'^\nu} v'^\nu. \tag{5.7}$$

Using this relation to eliminate the unprimed components from (5.4) we get

$$s = \Big(\eta_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\kappa}\frac{\partial x^\nu}{\partial x'^\lambda}\Big) u'^\kappa v'^\lambda.$$

If we define

$$g'_{\kappa\lambda} \equiv \eta_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\kappa} \frac{\partial x^\nu}{\partial x'^\lambda}, \tag{5.8}$$

we have

$$s = g'_{\kappa\lambda} u'^\kappa v'^\lambda. \tag{5.9}$$

Like $\eta_{\kappa\lambda}$ the general **metric tensor** $g'_{\kappa\lambda}$ is symmetric; $g'_{\kappa\lambda} = g'_{\lambda\kappa}$. However, it is not necessarily diagonal. It is called the metric tensor because it allows us to calculate the lengths of vectors such as v'^λ.

We may use $g'_{\kappa\lambda}$ to lower indices;

$$v'_\kappa \equiv g'_{\kappa\lambda} v'^\lambda. \tag{5.10}$$

Let $g'^{\mu\nu}$ be the tensor which raises indices. Then in order that the operations of raising and lowering be mutual inverses we require that for all v'^μ

$$\delta^\mu_\lambda v'^\lambda = v'^\mu = g'^{\mu\kappa} g'_{\kappa\lambda} v'^\lambda.$$

i.e. that $g'^{\mu\kappa} g'_{\kappa\lambda} = \delta^\mu_\lambda$ and hence that $g'^{\mu\kappa}$ is the inverse of $g'_{\kappa\lambda}$.

Exercise (17):

Show that this definition of $g'^{\mu\kappa}$ is equivalent to the definition

$$g'^{\kappa\lambda} = \frac{\partial x'^\kappa}{\partial x^\mu} \frac{\partial x'^\lambda}{\partial x^\nu} \eta^{\mu\nu}. \tag{5.11}$$

Similarly, if for any tensors **F** and **G** we define

$$F'^{\kappa\lambda} \equiv \frac{\partial x'^\kappa}{\partial x^\mu} \frac{\partial x'^\lambda}{\partial x^\nu} F^{\mu\nu} \qquad \text{and} \qquad G'_{\kappa\lambda} \equiv \frac{\partial x^\mu}{\partial x'^\kappa} \frac{\partial x^\nu}{\partial x'^\lambda} G_{\mu\nu}, \tag{5.12}$$

we ensure that the primed observer will be able to calculate the scalar quantities $F^{\mu\nu} v_\mu u_\nu$ and $G_{\mu\nu} v^\mu u^\nu$ from primed quantities. The generalization to tensors of arbitrary rank is obvious.

Exercise (18):

Show that if x'^μ and x''^μ are two non-inertial frames, the transformation rules

$$v''^\mu = \frac{\partial x''^\mu}{\partial x'^\nu} v'^\nu \quad ; \quad v''_\mu = \frac{\partial x'^\nu}{\partial x''^\mu} v'_\nu \tag{5.13a}$$

$$F''^{\mu\nu} = \frac{\partial x''^\mu}{\partial x'^\kappa} \frac{\partial x''^\nu}{\partial x'^\lambda} F'^{\kappa\lambda} \quad \text{etc} \tag{5.13b}$$

apply.

[Hint: divide (5.5) by $\mathrm{d}x''^\kappa$ to obtain a relation equivalent to $\dfrac{\partial x''^\mu}{\partial x^\kappa} \dfrac{\partial x^\kappa}{\partial x'^\nu} = \dfrac{\partial x''^\mu}{\partial x'^\nu}$].

Notice that there is an easy way to figure out whether to multiply by $\partial x^\mu/\partial x'^\nu$ or by $\partial x'^\mu/\partial x^\nu$ when transforming an object $G^{\mu\cdots}$ or $G_{\mu\ldots}$: If the prime are up on the left, put them up on the right by using $\partial x'^\mu/\partial x^\nu$; if the unprimes are up on the left put them on top on the right with $\partial x^\mu/\partial x'^\nu$. The other kind of index in the equation will "cancel out" just as in ordinary multiplication of fractions. These rules extend in the obvious way to down vectors.

The metric tensor $g'_{\mu\nu}$ enables us to calculate the length s of any curve $x'^\mu(\lambda)$ in space-time:

$$s \equiv \int_a^b \mathrm{d}\lambda \sqrt{\left| g'_{\mu\nu} \frac{\mathrm{d}x'^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x'^\nu}{\mathrm{d}\lambda} \right|}. \tag{5.14}$$

If the curve is time-like, s is just c times the elapse $\Delta\tau$ of time on the watch of the observer whose trajectory $x'^\mu(\lambda)$ is. If there is an inertial frame in which all the points on the curve have the same value of x^0, s coincides with the length of the curve as measured with meter rules etc by an observer who is stationary in that privileged frame. We shall call s the **affine parameter** along the curve and use it to characterize points on the curve; hence we write $x'^\mu(s)$.

5.1 Equation of motion in a non-inertial frame

We now use the principle of least action to obtain the equation of motion of a charged particle in a crazy coordinate system. In this frame the action (3.4) reads

$$S = \int_0^1 \mathrm{d}\lambda \left(-m_0 c \sqrt{-g'_{\mu\nu} \frac{\mathrm{d}x'^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x'^\nu}{\mathrm{d}\lambda}} + q \frac{\mathrm{d}x'^\mu}{\mathrm{d}\lambda} A'_\mu \right), \tag{5.15}$$

and the EL eqn to which it gives rise is

$$0 = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(m_0 g'_{\beta\mu} \frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau} + q A'_\beta \right) - \frac{m_0 c \dot{x}'^\mu \dot{x}'^\nu}{2\sqrt{-g'_{\kappa\lambda}\dot{x}'^\kappa \dot{x}'^\lambda}} \frac{\partial g'_{\mu\nu}}{\partial x'^\beta} - q \frac{\mathrm{d}x'^\mu}{\mathrm{d}\lambda} \frac{\partial A'_\mu}{\partial x'^\beta}, \tag{5.16}$$

where a dot denotes differentiation w.r.t. λ. As in §3.1, we multiply through by $\mathrm{d}\lambda/\mathrm{d}\tau$, obtaining

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\tau} \left(m_0 g'_{\beta\mu} \frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau} + q A'_\beta \right) - \tfrac{1}{2} m_0 \frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x'^\nu}{\mathrm{d}\tau} \frac{\partial g'_{\mu\nu}}{\partial x'^\beta} - q \frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau} \frac{\partial A'_\mu}{\partial x'^\beta} \\ &= m_0 \left\{ g'_{\beta\mu} \frac{\mathrm{d}^2 x'^\mu}{\mathrm{d}\tau^2} + \frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x'^\nu}{\mathrm{d}\tau} \left(\frac{\partial g'_{\beta\mu}}{\partial x'^\nu} - \tfrac{1}{2} \frac{\partial g'_{\mu\nu}}{\partial x'^\beta} \right) \right\} + q \frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau} \left(\frac{\partial A'_\beta}{\partial x'^\mu} - \frac{\partial A'_\mu}{\partial x'^\beta} \right) \end{aligned} \tag{5.17}$$

If we now define

$$F'_{\mu\beta} \equiv \left(\frac{\partial A'_\beta}{\partial x'^\mu} - \frac{\partial A'_\mu}{\partial x'^\beta} \right) \quad ; \quad \Gamma'_{\mu\nu,\beta} \equiv \tfrac{1}{2} \left(\frac{\partial g'_{\beta\mu}}{\partial x'^\nu} + \frac{\partial g'_{\beta\nu}}{\partial x'^\mu} - \frac{\partial g'_{\mu\nu}}{\partial x'^\beta} \right), \tag{5.18}$$

then (5.17) takes the suggestive form

$$0 = g'_{\beta\mu} \frac{\mathrm{d}^2 x'^\mu}{\mathrm{d}\tau^2} + \frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x'^\nu}{\mathrm{d}\tau} \Gamma'_{\mu\nu,\beta} + \frac{q}{m_0} \frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau} F'_{\mu\beta}. \tag{5.19}$$

Box 2: Calculating Christoffel Symbols

In the case $\mathbf{A} = 0$, the first line of eq (5.17) is exactly what we would get if we applied the EL equation to $\int \mathrm{d}\tau g'_{\mu\nu}(\mathrm{d}x'^\mu/\mathrm{d}\tau)(\mathrm{d}x'^\nu/\mathrm{d}\tau)$. This fact is worth remembering as it often provides the easiest way to calculate the Christoffel symbols, which are the coefficients of products of velocity components when the derivatives in (5.17) are worked through. Note, however, that we have no a priori justification for applying the EL eqn to this integral; the procedure is just an algebraic trick that is justified by our derivation of (5.17).

To clean up our act, we define $\mathbf{\Gamma}$ with an index up as

$$\Gamma'^{\mu}_{\ \alpha\beta} \equiv g'^{\mu\nu}\Gamma'_{\alpha\beta,\nu} = \tfrac{1}{2}g'^{\mu\nu}\left(\frac{\partial g'_{\nu\alpha}}{\partial x'^\beta} + \frac{\partial g'_{\beta\nu}}{\partial x'^\alpha} - \frac{\partial g'_{\alpha\beta}}{\partial x'^\nu}\right). \tag{5.20}$$

Notice the pattern of this important formula: the three terms in $(\ldots)$ are just the first derivative of g with the indices cyclically permuted. The minus assign attaches to the term which groups the indices in the same way as $\mathbf{\Gamma}$. Now multiplying equation (5.19) through by $g'^{\alpha\beta}$ and writing $v'^\mu \equiv \mathrm{d}x'^\mu/\mathrm{d}\tau$, we can write it

$$\frac{\mathrm{d}v'^\alpha}{\mathrm{d}\tau} = -\Gamma'^{\alpha}_{\mu\nu}v'^\mu v'^\nu + \frac{q}{m_0}F'^{\alpha}{}_{\mu}v'^\mu. \tag{5.21}$$

This equation relates the apparent acceleration in our non-inertial frame to the e.m. force given by the second term on the right, and a pseudo-force given by the first term. The principle of equivalence suggests that gravitational forces will take the same form as pseudo-forces. Thus $\mathbf{\Gamma}$ should play the same role for the gravitational field as $\mathbf{F}$ does for the e.m. field. Notice that where the e.m. force is obtained by contracting $\mathbf{F}$ with $\mathbf{v}$, the gravitational force is obtained by contracting $\mathbf{\Gamma}$ with *two* copies of $\mathbf{v}$: in quantum mechanics it follows from this that whereas photons are spin-one particles, gravitons (likely to be detected within 5 years!) are spin-two particles. As is required by the principle of equivalence, the charge-to-mass ratio for gravity is unity.

$\mathbf{\Gamma}$ is called the **Christoffel symbol**. From its definition (5.20) it is symmetric in its first two indices. $\mathbf{\Gamma}$ cannot be a tensor since all its components are zero in an inertial frame, so if it transformed like a third-rank tensor, all its components would be zero in any coordinate system. Notice from (5.20) that the relationship between $\mathbf{\Gamma}$ and $\mathbf{g}$ mirrors the relationship between $\mathbf{F}$ and $\mathbf{A}$; that is, $\mathbf{g}$ is the potential for gravity in the same way that $\mathbf{A}$ is the potential for electromagnetism.

Below we will find it useful to have an expression for $\mathbf{\Gamma}$ in terms of double derivatives of the inertial coordinates with respect to the non-inertial ones. From (5.8) and (5.18) we have

$$2\Gamma'_{\mu\nu,\beta} = \eta_{\kappa\lambda}\left\{\frac{\partial}{\partial x'^\nu}\left(\frac{\partial x^\kappa}{\partial x'^\beta}\frac{\partial x^\lambda}{\partial x'^\mu}\right) + \frac{\partial}{\partial x'^\mu}\left(\frac{\partial x^\kappa}{\partial x'^\beta}\frac{\partial x^\lambda}{\partial x'^\nu}\right) - \frac{\partial}{\partial x'^\beta}\left(\frac{\partial x^\kappa}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\nu}\right)\right\}.$$

When we differentiate these products, the two terms generated by the last product will each be cancelled by a term generated by one of the first two products. The two remaining terms will be identical. Thus we have

$$\Gamma'_{\mu\nu,\beta} = \eta_{\kappa\lambda}\frac{\partial x^\kappa}{\partial x'^\beta}\frac{\partial^2 x^\lambda}{\partial x'^\mu \partial x'^\nu}. \tag{5.22}$$

Raising the last index, we obtain

$$\Gamma'^\alpha_{\mu\nu} \equiv g'^{\alpha\beta}\Gamma'_{\mu\nu,\beta} = \eta^{\delta\epsilon}\frac{\partial x'^\alpha}{\partial x^\delta}\frac{\partial x'^\beta}{\partial x^\epsilon}\eta_{\kappa\lambda}\frac{\partial x^\kappa}{\partial x'^\beta}\frac{\partial^2 x^\lambda}{\partial x'^\mu \partial x'^\nu} = \frac{\partial x'^\alpha}{\partial x^\lambda}\frac{\partial^2 x^\lambda}{\partial x'^\mu \partial x'^\nu}. \tag{5.23}$$

5.2 Covariant differentiation

We shall need to compare vectors at different points on the curve. In an inertial frame this is easy: two vectors are the same iff all their components are the same. But in passing from an inertial to a non-inertial frame by equations (5.3), we change the components of vectors in a position-dependent way. So two vectors that are equal in the sense that in an inertial frame all their components are equal, can have different components in a non-inertial frame. We need a way of diagnosing this condition of hidden equality.

Suppose that in an inertial frame we have a vector field $\mathbf{A}(\mathbf{x})$. By (5.3) this gives rise to a vector field $\mathbf{A}'(\mathbf{x}')$ in a non-inertial frame. As we go along a curve $\mathbf{x}(s)$ the rate of change in the vector of the field is

$$\dot{\mathbf{A}} \equiv \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{A} = \frac{\mathrm{d}x'^\kappa}{\mathrm{d}s}\frac{\partial \mathbf{A}}{\partial x'^\kappa}, \tag{5.24}$$

where the affine parameter s is defined by (5.14). Using (5.7) we move the $\mathbf{A}$ on the right into the primed system and get

$$\begin{aligned}\dot{A}^\mu &= \frac{\mathrm{d}x'^\kappa}{\mathrm{d}s}\frac{\partial}{\partial x'^\kappa}\Big(\frac{\partial x^\mu}{\partial x'^\alpha}A'^\alpha\Big)\\ &= \frac{\mathrm{d}x'^\kappa}{\mathrm{d}s}\Big(\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial A'^\alpha}{\partial x'^\kappa} + \frac{\partial^2 x^\mu}{\partial x'^\kappa \partial x'^\alpha}A'^\alpha\Big).\end{aligned}$$

Finally, premultiplying by $\partial x'^\nu/\partial x^\mu$ and using (5.23) we get

$$\dot{A}'^\nu \equiv \frac{\partial x'^\nu}{\partial x^\mu}\dot{A}^\mu = \frac{\mathrm{d}x'^\kappa}{\mathrm{d}s}\Big(\frac{\partial A'^\nu}{\partial x'^\kappa} + \Gamma'^\nu_{\kappa\alpha}A'^\alpha\Big). \tag{5.25}$$

(Notice that $\dot{A}'^\nu$, the ν^{th} component in the primed system of the vector $\dot{\mathbf{A}}$, is *defined* by this equation. It must not be confused with the rate of change with s of the ν^{th} component of $\mathbf{A}'$. In (5.21), by contrast, $\mathrm{d}v'^\alpha/\mathrm{d}\tau$ is just the rate of change of the number v'^α.) If we define a new type of derivative, the **covariant derivative** by

$$A'^\nu_{;\kappa} \equiv \nabla_\kappa A'^\nu \equiv \frac{\partial A'^\nu}{\partial x'^\kappa} + \Gamma'^\nu_{\kappa\alpha}A'^\alpha, \tag{5.26}$$

then equation (25) can be written

$$\dot{A}'^{\nu} = \frac{\mathrm{d}x'^{\kappa}}{\mathrm{d}s}\nabla_{\kappa}A'^{\nu}. \tag{5.27}$$

The second term in the definition (5.26) of the covariant derivative has the following physical interpretation. For each value of κ, say $\kappa = 1$, we have a matrix $\Gamma'^{\nu}_{1\alpha}$. When we multiply this matrix by δx^1 we obtain the Lorentz transformation matrix $\mathbf{\Lambda}$ which tells us by how much the speed and orientation of the frame used at $\mathbf{x}$ differs from that used at $(x^0, x^1 + \delta x^1, x^2, x^3)$.[7]

If $\mathbf{A}$ is really the same all along the curve, and only seems to change because we are using a non-inertial coordinate system, we have $\dot{A}'^{\nu} = 0$, and thus that the "gradient" $\nabla_{\kappa}A'^{\nu}$ of A'^{ν} either vanishes or is "perpendicular" to the direction $\mathrm{d}x'^{\kappa}/\mathrm{d}s$ in which we are moving.

How does ∇ operate on down vectors? Consider

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}(A'^{\mu}B'_{\mu}) &= \frac{\mathrm{d}x'^{\kappa}}{\mathrm{d}s}\frac{\partial}{\partial x'^{\kappa}}(A'^{\mu}B'_{\mu})\\ &= \frac{\mathrm{d}x'^{\kappa}}{\mathrm{d}s}\Big[\Big(\frac{\partial A'^{\mu}}{\partial x'^{\kappa}}\Big)B'_{\mu} + A'^{\mu}\Big(\frac{\partial B'^{\mu}}{\partial x'^{\kappa}}\Big)\Big]\\ &= \frac{\mathrm{d}x'^{\kappa}}{\mathrm{d}s}\Big[\big(\nabla_{\kappa}A'^{\mu}\big)B'_{\mu} - \Gamma'^{\mu}_{\kappa\alpha}A'^{\alpha}B'_{\mu} + A'^{\mu}\frac{\partial B'_{\mu}}{\partial x'^{\kappa}}\Big]\\ &= \frac{\mathrm{d}x'^{\kappa}}{\mathrm{d}s}\Big[\big(\nabla_{\kappa}A'^{\mu}\big)B'_{\mu} + A'^{\mu}\frac{\partial B'_{\mu}}{\partial x'^{\kappa}} - \Gamma'^{\alpha}_{\kappa\mu}A'^{\mu}B'_{\alpha}\Big].\end{aligned}$$

This suggests that we define

$$\nabla_{\kappa}B'_{\mu} \equiv \frac{\partial B'_{\mu}}{\partial x'^{\kappa}} - \Gamma'^{\alpha}_{\kappa\mu}B'_{\alpha} \tag{5.28}$$

for then we will have $\nabla_{\kappa}(A'_{\mu}B'^{\mu}) = B'^{\mu}\nabla_{\kappa}A'_{\mu} + A'_{\mu}\nabla_{\kappa}B'^{\mu}$ and ∇ will operate on such products like any other derivative operator.

The same argument applied to quantities like $G'_{\mu\nu}A'^{\mu}B'^{\nu}$ leads to the rules

$$G'_{\mu\nu;\kappa} \equiv \nabla_{\kappa}G'_{\mu\nu} \equiv \frac{\partial G'_{\mu\nu}}{\partial x'^{\kappa}} - \Gamma'^{\alpha}_{\kappa\mu}G_{\alpha\nu} - \Gamma'^{\alpha}_{\kappa\nu}G_{\mu\alpha} \tag{5.29a}$$

$$G'^{\mu\nu}_{\ \ ;\kappa} \equiv \nabla_{\kappa}G'^{\mu\nu} \equiv \frac{\partial G'^{\mu\nu}}{\partial x'^{\kappa}} + \Gamma'^{\mu}_{\kappa\alpha}G^{\alpha\nu} + \Gamma'^{\nu}_{\kappa\alpha}G^{\mu\alpha}. \tag{5.29b}$$

Notice that each index requires a Γ-symbol, with a plus or a minus sign according as the index is up or down.

[7] In "gauge field theories" this idea is generalized to define covariant derivatives for objects ψ that live in spaces other than space-time. In the simplest case ψ lives in the two-dimensional space of complex numbers, for which the analogue of a Lorentz transformation is multiplication by another complex number, say iqA_1. The covariant derivative is now $\mathcal{D}_{\mu} \equiv \partial_{\mu} + iqA_{\mu}$. If ψ is the wavefunction of a spin-zero particle of charge q, A_{μ} proves to be the regular e.m. potential.

In the same spirit we define the operation of ∇ on scalars to be identical with partial differentiation:

$$\nabla_\kappa \psi = \frac{\partial \psi}{\partial x'^\kappa}$$

What action does ∇ have on the metric tensor? Suppose that **A** and **B** are two vector fields that everywhere have the same components in an inertial frame. Then $\nabla_\kappa A'^\mu = \nabla_\kappa B'^\mu = 0$. Also $A^\mu B_\mu = g'_{\mu\nu} A'^\mu B'^\nu$ is everywhere the same. Hence for all curves $x'(s)$

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\big(g'_{\mu\nu} A'^\mu B'^\nu\big).$$

Replacing $\mathrm{d}/\mathrm{d}s$ with $(\mathrm{d}x'^\kappa/\mathrm{d}s)\nabla_\kappa$ and differentiating each item in the bracket, we get

$$\begin{aligned} 0 &= \frac{\mathrm{d}x'^\kappa}{\mathrm{d}s}\Big\{\big(\nabla_\kappa g'_{\mu\nu}\big)A'^\mu B'^\nu + g'_{\mu\nu}\big[\big(\nabla_\kappa A'^\mu\big)B'^\nu + A'^\mu\big(\nabla_\kappa B'^\nu\big)\big]\Big\} \\ &= \frac{\mathrm{d}x'^\kappa}{\mathrm{d}s} A'^\mu B'^\nu \nabla_\kappa g'_{\mu\nu}. \end{aligned}$$

Since $\mathrm{d}x'^\kappa/\mathrm{d}s$, A'^μ and B'^ν are all arbitrary, it follows that

$$\nabla_\kappa g'_{\mu\nu} = 0. \tag{5.30}$$

In words, the *covariant derivative of the metric tensor is always zero.*

If $x^\mu(s)$ is a straight line, all components of the "tangent vector" $\mathrm{d}x^\mu/\mathrm{d}s$ are constant in an inertial frame. Hence in any coordinate system the tangent vector $x'^\mu(s)$ of a straight line satisfies the o.d.e.

$$0 = \frac{\mathrm{d}x'^\kappa}{\mathrm{d}s}\nabla_\kappa \frac{\mathrm{d}x'^\mu}{\mathrm{d}s}. \tag{5.31}$$

Substituting for ∇_κ this becomes

$$\begin{aligned} 0 &= \frac{\mathrm{d}x'^\kappa}{\mathrm{d}s}\Big(\frac{\partial}{\partial x'^\kappa}\frac{\mathrm{d}x'^\mu}{\mathrm{d}s} + \Gamma'^\mu_{\kappa\alpha}\frac{\mathrm{d}x'^\alpha}{\mathrm{d}s}\Big) \\ &= \frac{\mathrm{d}^2 x'^\mu}{\mathrm{d}s^2} + \Gamma'^\mu_{\kappa\alpha}\frac{\mathrm{d}x'^\kappa}{\mathrm{d}s}\frac{\mathrm{d}x'^\alpha}{\mathrm{d}s} \quad (x'^\mu(s) \text{ a straight line.}) \end{aligned} \tag{5.32}$$

Exercise (19):

Obtain (5.32) by extremizing the integral (5.14) with respect to variations of the path $x'^\mu(s)$; a straight line is the least distance between two points.

In terms of covariant derivatives, Newton's law of motion (1.44) and the Maxwell equations (1.49) become

$$m_0 v'^\kappa \nabla_\kappa v'^\mu = f'^\mu, \tag{5.33a}$$

$$F'^{\mu\nu}{}_{;\nu} = \mu_0 j'^\mu. \tag{5.33b}$$

The other laws of e.m. (3.32) (1.49) remain unchanged because the Christoffel symbols introduced in going over from partial to covariant derivatives magically cancel.

Exercise (20):

Prove that $A'_{\mu;\nu} - A'_{\nu;\mu} = A'_{\mu},_\nu - A'_{\nu},_\mu$.

5.3 Summary

The rules for transforming between non-inertial frames are the same as those for making regular Lorentz transformation with the substitutions

$$\Lambda_\mu{}^\nu \to \frac{\partial x'^\nu}{\partial x''^\mu} \quad ; \quad \Lambda^\mu{}_\nu \to \frac{\partial x''^\mu}{\partial x'^\nu}. \quad \text{Thus} \quad A''_\mu = \frac{\partial x'^\nu}{\partial x''^\mu} A'_\nu.$$

The Minkowski metric $\boldsymbol{\eta}$ is replaced by the metric tensor $\mathbf{g}$, which remains symmetric but is no longer its own inverse; consequently the up-up and down-down forms of $\mathbf{g}$ are in general distinct.

In a non-inertial frame $\mathbf{x}$ the partial derivative operator $\partial_\mu \equiv \partial/\partial x^\mu$ should be replaced with the covariant derivative operator ∇_μ:

$$\begin{aligned}
\nabla_\mu \psi &= \partial_\mu \psi \\
A^\nu_{;\mu} \equiv \nabla_\mu A^\nu &= \partial_\mu A^\nu + \Gamma^\nu_{\mu\alpha} A^\alpha \quad ; \quad \nabla_\mu B^{\nu\lambda} = \partial_\mu B^{\nu\lambda} + \Gamma^\nu_{\mu\alpha} B^{\alpha\lambda} + \Gamma^\lambda_{\mu\alpha} B^{\nu\alpha} \\
\nabla_\mu A_\nu &= \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu} A_\alpha \quad ; \quad \nabla_\mu B_{\nu\lambda} = \partial_\mu B_{\nu\lambda} - \Gamma^\alpha_{\mu\nu} B_{\alpha\lambda} - \Gamma^\alpha_{\mu\nu} B_{\nu\alpha}
\end{aligned}$$

The Christoffel symbol $\boldsymbol{\Gamma}$ is

$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\nu\alpha}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right).$$

The covariant derivative of $\mathbf{g}$ always vanishes: $\nabla \mathbf{g} = 0$

6 Gravity, Geometry & the Einstein Field Equations

Now that we have completed our programme for discovering what physics looks like in a non-inertial frame, it is a good idea to take a rest from all these acres of indices and summarize the physical content of our formulae.

We have defined quantities $g'_{\mu\nu}$, p'_μ, $F'_{\mu\nu}$, j'_μ, $\Gamma'^\mu_{\alpha\beta}$ etc which enable us to use a non-inertial coordinate system $\mathbf{x}'$ to find the space-time trajectory of a charged particle in an e.m. field. We defined these quantities in terms of the momenta, e.m. field tensor etc in an underlying inertial coordinate system $\mathbf{x}$ and the coordinate transformation $\mathbf{x}'(\mathbf{x})$

that couples the inertial and non-inertial systems. But we have found formulae (5.13) and (5.20) which enable us to calculate the values $g''_{\mu\nu}$ etc of all needful quantities in a second non-inertial coordinate system without reference back to the inertial system $\mathbf{x}$.

Since we shall no longer need to refer constantly to an inertial system, we now drop the convention that the unprimed system $\mathbf{x}$ is inertial; from here on *all systems are to be assumed to be non-inertial unless explicitly specified as inertial.*

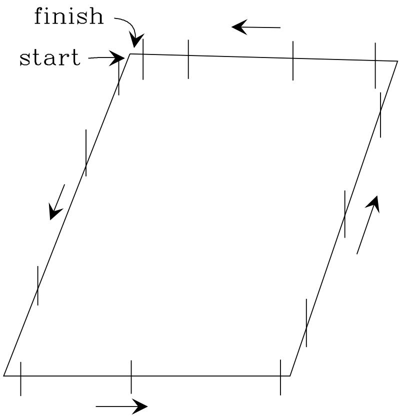
The principle of equivalence suggests that a gravitational field will look very much like a pseudo-force in an accelerating frame of reference. The Christoffel symbol $\mathbf{\Gamma}$ generates the pseudo-force associated with an accelerating frame, so when a gravitational field is present $\mathbf{\Gamma}$ will play the role of the Newtonian force $\boldsymbol{f}$. We have identified the metric $\mathbf{g}$ as the relativistic generalization of the Newtonian potential Φ on the ground that $\mathbf{\Gamma}$ can be written in terms of derivatives of $\mathbf{g}$ just as $\boldsymbol{f} = -\nabla\Phi$.

In Newton's theory $\boldsymbol{f}$ and Φ are related to the density ρ of gravitating matter via Poisson's equation (4.5). The relativistic generalization of (4.5) should be a second-order p.d.e. in $\mathbf{g}$, or equivalently, a first-order p.d.e. in $\mathbf{\Gamma}$. What is this equation?

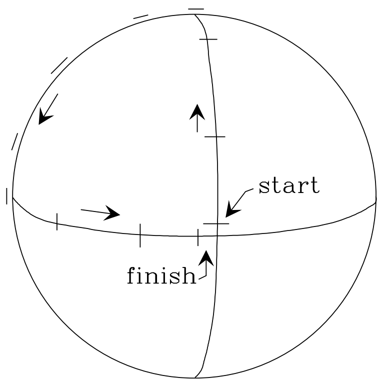
Since we can make $\mathbf{\Gamma}$ as large as we like simply by choosing a perverse coordinate system, it is clear that the trick in finding suitable field equations is to find a differential operator on $\mathbf{\Gamma}$ which differentiates away all the contribution to $\mathbf{\Gamma}$ that is caused by mere perversity of the coordinate system. The key to finding this operator proves to be an examination of the geometrical relationships between the lengths of lines and the magnitudes of angles between lines.

We have seen that the metric tensor enables us to define the length of any curve in space-time, and in particular to determine through (5.32) which curves $\mathbf{x}'(s)$ are straight. Now suppose we draw a straight line in a portion of space in which there is no gravitational field and then draw a unit circle around some point on this line. Then no matter what coordinate system we use for the calculations, we shall find that the length s of the circumference is exactly $\pi = 3.14159\ldots$ times the length of the circle's diameter. How come? By changing the coordinate system we can change $\mathbf{g}$ at any given point to almost any value [see (5.8)]. So how come that when we evaluate the integral (5.14) over two completely different sets of points, we always get answers in the same ratio? It must be that $\mathbf{g}$ at one point *is not independent of* $\mathbf{g}$ *at neighbouring points*: $\mathbf{g}$ must satisfy some differential equation. Einstein's idea, and it was pure magic, was that it is *this* differential equation which tells us that there is no gravitational field present, only a perverse coordinate system. Let us find this differential equation.

There are many geometrical relationships in addition to the one just discussed which $\mathbf{g}$ must furnish if there is no gravitational field present. For example, there are 180° in a triangle. But the key to the equation we are seeking turns out to be something slightly odd. It is to consider what happens when we slide a vector around a closed curve while being careful not to rotate the vector. If we do this on a table, the vector (a pencil, say) will be back in its old configuration at the end of the experiment:



But on a sphere things go differently:



In fact, on a sphere of radius r, the angle through which a pencil rotates on being "parallel-transported" around a curve is equal to the area enclosed by this curve divided by r^2.

6.1 The curvature tensor

If we parallel-transport a vector $\mathbf{A}$ around a closed curve $\mathbf{x}(s)$ in space-time, we have that at each point on the curve $\dot{\mathbf{x}} \cdot \nabla \mathbf{A} = 0$ (this is just a statement of the invariance along the curve of $\mathbf{A}$'s components in an inertial frame)

$$0 = \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \nabla_\alpha A^\mu = \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \Big(\frac{\partial A^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\beta} A^\beta \Big). \tag{6.1}$$

Consequently, the total change in each component A^μ on going around is

$$\Delta A^\mu = \oint \frac{\partial A^\mu}{\partial x^\alpha} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \,\mathrm{d}s = - \oint \Gamma^\mu_{\alpha\beta} A^\beta \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \,\mathrm{d}s. \tag{6.2}$$

In this integral both $\Gamma^\mu_{\alpha\beta}$ and A^β are functions of s through $\mathbf{x}(s)$. However, if we consider only infinitesimal loops we may expand each component of $\mathbf{\Gamma}$ and $\mathbf{A}$ in power series about some point, say $\mathbf{X}$, on the loop:

$$\begin{aligned}\Gamma^{\mu}_{\alpha\beta}(\mathbf{x}) &= \Gamma^{\mu}_{\alpha\beta}(\mathbf{X}) + (x^\nu - X^\nu)\frac{\partial \Gamma^{\mu}_{\alpha\beta}}{\partial x^\nu} + \cdots \\ A^\mu(\mathbf{x}) &= A^\mu(\mathbf{X}) + (x^\nu - X^\nu)\frac{\partial A^\mu}{\partial x^\nu} + \cdots\end{aligned} \tag{6.3}$$

Multiplying these two expansions together and substituting the result into (6.2), we get

$$\Delta A^\mu = -\oint \left\{ \left[\Gamma^{\mu}_{\alpha\beta} A^\beta\right]_{\mathbf{X}} + \left[\Gamma^{\mu}_{\alpha\beta}\frac{\partial A^\beta}{\partial x^\nu} + A^\beta \frac{\partial \Gamma^{\mu}_{\alpha\beta}}{\partial x^\nu}\right]_{\mathbf{X}} (x^\nu - X^\nu) + \cdots \right\} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\,\mathrm{d}s.$$

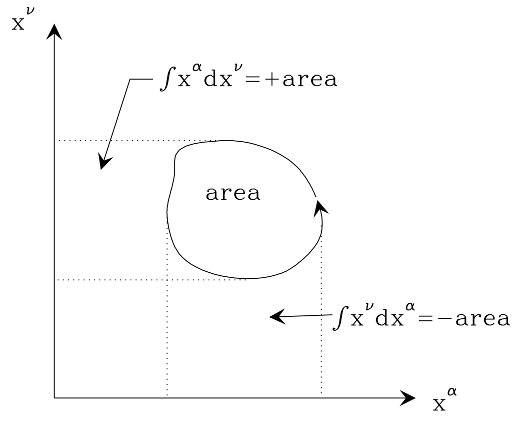
Since the first square bracket is constant, it can be taken outside the integral sign. Integrating its coefficient $\mathrm{d}x^\alpha/\mathrm{d}s$ around our closed contour we then obtain zero. The second square bracket may also be taken outside the integral sign. Integrating (6.1) along our contour we find

$$\sum_\alpha \frac{\partial A^\mu}{\partial x^\alpha}\bigg|_{\mathbf{X}} (x^\alpha - X^\alpha) = -\left(A^\beta \Gamma^{\mu}_{\alpha\beta}\right)_{\mathbf{X}} (x^\alpha - X^\alpha) + \mathrm{O}(s^2), \tag{6.4}$$

so we may eliminate $(\partial A^\mu/\partial x^\nu)$ from (6.3) and get

$$\begin{aligned}\Delta A^\mu &= \left[\Gamma^{\mu}_{\alpha\beta}\Gamma^{\beta}_{\nu\gamma}A^\gamma - \frac{\partial \Gamma^{\mu}_{\alpha\beta}}{\partial x^\nu} A^\beta\right]_{\mathbf{X}} \oint (x^\nu - X^\nu)\mathrm{d}x^\alpha + \cdots \\ &= \left[\Gamma^{\mu}_{\alpha\beta}\Gamma^{\beta}_{\nu\gamma} - \frac{\partial \Gamma^{\mu}_{\alpha\gamma}}{\partial x^\nu}\right]_{\mathbf{X}} A^\gamma \oint x^\nu\,\mathrm{d}x^\alpha + \cdots\end{aligned} \tag{6.5}$$

The integrals in (6.5) for which $\nu = \alpha$ vanish because each such integral is simply the change in $\frac{1}{2}(x^\alpha)^2$ on going around the loop. Furthermore, when $\alpha \neq \nu$, the integral $\oint x^\nu \mathrm{d}x^\alpha$ is equal in magnitude and opposite in sign to the integral $\oint x^\alpha \mathrm{d}x^\nu$ as this picture of the (x^α, x^ν) plane shows:



We define the directed area enclosed by the loop to be the antisymmetric tensor

$$\Delta S^{\nu\alpha} \equiv \oint x^\nu\,\mathrm{d}x^\alpha. \tag{6.6}$$

This done we may write

$$\Delta A^\mu = \left[\Gamma^\mu_{\alpha\beta}\Gamma^\beta_{\nu\gamma} - \frac{\partial \Gamma^\mu_{\alpha\gamma}}{\partial x^\nu}\right]_{\mathbf{X}} A^\gamma \Delta S^{\nu\alpha} + \dots \tag{6.7}$$

In the absence of a gravitational field, $\Delta A^\mu = 0$ for any A^μ. Furthermore, by an appropriate choice of loop $\Delta S^{\nu\alpha}$ can be set equal to any given antisymmetric tensor.[8] So it is tempting to conclude that the square bracket in the last equation vanishes. However, when we contract an antisymmetric tensor with a tensor of mixed symmetry, only the antisymmetric portion of the mixed tensor contributes to the sums. Hence from the vanishing of ΔA^μ for arbitrary A^μ and $\Delta S^{\nu\alpha}$ we can infer only the vanishing of the portion of the square bracket that is antisymmetric on exchange of ν and α. We therefore define the **curvature tensor** as minus twice this part of the square bracket in (6.7)

$$R^\mu_{\gamma\alpha\nu} \equiv \frac{\partial \Gamma^\mu_{\alpha\gamma}}{\partial x^\nu} - \frac{\partial \Gamma^\mu_{\nu\gamma}}{\partial x^\alpha} + \Gamma^\mu_{\nu\beta}\Gamma^\beta_{\alpha\gamma} - \Gamma^\mu_{\alpha\beta}\Gamma^\beta_{\nu\gamma}, \tag{6.8}$$

and rewrite (6.7) as

$$\Delta A^\mu = -\tfrac{1}{2} R^\mu_{\gamma\alpha\nu} A^\gamma \Delta S^{\nu\alpha} + \cdots \tag{6.9}$$

Since ΔA^μ is the difference between two vectors that are based at the same point, it is itself a vector. Furthermore, both A^γ and $\Delta S_{\nu\alpha}$ are tensors. Hence $R^\mu_{\gamma\alpha\nu}$ must also be a tensor as its name implies. In the absence of a gravitational field we have

$$R^\mu_{\gamma\alpha\nu} = 0. \tag{6.10}$$

This is the relativistic generalization of the Newtonian equation $\partial^2\Phi/\partial x^\alpha \partial x^\beta = 0$ of which Laplace's equation is the trace. As promised, it is first-order in $\mathbf{\Gamma}$ and second-order in $\mathbf{g}$. Notice that it is non-linear in both these quantities; this is highly significant (and very inconvenient!).

6.2 Derivation of the Field Equations

If we are to upgrade (6.10) into the relativistic generalization of Poisson's equation (4.5), we must replace the zero on the right with something that involves the density of mass-energy. We have seen [equations (1.29)] that the mass-energy density forms one component of a symmetric second-rank tensor $\mathbf{T}$. If we want a covariant theory of gravity we are going to have to allow the mass-energy density to bring along all its friends in $\mathbf{T}$ into the field equations. So consider replacing the zero in (6.10) with

$$\text{constant} \times T_{\alpha\beta}.$$

This has only two indices, whereas the left of (6.10) has four indices. Hence we must either use $\mathbf{g}$ (which is the only generally available tensor) to add two more indices on

[8] This is a lie, as the discussion of 6-tuples in §2.5 shows. However, the argument can be fixed up by adding the changes $\Delta\mathbf{A}$ around two non-coplanar paths.

the right, or we must contract away two indices on the left. It is not hard to see that these two courses are equivalent. We do it the second way.

Which two indices should we contract? Well, from the defining expression (6.8) one may show that $R_{\mu\nu\alpha\beta}$ has the following symmetries:

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu} \quad ; \quad R_{\mu\nu\beta\alpha} = -R_{\mu\nu\alpha\beta} = R_{\nu\mu\alpha\beta}. \tag{6.11}$$

In words; **R** is symmetric on interchange of the first pair of indices with the second pair, and antisymmetric under interchange of the indices within each of these pairs. Thus we get zero if we contract within any pair, and the same answer (to within a sign) if we contract between pairs. It is conventional to define the **Ricci tensor** by

$$R_{\alpha\beta} \equiv R^{\mu}_{\alpha\mu\beta}. \tag{6.12}$$

Note:

In terms of $\mathbf{\Gamma}$, $R_{\alpha\beta}$ is by (6.8)

$$R_{\alpha\beta} = \frac{\partial \Gamma^{\mu}_{\mu\alpha}}{\partial x^{\beta}} - \frac{\partial \Gamma^{\mu}_{\alpha\beta}}{\partial x^{\mu}} + \Gamma^{\lambda}_{\alpha\mu}\Gamma^{\mu}_{\beta\lambda} - \Gamma^{\mu}_{\lambda\mu}\Gamma^{\lambda}_{\alpha\beta}. \tag{6.13a}$$

Furthermore, by (5.20)

$$\Gamma^{\mu}_{\alpha\mu} = \Gamma^{\mu}_{\mu\alpha} = \tfrac{1}{2} g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}. \tag{6.13b}$$

While $R_{\alpha\beta}$ has the right number of indices to go on the left of our field equations, the law we seek is not $R_{\alpha\beta} = T_{\alpha\beta}$ because mass-energy conservation is expressed by the vanishing of the covariant divergence of **T**. Hence whatever goes on the left of our field equations must have zero divergence. Unfortunately, the divergence of $R_{\alpha\beta}$ is not always zero. However, it turns out that (see Appendix B)

$$R_{\alpha}{}^{\beta}{}_{;\beta} = \tfrac{1}{2} R_{;\alpha}, \tag{6.14}$$

where the **Ricci scalar** R is defined by

$$R \equiv R_{\beta}{}^{\beta}. \tag{6.15}$$

From (6.14) it follows that a tensor made out of $R^{\mu}_{\nu\alpha\beta}$ which *has* zero divergence is

$$G^{\alpha\beta} \equiv (R^{\alpha\beta} - \tfrac{1}{2} g^{\alpha\beta} R). \tag{6.16}$$

G is called the **Einstein tensor** because the p.d.e.'s which describe the generation of a gravitational field by matter are

$$G^{\alpha\beta} = -\frac{8\pi G}{c^4} T^{\alpha\beta}. \tag{6.17}$$

Here G is Newton's gravitational constant, as we shall shortly show. An alternative, and often handier, form of (6.17) is obtained by contracting both sides of the equation to obtain

$$G_\alpha{}^\alpha = (R_\alpha{}^\alpha - \tfrac{1}{2}\delta^\alpha_\alpha R) = -R = -\frac{8\pi G}{c^4}T_\alpha{}^\alpha.$$

Substituting this value of R into (6.17) we get

$$R^{\alpha\beta} = -\frac{8\pi G}{c^4}\big(T^{\alpha\beta} - \tfrac{1}{2}g^{\alpha\beta}T_\gamma{}^\gamma\big). \tag{6.18}$$

Equations (6.17) and (6.18) are the relativistic equivalents of Poisson's equation $\nabla^2\Phi = 4\pi G\rho$. As expected, these equations are second-order in the ten potentials $g^{\mu\nu}$ and involve all the energy-density's friends in $\mathbf{T}$.

There is a close analogy between (6.18) and its e.m. counterpart $F^{\mu\nu}{}_{;\nu} = \mu_0 j^\mu$ as may be seen by substituting for $\mathbf{R}$ from (6.13)

$$\frac{\partial\Gamma^\mu_{\mu\alpha}}{\partial x^\beta} - \frac{\partial\Gamma^\mu_{\alpha\beta}}{\partial x^\mu} + \Gamma^\lambda_{\alpha\mu}\Gamma^\mu_{\beta\lambda} - \Gamma^\mu_{\lambda\mu}\Gamma^\lambda_{\alpha\beta} = -\frac{8\pi G}{c^4}\big(T_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}T_\gamma{}^\gamma\big). \tag{6.19}$$

Worse still, the relationship (5.20) between $\mathbf{\Gamma}$ and the tensor potential $\mathbf{g}$ is a good deal more complex than the corresponding e.m. relation $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. So it is hardly surprising that not many exact solutions of the Einstein equations are known! But we shall be able to deduce some extremely interesting solutions nevertheless.

6.3 The Newtonian Limit

We now check that Einstein's theory agrees with Newton's when (i) the field is very weak and (ii) the field is generated by slowly-moving matter. The prototype of slowly-moving matter is 'dust': the matter at each event $\mathbf{x}$ has a well defined proper velocity $\mathbf{v}(\mathbf{x})$, and in the rest frame defined by $\mathbf{v}$ the matter density is ρ_0. Physically it is clear that in this rest frame the only non-zero component of $\mathbf{T}$ is $T^{00} = \rho_0 c^2$, and from this it follows easily that in a general frame a dust has

$$T^{\mu\nu} = \rho_0 v^\mu v^\nu. \tag{6.20}$$

Since the gravitational field is assumed very weak, we can find a nearly inertial coordinate system. In this system

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \quad \text{where} \quad |h_{\alpha\beta}| \ll 1. \tag{6.21}$$

We neglect squares and higher powers of $\mathbf{h}$. By (5.20) we then have

$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2}\eta^{\mu\nu}\Big(\frac{\partial h_{\nu\alpha}}{\partial x^\beta} + \frac{\partial h_{\nu\beta}}{\partial x^\alpha} - \frac{\partial h_{\alpha\beta}}{\partial x^\nu}\Big). \tag{6.22}$$

Consider the equation of motion (5.21) to which this gives rise for a non-relativistic free particle ($a'^\mu = 0$). The motion is governed by a gravitational force

$$f^\mu = -\Gamma^\mu_{\alpha\beta}v^\alpha v^\beta, \tag{6.23}$$

where $\mathbf{v}$ is the particle's 4-velocity. Since the zeroth component $v^0 = \gamma c$ of the 4-velocity of a non-relativistic particle is very much larger than any of $\mathbf{v}$'s spatial components, we expect the dominant term in the implied sum of (6.23) to be that for which $\alpha = \beta = 0$. Thus we expect

$$f^\mu \simeq -\gamma^2 c^2 \Gamma^\mu_{00}. \tag{6.24}$$

A typical spatial component of the equations of motion is then

$$\gamma \frac{\mathrm{d}}{\mathrm{d}t}\Big(\gamma \frac{\mathrm{d}x^j}{\mathrm{d}t}\Big) = -\gamma^2 c^2 \Gamma^j_{00} \simeq -c^2 \tfrac{1}{2}\Big(2\frac{\partial h_{j0}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^j}\Big).$$

If the field is stationary in our chosen coordinate system (and we are free to boost until it is), then $\partial h_{j0}/\partial x^0 = 0$ and to leading order in v/c

$$\frac{\mathrm{d}^2 x^j}{\mathrm{d}t^2} = \frac{\partial}{\partial x^j}\big(\tfrac{1}{2}c^2 h_{00}\big). \tag{6.25}$$

If this is to agree with Newton's theory, we require

$$\Phi = -\tfrac{1}{2}c^2 h_{00}, \tag{6.26}$$

where Φ is the Newtonian gravitational potential.

We now check whether Einstein's field equations (6.18) reduce in the same weak-field limit to Poisson's equation for Φ. We expect the source of Φ to be the energy density $\rho c^2 = T^{00}$, where $\mathbf{T}$ is the energy-momentum tensor, so we concentrate on the 00-component of (6.18).

From (6.13a,b), (6.21) and (6.22), $R_{\alpha\beta}$ is to first order in $\mathbf{h}$

$$R_{\alpha\beta} = \tfrac{1}{2}\eta^{\mu\nu}\Big(\frac{\partial^2 h_{\mu\nu}}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2 h_{\alpha\nu}}{\partial x^\mu \partial x^\beta} - \frac{\partial^2 h_{\nu\beta}}{\partial x^\mu \partial x^\alpha} + \frac{\partial^2 h_{\alpha\beta}}{\partial x^\nu \partial x^\mu}\Big). \tag{6.27}$$

In particular, for a time-independent field

$$R^{00} = R_{00} = \tfrac{1}{2}\nabla^2 h_{00} = -\frac{1}{c^2}\nabla^2\Phi.$$

If the only contributor to the energy-momentum tensor is a dust of stationary particles, $\mathbf{T}$ is given by (6.20) with $\mathbf{v} = (c, 0, 0, 0)$. Hence $T^\gamma_\gamma = -\rho_0 c^2$ and the 00-component of (6.18) is

$$R^{00} = -\frac{1}{c^2}\nabla^2\Phi = -\frac{4\pi G}{c^2}\rho_0$$

as expected.

6.4 Summary

The curvature tensor $R^{\mu}_{\nu\alpha\beta}$ tells us by how much a vector changes on being parallel transported around a small circuit. Hence we detect the use of a crazy coordinate system for flat space-time by seeing if the curvature tensor $\mathbf{R} = 0$. If $\mathbf{R} \neq 0$ there is a true gravitational field.

The presence of matter at $\mathbf{x}$ is signalled by $R_{\alpha\beta}(\mathbf{x}) \equiv R^{\mu}_{\alpha\mu\beta}(\mathbf{x}) \neq 0$.

Formally, there is a far-reaching analogy between g.r. and e.m.:

Parallelism of e.m. and g.r.

$$\begin{array}{rcl}
A_\mu & \leftrightarrow & g_{\mu\nu} \\
F_{\mu\nu} = -(A_{\mu,\nu} - A_{\nu,\mu}) & \leftrightarrow & \Gamma_{\mu,\alpha\beta} = \frac{1}{2}(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}) \\
f^\mu = \dfrac{q}{m_0} F^{\mu}{}_{\alpha} v^\alpha & \leftrightarrow & f^\mu = -\Gamma^\mu_{\alpha\beta} v^\alpha v^\beta \\
F^{\mu\nu}{}_{,\nu} = \mu_0 j^\mu & \leftrightarrow & \text{eq. (6.19)} \\
F^{\mu\nu}{}_{,\rho} + F^{\nu\rho}{}_{,\mu} + F^{\rho\mu}{}_{,\nu} = 0 & \leftrightarrow & R^\kappa_{\lambda\mu\nu;\rho} + R^\kappa_{\lambda\nu\rho;\mu} + R^\kappa_{\lambda\rho\mu;\nu} = 0 \\
 & & \text{(Bianchi identity)}
\end{array}$$

The parallel between Newton's theory and g.r. is less tight: $\Phi \leftrightarrow \mathbf{g}$, $\boldsymbol{f} \leftrightarrow \boldsymbol{\Gamma}$, $\nabla^2\Phi \leftrightarrow R_{\alpha\beta}$.

In a weak gravitational field we can have $\mathbf{g} \simeq \boldsymbol{\eta}$ with $-2\Phi/c^2$ as an estimate of $(g_{00} - \eta_{00})$.

7 Weak-field gravity

The simplest applications of GR are to weak gravitational fields, which are unbiquitous in the Universe at large as here on Earth.

7.1 Gravitational Redshift

We have just seen that in a weak gravitational field $g_{00} \simeq \eta_{00} - 2\Phi/c^2$ is closely related to the Newtonian gravitational potential. This conclusion has interesting physical consequences. Consider an observer at rest in a weak gravitational field. We choose spatial coordinates so that the field and the observer are stationary. Let the observer be at potential Φ_o and observe a stationary atom at potential Φ_a. Setting $\lambda = x^0$ in (5.14) and differentiating both sides of this equation we find that the observer's proper time elapses at a rate

$$\begin{aligned}
\frac{\mathrm{d}\tau_o}{\mathrm{d}t} &= \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}x^0}\frac{\mathrm{d}x^\nu}{\mathrm{d}x^0}} \\
&= \sqrt{-g_{00}} \simeq \sqrt{1 + 2\frac{\Phi_o}{c^2}} \qquad (7.1)\\
&\simeq 1 - \frac{|\Phi_o|}{c^2} \qquad \text{(because } \Phi_o < 0\text{)}.
\end{aligned}$$

Similarly, the atom's proper time elapses at a rate

$$\frac{d\tau_a}{dt} = 1 - \frac{|\Phi_a|}{c^2}. \tag{7.2}$$

If the atom is emitting e.m. radiation of frequency ν, then during an interval $\Delta\tau_o$ on the observer's clock it will emit $(\nu\Delta\tau_o) \times (d\tau_a/d\tau_o)$ wave fronts. Of course, these wavefronts will take some time (as measured by either clock) to reach the observer, but because our situation is static, the delay before each front reaches the observer is always the same. Hence the fronts will be received in time $\Delta\tau_o$ on the observer's clock and the observer measures frequency

$$\Big(\frac{d\tau_a}{d\tau_o}\Big)\nu = \frac{1 - |\Phi_a|/c^2}{1 - |\Phi_o|/c^2}\nu \simeq \Big(1 - \frac{|\Phi_a - \Phi_o|}{c^2}\Big)\nu. \tag{7.3}$$

In words: radiation that comes up out of a gravitational well is redshifted.

Exercise (21):

Consider a machine which lowers boxes full of excited atoms on ropes down a well, deexcites the atoms at the bottom, pulls the atoms back up and then reexcites the atoms with the photons released at the bottom and beamed up to the top. Show that this machine will violate energy conservation unless the photons' frequencies at top and bottom of the well satisfy (7.3).

7.2 Hydrodynamics

In GR the energy-momentum tensor of a perfect fluid (1.32) clearly becomes

$$T^{\mu\nu} = (\rho + P/c^2)u^\mu u^\nu + Pg^{\mu\nu}. \tag{7.4}$$

We now show how the equations of hydrodynamics emerge from $\nabla_\mu T^{\mu\nu} = 0$. We have

$$0 = u^\nu u^\mu \nabla_\mu(\rho + P/c^2) + (\rho + P/c^2)\nabla_\mu(u^\mu u^\nu) + g^{\mu\nu}\frac{\partial P}{\partial x^\nu}, \tag{7.5}$$

where we've used (5.30). To recover the familiar equations of hydrodynamics we assume that $c \simeq u^0 \gg u^i$ and that $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$ with $|h^{\mu\nu}| \ll 1$. Then the 0 equation may be approximated by

$$0 = cu^\mu \nabla_\mu(\rho + P/c^2) + (\rho + P/c^2)\nabla_\mu(cu^\mu) - \frac{\partial P}{\partial x^0}. \tag{7.6}$$

We multiply this equation by u^i/c and subtract it from the i equation of the set (7.5) to find

$$0 = (\rho + P/c^2)u^\mu \nabla_\mu u^i + \frac{\partial P}{\partial x^i} + \frac{u^i}{c}\frac{\partial P}{\partial x^0} \tag{7.7}$$

Now in view of (6.25) we can write

$$\begin{aligned} u^\mu \nabla_\mu u^i &= u^\mu \frac{\partial u^i}{\partial x^\mu} + \Gamma^i_{\mu\alpha} u^\mu u^\alpha \simeq \frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} + \Gamma^i_{00} u^0 u^0 \\ &\simeq \frac{\mathrm{d}u^i}{\mathrm{d}t} + c^2 \tfrac{1}{2}\left(2\frac{\partial h_{j0}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^j}\right), \end{aligned} \tag{7.8}$$

where the derivative of u^i is the Eulerian derivative of hydrodynamics. We neglect the derivative of h_{j0} w.r.t. x^0 on the grounds that the gravitational field is nearly static, and in (7.7) we neglect the derivative of P w.r.t. time on the grounds that it is smaller than the derivative w.r.t. x^i by a factor of order c_s/c, where c_s is the sound speed. Then substituting (7.8) into (7.7) and using $h_{00} = -2\Phi/c^2$ we obtain

$$(\rho + P/c^2)\frac{\mathrm{d}u^i}{\mathrm{d}t} = -\frac{\partial P}{\partial x^i} - (\rho + P/c^2)\frac{\partial \Phi}{\partial x^i}. \tag{7.9}$$

In the limit $P \ll \rho c^2$ this agrees with Euler's equation of hydrodynamics.

Equation (7.6) is a statement of energy conservation: ρc^2 contains both the fluid's rest-mass energy $\rho_0 c^2$ and its thermodynamic internal energy U. Since ρ_0 is contributed by a conserved number of baryons, we have an additional conservation equation $\nabla_\mu(\rho_0 u^\mu) = 0$, and this equation reduces in the Newtonian limit to the familiar equation of continuity

$$\frac{\mathrm{d}\rho_0}{\mathrm{d}t} + \rho_0 \frac{\partial u^j}{\partial x^j} = 0. \tag{7.10}$$

7.3 Harmonic coordinates & Gravitational Waves

We formulated the equations of physics in arbitrary coordinates as a mathematical ruse to extract the implications of the principle of equivalence. But in GR, as in every other branch of physics, it's politic, even vital, to use the best coordinates for the job. So we don't really want the freedom to use any old coordinates; we need a way of choosing sensible coordinates. When discussing black holes and cosmology we'll be guided to a coordinate system by the symmetries of the problem. But generic problems don't have much symmetry and then we should use coordinates that satisfy the **harmonic gauge condition**

$$g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0. \tag{7.11}$$

In Problem set 2 you can show that the harmonic gauge condition is satisfied when the coordinates share with Cartesian coordinates the property that they satisfy the wave equation: $\Box x^\alpha = 0$. To first order in $h_{\mu\nu}$ the gauge condition reads

$$0 \simeq \eta^{\mu\nu}\eta^{\alpha\beta}\left(\frac{\partial h_{\beta\nu}}{\partial x^\mu} + \frac{\partial h_{\mu\beta}}{\partial x^\nu} - \frac{\partial h_{\mu\nu}}{\partial x^\beta}\right) = 2\partial_\mu h^{\alpha\mu} - \partial^\alpha h \quad \text{where} \quad h \equiv h^\beta_\beta. \tag{7.12}$$

Equation (6.27) can be rewritten

$$R_{\alpha\beta} = \tfrac{1}{2}\left[\frac{\partial^2 h}{\partial x^\alpha \partial x^\beta} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial h^\mu_\alpha}{\partial x^\beta} + \frac{\partial h^\mu_\beta}{\partial x^\alpha}\right) + \Box h_{\alpha\beta}\right]. \tag{7.13}$$

When (7.12) is used three of the terms cancel and we have

$$R_{\alpha\beta} = \tfrac{1}{2}\Box h_{\alpha\beta}. \tag{7.14}$$

Let's see what happens when we use this form of the Ricci tensor in (6.18) when $T^{\mu\nu}$ on the right is for a perfect fluid [eq. (7.4)]. Since $u_\alpha u^\alpha = -c^2$ we have $T^\gamma_\gamma = 3P - \rho c^2$, so (6.18) reads

$$\Box h_{\alpha\beta} = -\frac{16\pi G}{c^4}\left[(\rho + P/c^2)u_\alpha u_\beta + \tfrac{1}{2}(\rho c^2 - P)\eta_{\alpha\beta}\right]. \tag{7.15}$$

From the occurrence of $\Box$ on the left it follows that GR predicts the existence of **gravitational waves** that propagate at the speed of light. The right side of this equation provides a source for these waves in the same way that the e.m. current j_μ provides the source for electromagnetic waves through the analogous equation $\Box A_\mu = \mu_0 j_\mu$.

If the gravitational field is static in the rest frame of the fluid, in this frame, and using $h_{00} = -2\Phi/c^2$, the 00 component of equation (7.15) reads

$$\nabla^2\Phi = 4\pi G(\rho + 3P/c^2). \tag{7.16}$$

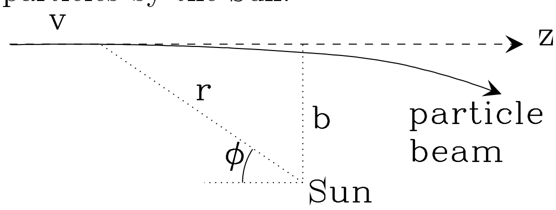
From this equation we see that GR predicts that pressure is a source of the gravitational field independently of the energy density that's associated with it. In the early Universe and inside very massive stars the energy density is dominated by black-body radiation, for which $P = \frac{1}{3}\rho c^2$. So for this fluid Poisson's equation reads

$$\nabla^2\Phi = 8\pi G\rho. \tag{7.17}$$

The factor 8 on the right implies that black body radiation is twice as powerful as a source of gravity as rest-mass energy.

7.4 Deflection of Light by Gravity

Naive treatment A simple back-of-the-envelope argument based on the Strong Principle of Equivalence shows that light must be deflected by the Sun and allows us to obtain a quick order-of-magnitude estimate of the magnitude of this effect: the S.P. of E. implies that the path of a photon beam must be approached by a particle beam in the limit as the particles' speed $v \to c$. So let's calculate the deflection of fast (but non-relativistic) particles by the Sun.



Since the beam is fast, its deflection will be small, and we can estimate the net gravitational impulse delivered to each particle by integrating the gravitational force along a straight line. We neglect variations in the particle's speed parallel to this line, so $z \simeq vt$. Hence after a fly-by to within distance b of the Sun, the particle has a component of velocity perpendicular to the original line of magnitude

$$v_\perp \simeq \frac{1}{m}\int_{-\infty}^{\infty} F_\perp \,\mathrm{d}t = 2\int_0^\infty \frac{G\,\mathrm{M}_\odot}{r^2}\frac{b}{r}\frac{\mathrm{d}z}{v} = \frac{c^2 r_s(\odot)}{bv}\int_0^\infty \frac{\mathrm{d}\zeta}{(1+\zeta^2)^{3/2}},$$

where

$$r_s(r) \equiv \frac{2GM}{c^2} \tag{7.18}$$

is the **Schwarzschild radius** and Pythagoras' useful result has been pressed into service. The substitution $\zeta = \sinh\theta$ enables one to show that the integral equals 1. So the beam is deflected through the small angle

$$\alpha \simeq \frac{v_\perp}{v} \simeq \frac{r_s c^2}{v^2 b}.$$

In the limit $v \to c$, this tends to $r_s/b \simeq 0.875''$ for $b = R_\odot$.

Relativistic treatment A proper calculation will show that our neglect of relativity has cost us a factor of 2, and Murphy's law notwithstanding, the true deflection is larger than our naive estimate predicts. In 1919 general relativity hit the headlines when its prediction for α was confirmed by measurements made during a solar eclipse.

Since 1979 observations of the gravitational deflection of light have become important astronomical tools for determining not only the structure of the Milky Way and the disposition of dark matter, but even the scale of the Universe. In these applications the gravitational field is always weak ($|\Phi|/c^2 \ll 1$), and for this case we can derive a general formula for deflection by an arbitrary weak gravitational field.

By imposing the harmonic gauge condition (7.11) the metric of a weak, static gravitational field can be put into the form (see Problem Set 2)

$$\mathrm{d}\tau^2 = \left(1 + 2\frac{\Phi}{c^2}\right)\mathrm{d}t^2 - c^{-2}\left(1 - 2\frac{\Phi}{c^2}\right)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{7.19}$$

Let $(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ be the change in the spatial coordinates of a photon in time $\mathrm{d}t$, then since $\mathrm{d}\tau$ has to vanish along a photon's world line, we have from (7.19) that

$$\begin{aligned} \mathrm{d}t &= \frac{1}{c}\left(\frac{1 - 2\Phi/c^2}{1 + 2\Phi/c^2}\right)^{1/2}\mathrm{d}s, \\ &\simeq \frac{1}{c}\left(1 - 2\frac{\Phi}{c^2}\right)\mathrm{d}s \end{aligned} \tag{7.20}$$

where $\mathrm{d}s \equiv \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}$ is the coordinate distance between two points on the ray. Since $\Phi \le 0$, equation (7.20) states that in our coordinate system light propagates precisely as it would if there were no gravitational field but space were filled by a medium of refractive index

$$n = 1 - 2\frac{\Phi}{c^2} \geq 1. \tag{7.21}$$

Thus looking through a region that is permeated by a gravitational field should be like looking through a sheet of bobbly glass: the images of background light sources will be shifted in position as well as distorted in shape and changed in brightness by refraction.

These effects are most readily quantified by use of Fermat's principle, which states that the paths taken by light rays between a source and an observer extremize the elapse of coordinate time as photons pass between source and observer.[9] Thus, we determine the paths for which the light-travel time

$$t = \frac{1}{c}\int \mathrm{d}s\, n \tag{7.22}$$

is stationary with respect to small changes in the path. Since we are interested in light paths that are nearly rectilinear, we may orient our coordinate system such that one coordinate, say z, increases monotonically along the path. When we employ z rather than s as the integration variable, (7.22) becomes

$$ct = \int \mathrm{d}z\, n(\boldsymbol{x}) \left[\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2 + 1\right]^{1/2} \equiv \int \mathrm{d}z\, f(x, y, x', y', z). \tag{7.23}$$

Finding the path $x(z)$ that extremizes this integral is a standard problem in the calculus of variations. The desired path satisfies the Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(n(\boldsymbol{x})\left[\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2 + 1\right]^{-1/2}\frac{\mathrm{d}x}{\mathrm{d}z}\right) = \frac{\partial n}{\partial x}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2 + 1\right]^{1/2} \tag{7.24}$$

We integrate both sides of this differential equation with respect to z between the source and the observer. Since

$$\mathrm{d}s = \left[\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2 + 1\right]^{1/2} \mathrm{d}z,$$

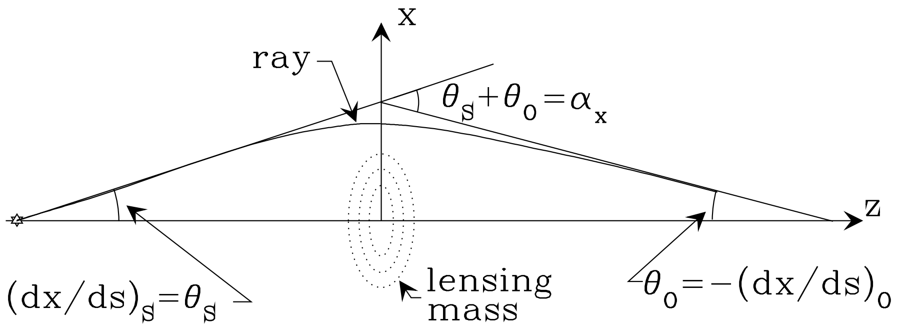
we find

$$\left[n(\boldsymbol{x})\,\frac{\mathrm{d}x}{\mathrm{d}s}\right]_{\text{source}}^{\text{obs}} = \int_{\mathrm{S}}^{\mathrm{O}} \mathrm{d}s\,\frac{\partial n}{\partial x}. \tag{7.25}$$

Both the source and the observer are assumed to lie far from the deflecting mass, in regions within which $n = 1$, so

$$\left.\frac{\mathrm{d}x}{\mathrm{d}s}\right|_{\text{obs}} - \left.\frac{\mathrm{d}x}{\mathrm{d}s}\right|_{\text{source}} = \int_{\mathrm{S}}^{\mathrm{O}} \mathrm{d}s\,\frac{\partial n}{\partial x}. \tag{7.26}$$

[9] Physically Fermat's principle applies because when the time elapse is not stationary, neighbouring paths allow the observer to 'see' the source at different times when it is in different phases of oscillation. These different 'views' average to zero.



The figure shows that the left-hand side of this equation is the angle through which the projection onto the xy-plane of the ray from source to observer is bent. We define α_x to be this angle and note that equivalent relations hold for the yz-plane. Hence the angle between the direction of the ray at the source S and at the observer O is given by

$$\boldsymbol{\alpha} = -\int_{\mathrm{S}}^{\mathrm{O}} \nabla_\perp n \, \mathrm{d}s, \tag{7.27}$$

where the integral is along the ray's path and $\nabla_\perp$ denotes the derivative perpendicular to the path. When we substitute from (7.21) for n, we have

$$\boldsymbol{\alpha} = \frac{2}{c^2}\int_{\mathrm{S}}^{\mathrm{O}} \nabla_\perp \Phi \, \mathrm{d}s = \frac{4}{c^2}\nabla_\perp \Phi_2, \tag{7.28a}$$

where

$$\Phi_2 \equiv \frac{1}{2}\int_{\mathrm{S}}^{\mathrm{O}} \Phi \, \mathrm{d}s. \tag{7.28b}$$

The potential Φ is related to the lens's mass-density ρ by Poisson's equation, $\nabla^2\Phi = 4\pi G\rho$. We orient our coordinate system so that the z axis passes through the observer and is tangent to the light path near the latter's point of closest approach to the deflector. Then we integrate Poisson's equation with respect to z:

$$\begin{aligned} 4\pi\Sigma \equiv 4\pi G \int_{z_1}^{z_2} \rho \, \mathrm{d}z &= \int_{z_1}^{z_2} \left(\nabla_\perp^2 \Phi + \frac{\partial^2 \Phi}{\partial z^2}\right) \mathrm{d}z \\ &= \nabla_\perp^2 \int_{z_1}^{z_2} \Phi \, \mathrm{d}z + \left[\frac{\partial \Phi}{\partial z}\right]_{z_1}^{z_2}. \end{aligned} \tag{7.29}$$

On account of the smallness of the deflection angle, the integral over z of Φ in the first term on the right side of this equation differs insignificantly from twice the quantity Φ_2 that is defined by (7.28b). Moreover, the square bracket in (7.29) represents the gravitational accelerations that the lensing object generates at source and observer. In practical applications these can be neglected because source and observer are extremely remote from the lens. Hence (7.29) yields

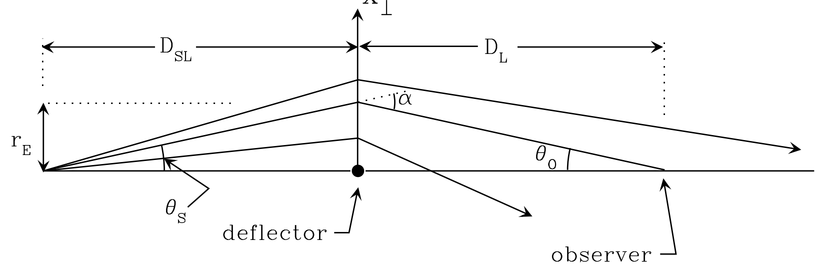
$$2\pi\Sigma \simeq \nabla_\perp^2 \Phi_2, \tag{7.30}$$

which is identical with the two-dimensional Poisson equation. Consequently, the potential Φ_2 that governs the deflection through equation (7.28a) is the gravitational potential that would be generated *in a 2-dimensional world* by the lens's projected density Σ.

An important special case is that in which the matter distribution is effectively that of a point mass M – that is, the deflecting matter distribution is confined well inside the impact parameter $\boldsymbol{x}_\perp$ of every ray of interest. Then $\Phi_2(\boldsymbol{x}_\perp) = GM \ln|\boldsymbol{x}_\perp|$ and

$$\alpha = \frac{4GM}{c^2 x_\perp}. \tag{7.31}$$

In particular, when light from a star that lies behind the Sun just grazes the Sun's limb, $x_\perp = R_\odot$ so $\alpha = 2r_s/R_\odot = 1.75$ arcsec is exactly twice our non-relativistic estimate (7.19). In the early 1990s the Hipparcos satellite measured the positions of a few $\times 10^5$ stars at various times of the year. Since the positions were accurate to of order one milliarcsec, the effects of light deflection by the Sun were apparent over the whole sky.



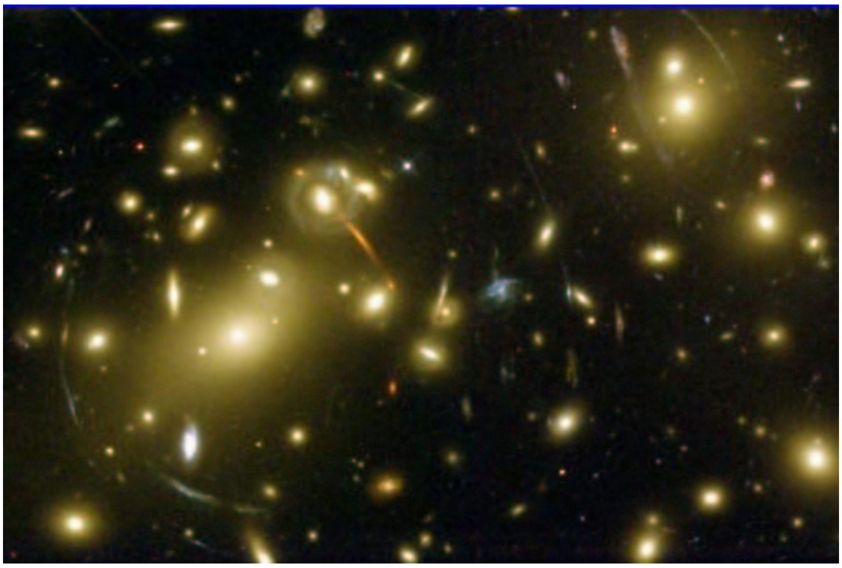
The situation when the source, mass and observer all lie on a straight line is described by the figure: rays that encounter the mass at small impact parameter cross the source–observer line in front of the observer, while rays that pass the mass at large impact parameters cross behind her. The ray that passes the mass with impact parameter $r_{\rm E}$ reaches the observer. Since, in the notation of the figure, $\theta_{\rm S} \simeq r_{\rm E}/D_{\rm SL}$, $\theta_{\rm O} \simeq r_{\rm E}/D_{\rm L}$ and $\alpha = \theta_{\rm S} + \theta_{\rm O}$, it follows with a little algebra from (7.31) that

$$r_{\rm E} = \sqrt{\frac{4GM}{c^2}}\sqrt{\frac{D_{\rm SL}D_{\rm L}}{D_{\rm SL}+D_{\rm L}}}.$$

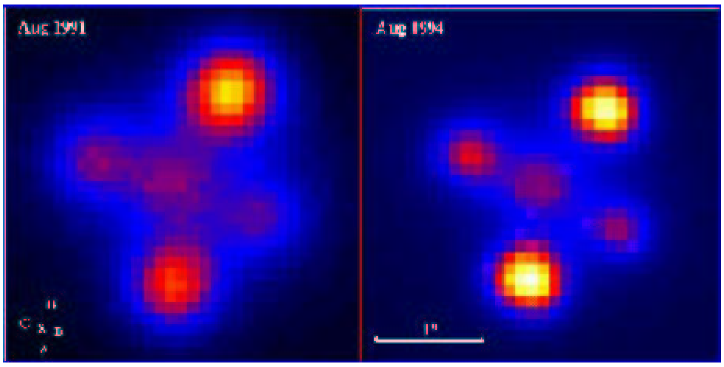
Although it depends on the relative positions of source, observer and deflector, $r_{\rm E}$ is usually called the **Einstein radius** of the deflector. If the source, deflector and observer are colinear as in the figure, the observer sees a bright ring of radius $r_{\rm E}$ around the deflector. The angular radius of this ring is the **Einstein angle**

$$\theta_{\rm E} \equiv \sqrt{\frac{4GM}{c^2}}\sqrt{\frac{D_{\rm SL}}{D_{\rm L}(D_{\rm SL}+D_{\rm L})}}. \tag{7.32}$$

When the source and lies to one side of the observer–deflector line, the Einstein ring degenerates into one or more arcs. An image of the cluster of galaxies Abell 2218 that was obtained by the Hubble Space Telescope provides several spectacular examples of this phenomenon.



When the deflecting mass distribution is not axisymmetric, several images of the source may form. The **Einstein Cross** consists of four images of a background quasar QSO 2237+0305 that happens to lie almost exactly behind a spiral galaxy at redshift $z = 0.04$.[10]



In general the time required for photons to pass from the source to the observer is different for each image of the source. Since the luminosity of a quasar typically varies on timescales of a year and even less, the differences between the times of flight to each image can be measured by cross-correlating as functions of time the measured brightnesses of each image. These time differences enable one to constrain the scale of the Universe, since they are clearly proportional to the distance to, and therefore the linear scale of, the deflector. For example a delay of $12 \pm 3\,$d between two images in B 0218+357 yields Hubble constant $H_0 \simeq 60\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.[11]

The Einstein radius is a dimensionally important quantity because lensing significantly modifies the appearance of a source that lies within about r_{E} of the deflector–observer line, while a source that lies further than r_{E} from this line will be seen very much as it would be if the deflector were not present. It is conventional to say that a source is lensed if it lies within r_{E} of a deflector.

[10] See Ostensen et al., 1996, *Astron. Astrophys*, 309, 59.
[11] Corbett et al., 1996, in proc. IAU Symp. 173; see also Saha & Williams 2003, *Astron. J.*, 125, 2769.

Consider the case in which both the source and the deflector are stars that lie within the Milky Way:

$$D_{\rm SL} = D_{\rm L} = 10\,{\rm kpc} = 3.08\times 10^{19}\,{\rm m} \quad\Rightarrow\quad \theta_{\rm E} = 0.9(M/M_\odot)^{1/2}\,{\rm mas}.$$

This angle is too small to be measured even with the Hubble Space Telescope. But it is easy to show that the relative motion of the source, deflector and the Sun will cause the amount by which deflection magnifies the background star to change within several weeks. So by monitoring the brightnesses of millions of stars lensing events can be detected even though their constituent images cannot be resolved. This technique has proved to be a powerful way of detecting faint objects in the Milky Way – the objects themselves are too faint to be seen, but they are detected by the effect they have on luminous background stars.[12]

The effects of gravitational deflection can also be important well outside $r_{\rm E}$. Specifically, when light from an extended source such as a galaxy passes to one side of a large mass concentration, differences in the deflections suffered by rays that come to the observer from different points on the source will distort the observer's image of the source. In particular, the image will tend to be stretched in the direction perpendicular to the line on the sky that runs from the source to the mass concentration. This effect is called **weak lensing**. By measuring the shapes of galaxy images in the vicinity of a cluster of galaxies, one can constrain the cluster's gravitational field.[13]

7.5 Summary

GR predicts that a gravitational field makes clocks run slow by a factor $1-|\Phi|/c^2$ that manifests itself in gravitational redshifts. The equations of hydrodynamics, recovered from $\nabla_\mu T^{\mu\nu} = 0$, predict that pressure augments the inertia of matter: in Euler's equation ρ is replaced by $\rho+P/c^2$. The harmonic gauge condition $g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0$ simplifies the equations by guiding us to sensible "near Cartesian" coordinates. With its help we see that GR predicts the existence of gravitational waves, and predicts that pressure is a source of gravity in its own right, so Poisson's equation has to be modified to $\nabla^2\Phi = 4\pi G(\rho + 3P/c^2)$. In the case of ultrarelativistic matter such as black-body radiation, $P = \frac{1}{3}\rho c^2$ so the strength of gravity is effectively doubled. Gravity appears to endow the vacuum with a non-trivial refractive index $n = 1 - 2\Phi/c^2 > 1$ – this phenomenon is an aspect of the slowing of clocks that are gravitational potential wells. The distortion of the images of distant objects to which $n \neq 1$ gives rise now provides a crucial probe of the Universe.

8 The Schwarzschild Solution

Now that we have the field equations (6.18) it is natural to seek the solution $\mathbf{g}$ that describes the gravitational field in the solar system. A useful step in this direction would be to find the metric associated with a point mass in an otherwise empty universe.

[12] See Popowski et al., 2004, (astro-ph/0410319)

[13] See Kaiser & Squires, 1993, *Astrophys. J.*, 404 441; also Cypriano et al. 2004, *Astrophys. J.* 613, 95.

The way we derive most solutions to Einstein's equations is at root the same as that by which we are accustomed to solve other partial differential equations, for example Maxwell's equations. If we want to find the electrostatic potential inside a charged spherical surface, we start by looking for potentials of the special form $\Phi(r,\theta,\phi) = R(r)\Theta(\theta)\mathrm{e}^{\mathrm{i}m\phi}$. We are not initially certain that such solutions exist, but we try the idea out anyway in the knowledge that if there are no such solutions we shall derive inconsistent conditions on R and Θ and thus discover our mistake, but if no inconsistencies arise, we shall get a valid solution and it will not matter that we found it by leaping into the dark.

Proceeding in this spirit towards the metric outside a point mass, we first argue that we should be able to find coordinates in which the metric is diagonal. To see why this is so, suppose we are given a metric tensor $\mathbf{g}$ for some two-dimensional space. Then from simple matrix algebra we know that at any point in the space we can find two mutually perpendicular directions, the eigenvectors $\mathbf{u}$ and $\mathbf{v}$ of $\mathbf{g}$, such that $\mathbf{g}$ would be a diagonal matrix if our coordinate directions coincided with $\mathbf{u}$ and $\mathbf{v}$. Now imagine marking the directions $\mathbf{u}$, $\mathbf{v}$ as small crosses on a grid of points in the space. Since $\mathbf{g}$ is a smoothly varying function of position, the orientation of neighbouring crosses will be similar. Hence we may draw smooth curves through neighbouring crosses, thus covering the space with a curvilinear grid. Finally, if we are able to label each curve of this doubly infinite family of curves with numbers (a,b), these numbers will constitute a valid coordinate system for the space and $\mathbf{g}$ will be diagonal in this coordinate system.

If we start from the metric tensor of a 4-space, the situation is fundamentally the same as in our two-dimensional example; the only difference is that there are now four special directions at each point. So it is reasonable to conjecture that we can find coordinates in which the metric of any simple spacetime is everywhere diagonal.

Furthermore, since the gravitational field we seek to describe is time-independent, we should be able to choose coordinates in such a way that none of the metric coefficients depends on time. Also the gravitational field will be spherically symmetric, so there must be closed 2-surfaces on which the geometry is that of a sphere. If we label these surfaces with the coordinates (r,t) and indicate position on each surface with the angle variables (θ,ϕ), we have

$$\begin{aligned} \mathrm{d}s^2 &\equiv g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -D(r)c^2\mathrm{d}t^2 + A(r)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) + B(r)\mathrm{d}r^2. \end{aligned} \tag{8.1}$$

We next fix the meaning of r by determining that the sphere with labels (r,t) should have area $4\pi r^2$. This yields

$$\mathrm{d}s^2 = -D(r)c^2\mathrm{d}t^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) + B(r)\mathrm{d}r^2. \tag{8.2}$$

The metric now takes the form

$$g_{\mu\nu} = \begin{matrix} t \\ r \\ \theta \\ \phi \end{matrix} \begin{pmatrix} -c^2D & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{8.3}$$

Exercise (22):

By making an appropriate coordinate transformation $\mathbf{x}'(\mathbf{x})$ show that when, as here, one uses t rather than ct for the 0^{th} coordinate, the 4-vector of a photon becomes $k^\mu = (\omega/c^2, \boldsymbol{k})$.

We next calculate the Christoffel symbols. We could proceed directly from (5.20), but when one wants to calculate large numbers of Christoffel symbols it is generally more cost-effective to use the procedure described in Box 2. We apply the EL eqn to the Lagrangian

$$L \equiv -c^2 D\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + B\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right] \tag{8.4}$$

finding

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(D\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)\\
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(B\frac{\mathrm{d}r}{\mathrm{d}\tau}\right) + \tfrac{1}{2}c^2 D'\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \tfrac{1}{2}B'\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - r\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right]\\
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right) - r^2\sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\\
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\sin^2\theta\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right).
\end{aligned} \tag{8.5}$$

After differentiating the products in these equations we can read off the Christoffel symbols by comparing the resulting equations of motion with (5.32):

$$\begin{aligned}
&\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{D'}{2D}\\
&\Gamma^r_{rr} = \frac{B'}{2B} \qquad \Gamma^r_{\theta\theta} = -\frac{r}{B} \qquad \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{B} \qquad \Gamma^r_{tt} = \frac{c^2 D'}{2B}\\
&\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r}\\
&\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{r} \qquad \Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta.
\end{aligned} \tag{8.6}$$

Hence

$$\Gamma^\mu_{r\mu} = \frac{B'}{2B} + \frac{2}{r} + \frac{D'}{2D}, \qquad \Gamma^\mu_{\theta\mu} = \cot\theta, \qquad \Gamma^\mu_{\phi\mu} = 0, \qquad \Gamma^\mu_{t\mu} = 0. \tag{8.7}$$

By hard slog and (6.13) one can now obtain

$$R_{tt} = -\frac{c^2 D''}{2B} + \frac{c^2 D'}{4B}\left(\frac{B'}{B} + \frac{D'}{D}\right) - \frac{c^2 D'}{rB} \tag{8.8a}$$

$$R_{rr} = \frac{D''}{2D} - \frac{D'}{4D}\left(\frac{B'}{B} + \frac{D'}{D}\right) - \frac{B'}{rB} \tag{8.8b}$$

$$R_{\theta\theta} = -1 + \frac{r}{2B}\left(-\frac{B'}{B} + \frac{D'}{D}\right) + \frac{1}{B} \tag{8.8c}$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta} \tag{8.8d}$$

$$R_{\mu\nu} = 0 \qquad \mu \neq \nu. \tag{8.8e}$$

We require $R_{\mu\nu} = 0$ everywhere except at $r = 0$ (where these expressions fail anyway). Multiplying (8.8a) by B/c^2D and adding the result to (8.8b) yields

$$\frac{B'}{B} = -\frac{D'}{D} \qquad \Rightarrow \qquad BD = \text{constant}. \tag{8.9}$$

As $r \to \infty$ the metric should become that of flat spacetime for which $B = D = 1$. Thus

$$B(r) = \frac{1}{D(r)} \quad \forall \quad r > 0. \tag{8.10}$$

By (8.8c) the equation $R_{\theta\theta} = 0$ now becomes

$$0 = R_{\theta\theta} = -1 + rD' + D \qquad \Rightarrow \qquad D = 1 + \text{constant}/r. \tag{8.11}$$

By (6.26) we know that as $r \to \infty$ and the field becomes weak, $D \to 1 + 2\Phi/c^2 = 1 - r_s/r$, where M is the mass at the centre and the **Schwarzschild radius** r_s is defined by

$$r_s \equiv \frac{2GM}{c^2}. \tag{8.12}$$

Hence we may identify the constant in (8.11) as $-r_s$, giving

$$D = 1 - \frac{r_s}{r}. \tag{8.13}$$

Collecting everything together we have the **Schwarzschild metric**

$$g_{\mu\nu} = \begin{matrix} t \\ r \\ \theta \\ \phi \end{matrix} \begin{pmatrix} -c^2D & & & \\ & D^{-1} & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix} = \begin{pmatrix} -c^2(1 - r_s/r) & & & \\ & (1 - r_s/r)^{-1} & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix}. \tag{8.14}$$

The metric (8.14) deviates markedly from the metric associated with spherical polar coordinates (which has $g_{tt} = -c^2$ and $g_{rr} = 1$) for values of r up to a few times larger than r_s. If M has the same mass as the Sun, $\mathrm{M}_\odot = 1.99 \times 10^{30}\,\mathrm{kg}$, we find $r_s = 2.95\,\mathrm{km}$.

8.1 Constants of Motion

It is clear that a possible solution to the θ-equation of the set (8.5) is $\theta = \frac{\pi}{2}$; that is, a particle can move always in the equatorial plane of the coordinate system. We shall assume that our coordinate system has been oriented to ensure $\theta = \frac{\pi}{2}$. The t equation of the set (8.5) implies that $\mathrm{d}t/\mathrm{d}\tau = \text{constant}/D$. In special relativity, $\mathrm{d}t/\mathrm{d}\tau$ is constant and we call this constant γ. So let's call the constant of integration that arises here γ too. Then we have

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\gamma}{D}. \tag{8.15}$$

Similarly, the ϕ equation of the set (8.5) implies that $r^2(\mathrm{d}\phi/\mathrm{d}\tau) =$ constant. Calling this constant γL, we obtain the statement of angular-momentum conservation

$$r^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \gamma L. \tag{8.16}$$

The physical interpretation of γ is clarified by going back to the definition $-c^2\mathrm{d}\tau^2 = -c^2D\mathrm{d}t^2 + D^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\phi^2$ of proper time: dividing both sides by $\mathrm{d}\tau^2$ and using equations (8.15) and (8.16) we obtain

$$-c^2 = -\frac{c^2\gamma^2}{D} + \frac{1}{D}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{\gamma^2L^2}{r^2}. \tag{8.17}$$

Rearranging, we have

$$\frac{c^2}{\gamma^2} = \frac{c^2}{D} - \frac{1}{D^3}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{L^2}{r^2}. \tag{8.18}$$

Expanding the r.h.s. in powers of r_s/r and then using the binomial theorem to take a square root, we find

$$\gamma = 1 + c^{-2}\left(-\frac{c^2r_s}{2r} + \tfrac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\left[1 + 3\frac{r_s}{r} + \cdots\right] + \frac{L^2}{2r^2} + \cdots\right). \tag{8.19}$$

Since $-c^2r_s/2r$ is the Newtonian potential energy $-GM/r$, it is clear that γc^2 is just the energy per unit mass of the orbiting particle, as we might have anticipated by analogy with the special-relativistic case.

With $\theta = \frac{\pi}{2}$ the r-equation of motion is

$$0 = \frac{\mathrm{d}^2r}{\mathrm{d}\tau^2} + \tfrac{1}{2}\frac{c^2D'}{B}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \tfrac{1}{2}\frac{B'}{B}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{r}{B}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2.$$

With (8.10), (8.18) and (8.19) this becomes

$$0 = \frac{\mathrm{d}^2r}{\mathrm{d}\tau^2} + \tfrac{1}{2}\gamma^2c^2\frac{D'}{D} - \frac{D\gamma^2L^2}{r^3} - \tfrac{1}{2}\frac{D'}{D}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2. \tag{8.20}$$

We shall see that in Newton's theory slightly modified forms of the first, second and third terms occur. The last represents a new, speed dependent force.

Exercise (23):

From (8.16) and (8.18) show that $L^2 = r^3c^2D'/(2D^2)$ and hence that the angular frequency of a circular orbit as seen by an observer at infinity is

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \sqrt{\frac{GM}{r^3}}$$

exactly as in Newton's theory.

Exercise (24):

Multiply (8.20) by $\dfrac{2}{D}\dfrac{\mathrm{d}r}{\mathrm{d}\tau}$ and integrate the result to rederive the energy equation (8.18).

8.2 The Perihelion of Mercury

When Einstein introduced g.r. in 1916, the only significant discrepancy between Newtonian dynamics and solar system observations was the rate of advance of the perihelion of Mercury. One of g.r.'s early triumphs was to account for this discrepancy. We start by reviewing Newton's results for motion in the gravitational field of a point mass.

Newtonian motion around a point mass The equation of motion of a particle in the Newtonian field of a mass M located at the origin is $\ddot{\boldsymbol{r}} = -GM\boldsymbol{r}/r^3 = -\frac{1}{2}c^2 r_s \boldsymbol{r}/r^3$. On crossing this equation through by $\boldsymbol{r}$ we obtain $\dot{\boldsymbol{L}} = 0$ where $\boldsymbol{L}$ is the angular momentum vector $\boldsymbol{L} \equiv \boldsymbol{r} \times \dot{\boldsymbol{r}}$. From the constancy of $\boldsymbol{L}$ we deduce that the motion is confined to the plane $\boldsymbol{L} \cdot \boldsymbol{r} = 0$ perpendicular to the angular momentum vector $\boldsymbol{L}$. Let r and ϕ be polar coordinates for this plane. Conservation of angular momentum requires $r^2\dot{\phi} = L$, while the equation of motion of r is $\ddot{r} - r\dot{\phi}^2 = -\frac{1}{2}c^2 r_s/r^2$. Eliminating $\dot{\phi}$ in favour of L the latter reads

$$0 = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} + \frac{c^2 r_s}{2r^2} - \frac{L^2}{r^3}. \tag{8.21}$$

This is the Newtonian analogue of (8.20): to see this recall that $D = 1 - r_s/r$ and $D'/D \simeq r_s/r^2$.

We obtain the shape of Newtonian orbits by eliminating t from (8.21) through the substitution $\mathrm{d}t = (r^2/L)\mathrm{d}\phi$, and eliminating r in favour of a new variable $u \equiv 1/r$. We then find

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{c^2 r_s}{2L^2}. \tag{8.22}$$

This is just the equation of motion of a simple harmonic oscillator. So the orbit is given by

$$r(\phi) = \frac{1}{u} = \frac{1}{A\cos(\phi - \phi_0) + \frac{1}{2}c^2 r_s/L^2}, \tag{8.23}$$

where A and ϕ_0 are suitable constants of integration. This is actually the equation of an ellipse with one focus at the origin. But the most important point is that since the right side of (8.23) is periodic in ϕ with period 2π, $r(\phi + 2\pi) = r(\phi)$ for any ϕ and thus (8.23) defines a *closed* curve. Consequently, a planet in undisturbed orbit around the Sun would always come closest to the Sun (in the jargon, "move through perihelion") at the same value of ϕ. Actually the perihelia of all the planets precess, that is, they move very slowly around the plane of the planet's orbit.

The planet with the most rapidly precessing perihelion is Mercury because it is the planet with the shortest year. Its perihelion precesses by 576 seconds of arc (576″) per century. Most of this precession is caused by the gravitational field of Jupiter.[14]

[14] One may understand how Jupiter causes Mercury's perihelion to precess by imagining Jupiter's mass to be uniformly distributed in an annulus centred on Jupiter's orbit. This material pulls Mercury outwards. Hence Mercury's net acceleration towards the Sun falls off with r more steeply than as r^{-2}. This in turn slightly depresses the frequency at which Mercury's radius oscillates around its mean value, and these radial oscillations gradually get out of phase with the overall rotation about the Sun.

In the late 19th century Bessel showed that disturbance of Mercury's orbit by all the planets gives rise to a net precession of 532″ per century. Thus Bessel was able to account for all but 44″ per century of Mercury's precession. Since Mercury's year is 0.24 siderial years long, 44″ per century corresponds to 0.106″ per Mercury year.

Relativistic precession Working from (8.20) in close analogy with the our Newtonian calculation, we eliminate τ between (8.16) and (8.20) to obtain

$$0=\frac{\gamma L}{r^2}\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{\gamma L}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)+\tfrac{1}{2}\gamma^2c^2\frac{D'}{D}-\frac{D\gamma^2L^2}{r^3}-\tfrac{1}{2}\frac{D'}{D}\frac{\gamma^2L^2}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2.$$

We define $u\equiv 1/r$, substitute for D and divide through by $-\gamma^2L^2u^2$ to obtain

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}+u(1-r_su)+\tfrac{1}{2}\frac{r_s}{1-r_su}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2=\frac{c^2r_s}{(1-r_su)2L^2}. \qquad (8.24)$$

With the help of (8.17) we can simplify this significantly: from (8.17) we have that

$$\frac{1}{D}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2=\frac{c^2}{DL^2}-\frac{c^2}{\gamma^2L^2}-u^2, \qquad (8.25)$$

so (8.24) simplifies to

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}+u-\tfrac{3}{2}r_su^2=\frac{r_sc^2}{2\gamma^2L^2}. \qquad (8.26)$$

The only essential difference between this equation and its Newtonian equivalent, (8.22), is the appearance of a term in u^2 on the left. So the oscillations in u are now nonlinear rather than harmonic, and we can no longer write down an analytic solution. It is, however, apparent that solutions to (8.26) are unlikely to be periodic with period 2π and thus we do not expect relativistic orbits around a point mass to be closed. Let us calculated the angle between successive perihelia and compare it with Bessel's discrepancy of 0.106″.

With (8.25) we have that the angle $\Delta\phi$ between apo- and perihelion is therefore

$$\Delta\phi=\int_{u_1}^{u_2}\frac{\mathrm{d}u}{\mathrm{d}u/\mathrm{d}\phi}=\int_{u_1}^{u_2}\frac{\mathrm{d}u}{\sqrt{c^2/L^2-K(1-r_su)-u^2(1-r_su)}}, \qquad (8.27)$$

where $K\equiv c^2/\gamma^2L^2$ and u_1, u_2 are the smallest and largest values of u along the orbit. The denominator in (8.27) involves a cubic in u. Two roots of the cubic are u_1 and u_2, so if the third root is u_3 the cubic may be written

$$H(u-u_1)(u_2-u)(1-u/u_3), \qquad (8.28)$$

where H is a constant to be determined. Comparing coefficients of u^2 and u^3 in (8.28) and the denominator of (8.27) we find

$$u^2: \qquad -H\left(1+\frac{u_1+u_2}{u_3}\right)=-1 \qquad\qquad u^3: \qquad \frac{H}{u_3}=r_s,$$

so

$$u_3 = \frac{1}{r_s} - (u_1 + u_2) \simeq \frac{1}{r_s} \qquad \text{and} \qquad H = 1 - r_s(u_1 + u_2). \tag{8.29}$$

Thus $u_3 \gg \max(u_1, u_2)$ and with equations (8.28) and (8.29) we can rewrite equation (8.27) as

$$\begin{aligned}
\Delta\phi &= \frac{1}{\sqrt{H}} \int_{u_1}^{u_2} \frac{\mathrm{d}u}{\sqrt{(u-u_1)(u_2-u)}} \left(1 + \tfrac{1}{2}\frac{u}{u_3} + \cdots\right) \\
&\simeq [1 + \tfrac{1}{2} r_s(u_1+u_2)] \int_{u_1}^{u_2} \frac{\mathrm{d}u}{\sqrt{(u-u_1)(u_2-u)}} (1 + \tfrac{1}{2} u r_s) \\
&\simeq \pi[1 + \tfrac{3}{2} r_s \tfrac{1}{2}(u_2 + u_1)].
\end{aligned} \tag{8.30}$$

For Mercury $\frac{1}{2}(u_1 + u_2) \simeq 1/r_{\rm Merc} = 1/(5.83 \times 10^7\,\mathrm{km})$, so the perihelion of Mercury should advance in one Mercury year by

$$3\pi \frac{r_s}{r_{\rm Merc}} \simeq 0.0983''$$

in excellent agreement with Bessel's discrepancy.

In 1975 Hulse & Taylor discovered a pulsar, PSR 1913+16, that proved to be one component of a tight and eccentric binary: the binary period is $7\frac{3}{4}$ h and the eccentricity is $e = 0.617$. The periastron of this orbit has been shown to precess by $4.22°\,\mathrm{yr}^{-1}$. Both components have mass close to $M = 1.42\,\mathrm{M}_\odot$ and are presumably neutron stars, although pulses are detected from only one of them. Thus PSR 1913+16 is a system in which general relativity is of prime importance rather than a marginal correction.

Exercise (25):

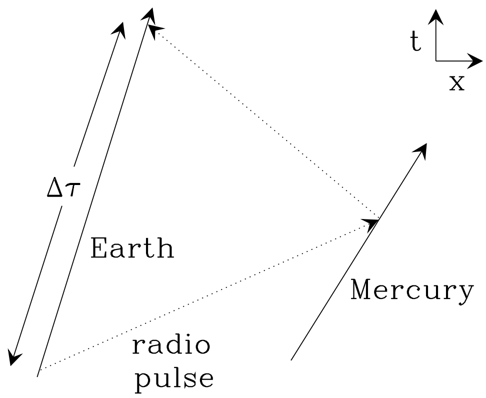
Show that the semi-major axis of the orbit of PSR 1913+16 is $a = 1.9 \times 10^9$ m, about three times the radius of the Sun, and that the each neutron star moves with a speed of order $220\,\mathrm{km\,s^{-1}}$.

8.3 Tests based on Planetary and Pulsar Dynamics

If one claims to know the orbits of the planets and **g** in the intervening space, one can calculate the time for a signal to pass from one planet to another or the time required for an e.m. signal to reach us from a specified point outside the Solar System. G.R. can be tested by comparing these calculated times with observed delays. There are two main types of experiment to consider: (i) a signal goes out from Earth, bounces off a planet or satellite within the Solar System and returns to us; (ii) a steady stream of signals reaches us from a pulsar after traversing the Solar System.

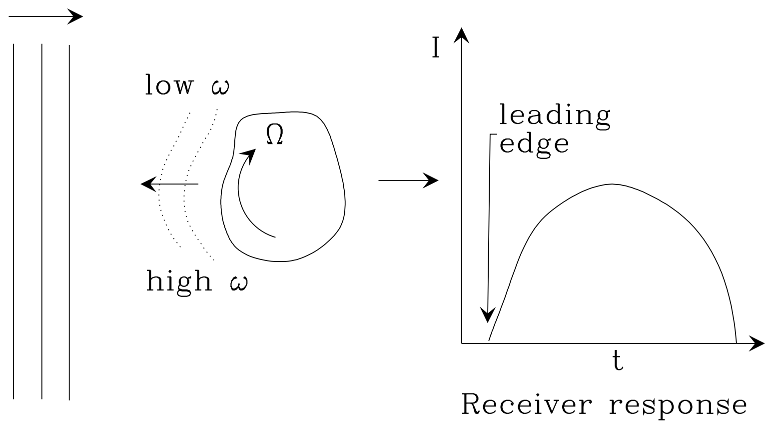
In each case the time $\Delta\tau(t)$ required for the signal to reach us is a complex function of the parameters ("orbital elements") that define planetary orbits and any relevant pulsar orbits. In practice these parameters have to be adjusted to optimize the fit between the calculated and observed values of $\Delta\tau$. Thus these experiments not only test g.r.; they also refine our knowledge of the structure of the Solar System and certain pulsars.

Bouncing signals within the solar system The earliest work involved bouncing radar signals off the inner planets. One measures the delay before the first signals return. This gives $\Delta\tau$



There are two important difficulties:

(i) The reflecting planetary surface is not a smooth mirror. Hence the returning pulse has a complex shape. One looks for the leading edge of the pulse and tries to use frequency information:



(ii) The most interesting lines of sight pass close to the Sun. Free electrons near the Sun cause the refractive index to differ from unity.

Later experiments concentrated on timing signals sent to artificial satellites. Since a satellite is too small to give a detectable radar reflection, one programmes the satellite to respond to a pulse from Earth by emitting a similar pulse after a known small delay. With this technique one does not have to worry about planetary topography. By sending signals at several frequencies one can eliminate the effect of dispersion by free electrons along the line of sight.

Analysis of these data has to proceed via a computer program which adjusts orbital elements, the masses of the planets and asteroids, the oblateness of the Sun, the orientation of an inertial coordinate system, etc., until the fit of the predicted $\Delta\tau$'s to the observed $\Delta\tau$'s is optimized. One finds that the agreement with g.r. is excellent.

The quality of the fit is normally judged by calculating predictions from the metric[15]

$$ds^2 = -\left[1 - \alpha\frac{r_s}{\rho} + \tfrac{1}{2}\beta\left(\frac{r_s}{\rho}\right)^2\right]c^2\,dt^2 + \left(1 + \gamma\frac{r_s}{\rho}\right)[d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)], \quad (8.31)$$

where α, β and γ are dimensionless parameters to be determined by fitting the calculated to the observed $\Delta\tau$'s. If we identify ρ with

$$\rho \equiv \tfrac{1}{2}\left[r - \tfrac{1}{2}r_s + \sqrt{r(r - r_s)}\,\right], \quad (8.32)$$

this metric agrees with the Schwarzschild metric (8.14) up to order r_s/r in space and $(r_s/r)^2$ in time when $\alpha = \beta = \gamma = 1$. (In the equations of motion the tt-component of $g_{\mu\nu}$ is multiplied by the largest components of v^μ.) Hence if Einstein was right, the observations should lead to $\alpha \simeq 1$ etc. Data from missions to Mercury & Mars give

$$\begin{aligned}\alpha - 1 &= (2.1 \pm 1.9) \times 10^{-4}\\ \beta - 1 &= (-2.9 \pm 3.1) \times 10^{-3}\\ \gamma - 1 &= (-0.7 \pm 1.7) \times 10^{-3}\\ J_2 &= (-1.4 \pm 1.5) \times 10^{-6}\end{aligned}$$

where J_2 is a parameter describing the oblateness of the Sun.

It is interesting that the precision of these measurements is such that

(i) they determine the inertial frame of reference as accurately as can be done by looking right across the Universe at quasars with redshift $z = 2$ (see below);

(ii) they furnish the best estimates of the mass of the asteroid Ceres (the old value proved to be in error by 15%);

(iii) Dirac speculated that Newton's "constant" might decrease as the Universe expands. These measurements yield $\dot{G}/G = (0.2 \pm 0.4) \times 10^{-11}\,\mathrm{yr}^{-1}$.

Pulsar timing The discovery of PSR 1913+16 in 1975 facilitated a dramatic extension and refinement of results based on solar-system dynamics. By virtue of its spin, the pulsar is an accurate clock that is carried around a fast and eccentric orbit in a strong gravitational field. The time taken for the electromagnetic pulses it emits to reach Earth is affected by

(i) the positions as functions of time of PSR 1913+16 and the Earth – g.r. has to be used to calculate these to the required accuracy;

(ii) variations in the gravitational redshift of the pulsar as it moves closer to and further from its companion;

(iii) variations in the effective refractive index of the vacuum along the line of sight from Earth to the pulsar – the moving gravitational fields of PSR 1913+16's

[15] This may be thought of as generated by expanding the functions B and D of (8.2) in powers of r_s/r.

companion and objects in the solar system all make non-negligible contributions to the measured delays;

(iv) evolution of the pulsar orbit that is driven by the radiation of gravitational waves.

The evolution of the orbit is predicted by calculating the energy and angular momentum that the waves should carry away in a given time, and then adjusting the orbit to ensure global conservation of E and L. Different variants of g.r. predict different rates of E and L loss. Only Einstein's original (and simplest) theory successfully predicts the observed evolution of the period $\dot{P} = -2.4 \times 10^{-12}$.

8.4 The Schwarzschild Singularity

For $r = r_s \equiv 2GM/c^2$, the component g_{tt} of the Schwarzschild metric (8.14) vanishes. Hence the trajectory $r = r_s$ is null rather than time-like. Furthermore, since g_{tt} changes sign at $r = r_s$, the trajectory $r = \text{constant} < r_s$ is space-like. Consequently an explorer who penetrates to $r < r_s$ is doomed: no matter how hard he fires his rockets, his trajectory must remain time-like. Hence he cannot pass from the condition $\mathrm{d}r/\mathrm{d}\tau < 0$ through the condition $\mathrm{d}r/\mathrm{d}\tau = 0$ as he must if he is to escape. He is carried down to $r = 0$ as surely as you and I are carried into next year.

It is interesting to investigate this predicament more closely. Suppose for simplicity that our explorer's angular momentum L is zero and that at $t = \tau = 0$ he is falling towards the centre at radius r_0 with the speed he would have picked up had he fallen all the way from rest at infinity. Then evaluating (8.17) at infinity we find that the constant γ is one. Hence, by (8.17) the elapse of time on his watch as he falls to r_s is

$$\begin{aligned}\Delta\tau &= \int_{r_0}^{r_s} \frac{\mathrm{d}\tau}{\mathrm{d}r}\,\mathrm{d}r = \frac{1}{c}\int_{r_s}^{r_0} \frac{\mathrm{d}r}{\sqrt{1-D}} \\ &= \frac{1}{c\sqrt{r_s}}\int_{r_s}^{r_0} \sqrt{r}\,\mathrm{d}r = \frac{2}{3c\sqrt{r_s}}\left(r_0^{3/2} - r_s^{3/2}\right),\end{aligned} \tag{8.33}$$

which is perfectly finite. Furthermore, he clearly reaches $r = r_s$ with $\mathrm{d}r/\mathrm{d}\tau < 0$. Hence he would be well advised to fire his rockets before he reaches r_s.

Why does g_{rr} diverge at $r = r_s$? Is this divergence caused by gravity or our choice of coordinates? It is straightforward, if tedious, to check that no components of the curvature tensor $R^{\mu}{}_{\nu\alpha\beta}$ diverge at r_s. So our explorer can endure the tidal forces he experiences if he is stocky enough. The reason g_{rr} diverges at r_s turns out to be that Schwarzschild's coordinate system assigns to all events that occur at r_s the time coordinate $t = \infty$. As a specific example, let us calculate the time coordinate at which our explorer crosses $r = r_s$:

$$t = \int_0^{\tau} \frac{\mathrm{d}t}{\mathrm{d}\tau}\,\mathrm{d}\tau = \int_{r_0}^{r_s} \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}r}\,\mathrm{d}r.$$

With (8.15) and (8.33) this becomes

$$t = \int_{r_0}^{r_s} \frac{\mathrm{d}r}{D\sqrt{1-D}} = \frac{1}{\sqrt{r_s}}\int_{r_0}^{r_s} \frac{r^{3/2}\mathrm{d}r}{r - r_s} = \infty. \tag{8.34}$$

Thus no matter when our explorer sets off, an observer who uses Schwarzschild's coordinates always assigns $t = \infty$ to the event at which the explorer crosses $r = r_s$. We should not be surprised that such a foolish convention leads to a singular metric; if we choose coordinates q_i in ordinary space in such a way that all points on the edge of a ruler are assigned the same three numbers q_i, an expression for the length of the ruler in terms of the coordinates of the ruler's ends is going to involve multiplication by some awfully big numbers!

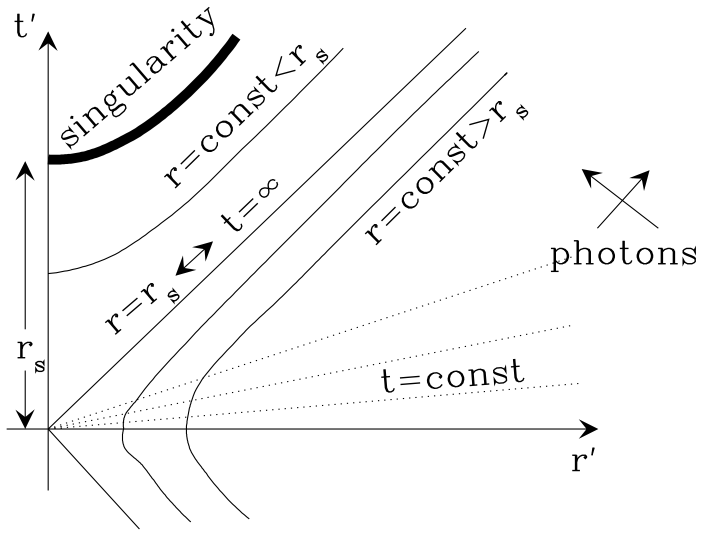
To bring this problem under control we need to choose a new coordinate system. In 1960 M. Kruskal showed that when new coordinates (r', t') are defined through

$$\begin{aligned} r'^2 - t'^2 &= r_s^2\left(\frac{r}{r_s} - 1\right)e^{r/r_s} \\ t' &= r'\frac{\cosh(ct/r_s) - 1}{\sinh(ct/r_s)} = r'\tanh\left(\frac{ct}{2r_s}\right) \end{aligned} \tag{8.35a}$$

the metric takes the non-singular form

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) + 4(dr'^2 - dt'^2)\frac{r_s}{r}e^{-r/r_s}. \tag{8.35b}$$

The lines $r' =$ constant are always timelike. Radially directed photons move along the 45° lines $dr' = \pm dt'$ in the (r', t') plane. In particular, the null line $r = r_s$ becomes $r' = t'$. If we plot curves of constant r and t in the (r', t') plane, we get a picture like this



It is now obvious that Schwarzschild's coordinates (r, t) break down as $r' = t'$ is approached. To first order in ct/r_s (8.35a) becomes $t' \simeq \frac{1}{2}ctr'/r_s$, so t' may be considered a stretched form of t at $r = \infty$. Near $r = r_s$, $t' \simeq r'$ and by (8.35a) all events correspond to large t as expected. The region $t' > r'$ corresponds to $r < r_s$. At $r = 0$, corresponding to $t'^2 - r'^2 = r_s^2$, there is a bona-fide singularity in the gravitational field.

The Schwarzschild radius r_s corresponding to the mass of the Sun is 2.96 km. The black holes that power quasars and other very active galactic nuclei have Schwarzschild radii between the radius of the Sun and that of the Earth's orbit.

Exercise (26):

Show that a cubic light-year of water (supposed incompressible) would be contained within its Schwarzschild radius.

8.5 Summary

The metric outside a point mass can be written to look like that of ordinary spherical polar coordinates with $1 \to (1-r_s/r)$ in the tt slot and $1 \to 1/(1-r_s/r)$ in the rr slot. The singularity of these correction factors when $r = r_s = 2GM/c^2$ is not physically interesting. However the geometry of spacetime *is* singular at $r = 0$ and $r = r_s$ *is* special in that an "outward" running photon on this sphere would actually not move away from the centre.

The Schwarzschild metric accounts for the last 10% of the precession of Mercury's perihelion and for the measured bending of light by the Sun. The magnitude of both these effects is of order $n \times r_s/r$, where $n \sim 4$ and r is the smallest distance of the test body from the Sun. Detailed studies of the Solar System's dynamics show that any errors in the g.r.'s corrections to Newtonian dynamics are smaller that $\sim 0.1\%$.

9 Cosmology

9.1 Empirical Basis

Between 1920 and 1928 it became clear that the Universe is populated by countless galaxies like the Milky Way, and that these are receding from one another with velocities that are proportional to separation. If we follow the trajectories of these galaxies back in time, we find that some 10^{10} yr ago the mean density of the Universe must have been extremely high. Indeed, a naive extrapolation leads to the conclusion that a finite time in the past any density was reached, no matter how great.

In 1946 G. Gamow at Cornell, and 20 years later R. Dicke in Princeton, argued that the large abundance (about 25% by weight) of He in the present Universe could have been generated some minutes after the formation of the Universe if a black-body radiation field fills the present Universe. The first estimate of the current temperature of this radiation field was 25 K, but this later fell to ≈ 3 K. In 1964 A. Penzias & R. Wilson at Bell Labs discovered this cosmic background serendipitously. This triumph of the big-bang theory quickly killed all interest in attempts to construct a steady-state cosmology.

It is now known that the spectrum of the cosmic background is accurately Planckian with $T = 2.7 \pm 0.1$ K. An observer who moves with respect to the centre of our Galaxy at $\approx 400\,\mathrm{km\,s^{-1}}$ in a certain direction would see the same spectrum in all directions, to within a few parts in 10^5. At any point in the Universe a natural standard of rest is defined as that of an observer whose cosmic background is isotropic. Such observers are called **fundamental observers**. Any two fundamental observers recede from one another with a speed $v \approx D/13.6$ Gyr, where D is their separation.[16]

[16] Astronomers write $v = H_0 D$ with $H_0 = 72 \pm 5\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

Constructing the unit n-sphere

1-sphere:	$(x_1, x_2) = (\sin\phi, \cos\phi)$
2-sphere:	$(x_1, x_2, x_3) = (\sin\phi\sin\theta, \cos\phi\sin\theta, \cos\theta)$
3-sphere:	$(x_1, x_2, x_3, x_4) = (\sin\phi\sin\theta\sin\eta, \cos\phi\sin\theta\sin\eta, \cos\theta\sin\eta, \cos\eta)$
...	
n-sphere:	$(x_1, \dots, x_{n+1}) = (\sin\theta_1\sin\theta_2\dots\sin\theta_n, \dots, \cos\theta_{n-1}\sin\theta_n, \cos\theta_n)$

As the Universe expands, the photons of the cosmic background are doppler shifted to lower frequencies and the temperature characterizing their distribution falls.

9.2 Friedmann Metrics

The first step towards finding a solution of Einstein's equations to describe the expanding Universe is to choose a good coordinate system. The cosmic radiation background is a great help in this: we may say that two events occur at the same place if they occur on the world-line of a single fundamental observer. Similarly, two events that occur at different places may be said to occur simultaneously if the background temperature measured by fundamental observers local to those events are the same. With this natural division into space and time we would expect $\mathrm{d}s^2$ to be of the form

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{9.1}$$

$\boldsymbol{g}$ is the metric of a 3-space of simultaneous events.

The structure of $\boldsymbol{g}$ is strongly restricted by the fact that fundamental observers observe the cosmic background to be highly isotropic: the photons they receive were last scattered at a point several thousands of millions of light years away, at a time when the mean density of the Universe was about 10^9 times its present value. In fact, until these photons collide with an observer's telescope they have been flying freely through space since the Universe was a mere 10^{-4} of its present age. Consequently, when a fundamental observer compares the temperature he sees in the forward and backward directions, he is comparing physical conditions in the early Universe at points that are now separated by thousands of millions of light years. Since these conditions are found to be identical to within a few parts in 10,000 we conclude that the Universe is extremely homogeneous on any time-slice $t=$ constant. Hence the geometry of such a space, which is described by $\boldsymbol{g}$, should be extremely homogeneous too.

A theorem in differential geometry states that any homogeneous and isotropic 3-space must be a scaled version of one of three basic models:

(i) Flat space Obviously this admits spherical polar coordinates in which the line element can be written

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{9.2}$$

(ii) The 3-sphere Suppose we parameterize the coordinates of points $\mathbf{x}$ in a 4-dimensional Euclidean space (nothing to do with spacetime) by

$$(x_1, x_2, x_3, x_4) = a(\sin\psi\sin\theta\cos\phi, \sin\psi\sin\theta\sin\phi, \sin\psi\cos\theta, \cos\psi).$$

Then it is easy to show that $\sum_\mu x_\mu^2 = a^2$. Hence as we vary the three angles (ψ, θ, ϕ) the point $\mathbf{x}$ moves over a 3-sphere. The small vector $\mathbf{\Delta}^{(\phi)}$ that joins two points whose coordinates differ only by a small change $\delta\phi$ in ϕ is

$$\begin{aligned}\mathbf{\Delta}^{(\phi)} &= \frac{\partial\mathbf{x}}{\partial\phi}\,\delta\phi \\ &= a(-\sin\psi\sin\theta\sin\phi, \sin\psi\sin\theta\cos\phi, 0, 0)\,\delta\phi.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbf{\Delta}^{(\theta)} &= a(\sin\psi\cos\theta\cos\phi, \sin\psi\cos\theta\sin\phi, -\sin\psi\sin\theta, 0)\,\delta\theta \\ \mathbf{\Delta}^{(\psi)} &= a(\cos\psi\sin\theta\cos\phi, \cos\psi\sin\theta\sin\phi, \cos\psi\cos\theta, -\sin\psi)\,\delta\psi.\end{aligned}$$

It is straightforward to check that these three small vectors are mutually perpendicular. Hence when we move by an arbitrary small amounts $(\delta\psi, \delta\theta, \delta\phi)$ over the sphere, the distance traversed δs is given by

$$\begin{aligned}\delta s^2 &= |\mathbf{\Delta}^{(\psi)}|^2 + |\mathbf{\Delta}^{(\theta)}|^2 + |\mathbf{\Delta}^{(\phi)}|^2 \\ &= a^2(\delta\psi^2 + \sin^2\psi\delta\theta^2 + \sin^2\psi\sin^2\theta\delta\phi^2).\end{aligned} \tag{9.3}$$

If we introduce a new coordinate in place of ψ

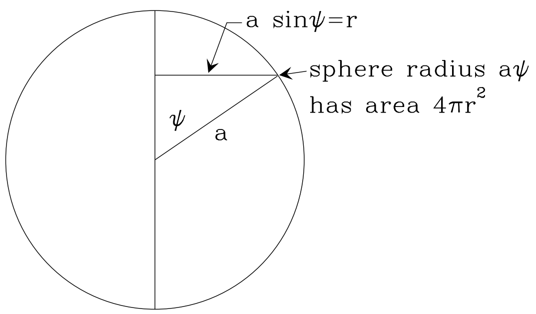
$$r \equiv a\sin\psi \quad \Rightarrow \quad \mathrm{d}r^2 = (a^2 - r^2)\,\mathrm{d}\psi^2, \tag{9.4}$$

and define the **curvature** K of the sphere as

$$K \equiv \frac{1}{a^2}, \tag{9.5}$$

then (9.3) becomes

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - Kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{9.6}$$



Notice that the 2-sphere with area $4\pi r^2$ has radius $a\psi > r$. Thus within the 3-sphere the areas of the members of a nested sequence of 2-spheres increase more slowly than they would in Euclidean space. (Similarly, for concentric small circles on a two sphere circumference/2π increases more slowly than radius.)

(iii) Hyperbolic space If we set $K = 0$, the line element (9.6) of the 3-sphere becomes the line-element (9.2) of flat Euclidean space. The line element of the only other homogeneous, isotropic 3-space is given by (9.6) with K set equal to a negative number. This space is called **hyperbolic space**, and is harder to visualize than the 3-sphere. The characteristic property of hyperbolic space is that in it a 2-sphere with area $4\pi r^2$ has radius

$$R = \int_0^r \frac{\mathrm{d}r}{\sqrt{1+|K|r^2}} = \frac{1}{\sqrt{|K|}} \sinh^{-1}\left(r\sqrt{|K|}\right) < r.$$

That is, in this space the areas of a sequence of nested 2-spheres increase *faster* than in Euclidean space.

In summary, a spatial section of simultaneous events must form either a 3-sphere, flat space or hyperbolic space. In each case the line element may be expressed in the form (9.6) with an appropriate value of K.

We want to use coordinates on these spatial sections such that the coordinates of each fundamental observer are constant. These are called **comoving coordinates**. Since fundamental observers are receding from one another, it follows that our desired coordinates cannot at all times coincide with those in which the line element takes the form (9.6). However, if at one time, for example now, the comoving coordinates (r, θ, ϕ) are such that the line element is of this form, then at an earlier time, when fundamental observers were closer to one another, the separation δs between neighbouring observers was some fraction $a(t)$ of their current separation. Hence at all times the metric of spacetime can be written

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right], \tag{9.7}$$

where K is the curvature of the *current* time-slice $t = t_0$ and $a(t_0) = 1$.

Using the trick of Box 2 we obtain the eqns of motion by applying the Euler-Lagrange equations to

$$L = -c^2\dot{t}^2 + a^2\left[\frac{\dot{r}^2}{1-Kr^2} + r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)\right].$$

where a dot denotes $\mathrm{d}/\mathrm{d}\tau$. Using the convention that a prime denotes $\mathrm{d}/\mathrm{d}t$ the equations of motion are

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}(-c^2\dot{t}) - aa'\left[\frac{\dot{r}^2}{1-Kr^2} + r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)\right] \\
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{a^2\dot{r}}{1-Kr^2}\right) - a^2\left[\frac{\dot{r}^2Kr}{(1-Kr^2)^2} + r\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)\right] \\
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}(a^2r^2\dot{\theta}) - a^2r^2\sin\theta\cos\theta\dot{\phi}^2 \\
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(a^2r^2\sin^2\theta\dot{\phi}\right).
\end{aligned} \tag{9.8}$$

From the first equation we read off the non-vanishing Γs with top index t:

$$\Gamma^t_{rr} = \frac{aa'}{c^2(1-Kr^2)} \quad ; \quad \Gamma^t_{\theta\theta} = \frac{aa'r^2}{c^2} \quad ; \quad \Gamma^t_{\phi\phi} = \frac{aa'r^2\sin^2\theta}{c^2}. \tag{9.9a}$$

The equation of motion for r cleans up to

$$0 = \ddot{r} + \frac{2a'}{a}\dot{t}\dot{r} + \frac{Kr}{1-Kr^2}\dot{r}^2 - r(1-Kr^2)(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

from which we read off the non-vanishing Γs with top index r:

$$\Gamma^r_{tr} = \frac{a'}{a} \;;\; \Gamma^r_{rr} = \frac{Kr}{1-Kr^2} \;;\; \Gamma^r_{\theta\theta} = -r(1-Kr^2) \;;\; \Gamma^r_{\phi\phi} = -r(1-Kr^2)\sin^2\theta \tag{9.9b}$$

The angular equations of motion are

$$0 = \ddot{\theta} + \frac{2a'}{a}\dot{t}\dot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 \quad ; \quad 0 = \ddot{\phi} + \frac{2a'}{a}\dot{t}\dot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi}$$

so the remaining non-vanishing Γs are

$$\begin{aligned} \Gamma^\theta_{t\theta} &= \frac{a'}{a} \quad ; \quad \Gamma^\theta_{r\theta} = \frac{1}{r} \quad ; \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \\ \Gamma^\phi_{t\phi} &= \frac{a'}{a} \quad ; \quad \Gamma^\phi_{r\phi} = \frac{1}{r} \quad ; \quad \Gamma^\phi_{\theta\phi} = \cot\theta. \end{aligned} \tag{9.9c}$$

9.3 The Cosmological Redshift

We know that the Universe is expanding because we observe the frequencies of spectral lines from distant galaxies to be shifted towards lower frequencies. It turns out that the magnitude of this spectral shift is related in a remarkably simple way to the scale of the Universe when the light by which we see galaxies set out towards us.

The **redshift** z is defined by

$$1 + z \equiv \frac{\omega_{\text{emit}}}{\omega_{\text{observe}}}.$$

If we elevate our status to that of a fundamental observer, and suppose that the atoms that emit the radiation we receive were stationary with respect to a local fundamental observer, then $k^0 = \omega_{\text{emit}}/c^2$ on emission of a photon and $k^0 = \omega_{\text{obs}}/c^2$ on its observation.[17] The definition (5.14) of the affine parameter s fails when applied to a trajectory $x^\mu(\lambda)$ of a photon. Instead we define s by requiring that

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = k^\mu(s), \tag{9.10}$$

[17] See Exercise (22).

where k^μ is the wavevector $(\omega/c^2, \boldsymbol{k})$ of the photon. The equation of motion of the photon is $0 = k^\mu \nabla_\mu k^\nu$. Multiplying this equation through by $\mathrm{d}s/\mathrm{d}t$, we find for the time component of the resulting equation

$$\begin{aligned} 0 &= \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\left[\frac{\partial \omega/c^2}{\partial x^\mu} + \Gamma^t_{\mu\gamma} k^\gamma\right] \\ &= \frac{\mathrm{d}\omega/c^2}{\mathrm{d}t} + \frac{\mathrm{d}s}{\mathrm{d}t}\Gamma^t_{\mu\gamma} k^\mu k^\gamma. \end{aligned} \tag{9.11}$$

We evaluate this for a radially propagating photon. Henceforth using the convention that $\dot{a} = \mathrm{d}a/\mathrm{d}t$, (9.9a) states that $\Gamma^t_{rr} = \dot{a} g_{rr}/(ac^2)$ while (9.10) gives $\mathrm{d}s/\mathrm{d}t = 1/k^0 = c^2/\omega$, so (9.11) yields

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{\dot{a}}{a}\left(g_{rr} k^r k^r\right)\frac{c^2}{\omega} = -\frac{\dot{a}}{a}\omega,$$

where we have used the null property of k^μ in the form $g_{rr} k^r k^r + g_{tt}(\omega/c^2)^2 = 0$. Integrating we get

$$1 + z = \frac{\omega_{\mathrm{emit}}}{\omega_{\mathrm{obs}}} = \frac{a(t_{\mathrm{obs}})}{a(t_{\mathrm{emit}})}.$$

In words, $1+z$ gives the factor by which the Universe has expanded since the photons we receive were emitted. Notice that this result has been obtained without using Einstein's equations to determine the dynamics of the Universe.

9.4 Field Equations for Friedmann Cosmologies

When using equations (9.9) in (6.13) to calculate $R_{\alpha\beta}$, it is helpful to isolate all terms that involve a t index. One finds

$$\begin{aligned} R_{it} &= R_{ti} = 0 \qquad R_{tt} = \frac{\partial \Gamma^\mu_{t\mu}}{\partial t} + \Gamma^j_{tk}\Gamma^k_{tj} = 3\frac{\ddot{a}}{a} \\ R_{ij} &= \widetilde{R}_{ij} - \frac{\partial \Gamma^t_{ij}}{\partial t} + 2\Gamma^t_{ik}\Gamma^k_{jt} - \Gamma^t_{ij}\Gamma^k_{tk} \\ &= \widetilde{R}_{ij} - \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2\right]\frac{g_{ij}}{c^2}, \end{aligned}$$

where $\widetilde{R}_{ij}$ is the Ricci tensor of the 3-space whose metric is

$$g_{ij} = a^2 \mathrm{diag}\left(\frac{1}{1 - Kr^2}, r^2, r^2 \sin^2\theta\right).$$

Since the 3-space is homogeneous and isotropic, it is obvious that $\widetilde{\boldsymbol{R}} \propto \boldsymbol{g}$. Hence it is only necessary to calculate one non-zero component of $\widetilde{\boldsymbol{R}}$, say $\widetilde{R}_{rr}$. A tedious calculation yields

$$\widetilde{R}_{ij} = -\frac{2K}{a^2} g_{ij}. \tag{9.12}$$

Hence

$$R_{\alpha\beta} = \begin{pmatrix} \dfrac{3\ddot{a}}{a} & & & \\ & -f(t)g_{rr}/c^2 & & \\ & & -f(t)g_{\theta\theta}/c^2 & \\ & & & -f(t)g_{\phi\phi}/c^2 \end{pmatrix}, \tag{9.13a}$$

where

$$f(t) \equiv \frac{2Kc^2}{a^2} + \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2. \tag{9.13b}$$

We now turn our attention to the right side of the Einstein equations (6.18). We take $\mathbf{T}$ to be the energy-momentum tensor (7.4) of a fluid that is at rest in the frame of the local fundamental observer. With $\mathbf{T}$ of the form (7.4), $T^\alpha_\alpha = 3P - \rho c^2$. With our (t, r, θ, ϕ) coordinates, $u^\alpha = (1, 0, 0, 0)$, $u_\alpha = (-c^2, 0, 0, 0)$, and the tt-equation of the set (6.18) reads

$$\frac{3\ddot{a}}{a} = -\frac{8\pi G}{c^2}(\tfrac{3}{2}P + \tfrac{1}{2}\rho c^2). \tag{9.14a}$$

The rr-equation reads

$$-\left[\frac{2Kc^2}{a^2} + \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2\right]\frac{g_{rr}}{c^2} = -\frac{8\pi G}{c^4}\tfrac{1}{2}(\rho c^2 - P)g_{rr}. \tag{9.14b}$$

Eliminating $\ddot{a}$ between these equations yields the cosmic energy equation

$$\dot{a}^2 + Kc^2 = \tfrac{8}{3}\pi G\rho a^2. \tag{9.15}$$

We also have the equation of mass-energy conservation

$$0 = \nabla_\beta T^{\alpha\beta} = u^\alpha u^\beta \nabla_\beta \rho + \left(g^{\alpha\beta} + \frac{u^\alpha u^\beta}{c^2}\right)\nabla_\beta P + (\rho + P/c^2)\nabla_\beta(u^\alpha u^\beta), \tag{9.16}$$

where we've used (5.30). Now

$$\nabla_\beta(u^\alpha u^\beta) = \partial_\beta(u^\alpha u^\beta) + \Gamma^\alpha_{\gamma\beta}u^\gamma u^\beta + \Gamma^\beta_{\gamma\beta}u^\alpha u^\gamma.$$

With $\alpha = t$ this yields $\nabla_\beta(u^t u^\beta) = \Gamma^\beta_{\beta t} = 3\dot{a}/a$. For $\alpha = t$ the first term in (9.15) is $\dot{\rho}$ and the second vanishes, so we find

$$0 = \dot{\rho} + (\rho + P/c^2)\frac{3\dot{a}}{a} \quad \Rightarrow \quad \frac{\mathrm{d}\rho a^3}{\mathrm{d}a} = -\frac{3a^2 P}{c^2}. \tag{9.17}$$

There are three possible contributers to the cosmic energy density.

Rest-mass energy The random motions of galaxies with respect to the cosmic background radiation are $\lesssim 1000\,\mathrm{km\,s^{-1}} \ll c$, as are the random motions of particles within galaxies. So in the frame of the local Fundamental Observer, the energy of such matter is dominated by its rest mass and we may adopt for $\mathbf{T}$ the formula (6.20) for dust, or equivalently (7.4) for a perfect fluid with $P = 0$. Numerically, $\rho_{\rm dust} \gtrsim 10^{-27}\,\mathrm{kg\,m^{-3}} = 5.6 \times 10^8\,\mathrm{eV\,m^{-3}}$. Given that $P = 0$, equation (9.17) implies $\rho_{\rm dust} \sim 1/a^3$ as we would expect naively.

Relativistic matter At early times the Universe was so hot that its constituent particles had thermal velocities near c. Moreover, even at the present time photons of the cosmic background radiation form such a relativistic gas. We know from thermodynamics that in its rest frame the pressure of a photon gas is one third of its energy density. Hence, in the frame of a Fundamental Observer the energy-momentum tensor of such a relativistic gas is given by (7.4) with $\rho = 3P = \rho_{\rm rad}$. Eliminating P from (9.17) we find that $\rho_{\rm rad} \sim 1/a^4$.

Exercise (27):

Recover $\rho_{\rm rad} \sim 1/a^4$ by considering the adiabatic expension of a gas with ratio of principal specific heats $\gamma = \frac{4}{3}$.

At the present epoch the energy density contributed by the cosmic background is $a_s(2.7)^4 \simeq 1.9 \times 10^5\,{\rm eV\,m^{-3}}$, which is significantly smaller than the rest-mass energy density of dust. However, since $\rho_{\rm rad}/\rho_{\rm dust} \sim 1/a$, for $a \lesssim 10^{-4}$ radiation will have been dominant.

Vacuum energy The vacuum is a complex, non-linear dynamical system: it carries fields (electro-magnetic, electron, muon, quark...) that obey field equations that are sometimes non-linear and are always coupled to one another by non-linear terms. According to quantum-field theory, even in the ground state the fields have non-zero mean-square values by virtue of zero-point fluctuations. When you calculate the energy-density to which these fluctuations give rise, you obtain the answer infinity. While this result is not encouraging, it does lead to a valid experimental prediction: if you calculate the zero-point energy per unit volume in the space between two grounded capacitor plates, you obtain a (formally infinite) expression that depends on the separation between the plates, s. The differential of this energy w.r.t. s is finite and positive. Thus the energy density between the plates rises as the plates move apart. By conservation of energy, you have to work on the plates to get them apart – the plates attract one another (the **Casimir effect**). This prediction has been confirmed experimentally.

The Casimir effect suggests that differences in zero-point vacuum energy are physical, even if baseline values are not, and we conjecture that when the energy-density of the vacuum is for any reason greater than its minimum value, the excess energy is classically manifest. A vacuum with excess energy is called a 'false vacuum'. Obviously a vacuum must be Lorentz invariant, so the energy-momentum tensor of a false vacuum must be a multiple of the metric tensor. Thus

$$T_{\mu\nu} = -\lambda g_{\mu\nu} \qquad (\lambda \text{ a constant}). \tag{9.18}$$

In a locally freely-falling frame $g_{\mu\nu} = \eta_{\mu\nu}$, so a positive energy density corresponds to $\lambda > 0$. It follows that a false vacuum exerts a negative pressure; $P = -\lambda$. When we plug $P = -\rho c^2 = \lambda$ into (9.17) we find $\rho =$ constant, so the energy-density of the vacuum is unchanged by cosmic expansion. A simple physical argument shows the connection between negative pressure and constant energy-density: Imagine what happens when we increase by ${\rm d}V$ the volume of a cylinder containing a false vacuum.

The false vacuum's mass increases by $\rho_{\rm vac}{\rm d}V$, so its energy increases by $\rho_{\rm vac}c^2{\rm d}V$. The latter increase must equal the work done on the piston, $-P{\rm d}V$. Thus the pressure of the false vacuum is $P = -\rho_{\rm vac}c^2$.

Note:

The constant λ has units of energy density. In 1917, from a desire to construct a static universe, Einstein replaced $G_{\mu\nu}$ in the field equations by $G_{\mu\nu} - \Lambda g_{\mu\nu}$. He called Λ, which has units of length^{-2} the **cosmological constant**. It is easy to see that $\Lambda = 8\pi G\lambda/c^4$.

We now return to equation (9.15) and replace $\rho(t)$ by $\rho(t_0)$ times $a(t)^{-n}$, where $a(t_0) = 1$ and $n = 3, 4, 0$ for the cases of dust, radiation and vacuum energy, respectively. We find

$$\dot a^2 = \begin{cases} \dfrac{8\pi G}{3a}\rho(t_0) - Kc^2 & \text{(dust)} \\ \dfrac{8\pi G}{3a^2}\rho(t_0) - Kc^2 & \text{(radiation)} \\ \dfrac{8\pi G}{3}a^2\rho(t_0) - Kc^2 & \text{(vacuum).} \end{cases} \tag{9.19}$$

Currently the Universe is expanding, so $\dot a > 0$. Equation (9.19) states that, if it is matter dominated, it will expand for ever if $K \le 0$. But if $K > 0$ (the case in which spatial sections are 3-spheres), the expansion will cease when

$$a = \frac{8\pi G\rho(t_0)}{3c^2K} = \frac{1}{(7.5\times 10^{10}\,\text{light yr})^2 K} \times \frac{\rho(t_0)}{10^{-27}\,\text{kg}\,\text{m}^{-3}}.$$

Thus our longevity hangs ultimately on how the radius of curvature of the Universe compares with some tens of billions of light years.

Exercise (28):

Integrate (9.19) in the case of dust to show

$$\frac{c\sqrt{|K|}}{a_m}t(a) = \begin{cases} \theta - \frac12\sin 2\theta & \text{when } K > 0 \quad [\theta \equiv \arcsin(\sqrt{a/a_m})] \\ \frac12\sinh 2\theta - \theta & \text{when } K < 0 \quad [\theta \equiv \operatorname{arcsinh}(\sqrt{a/a_m})] \end{cases}$$

Sketch $a(t)$ in the two cases.

The special case $K = 0$ divides a doom-laden future from one of ultimate boredom. In this case the present density is given by

$$\rho_{\rm crit}(t_0) = \left.\frac{3\dot a^2}{8\pi G a^2}\right|_{t_0}. \tag{9.20}$$

The distance between nearby fundamental observers, $\Delta s \simeq a(t)\Delta r$, increases at a rate $\dot a\Delta r = (\dot a/a)\Delta s$. Thus $(\dot a/a)$ is the quantity H in Hubble's relation $v = Hs$. Its current value lies near $75\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$ in idiotic astronomical units; this translates to $2.43\times 10^{-18}\,\text{s}^{-1}$, so

$$\rho_{\rm crit}(t_0) = 1.06\times 10^{-26}\,\text{kg}\,\text{m}^{-3}. \tag{9.21}$$

The best observational evidence suggests that the actual density of matter is a factor of several lower than this: unless vacuum energy is significant, the future is more likely to be boring than otherwise. Note that if $\rho \leq \rho_{\rm crit}$, the Universe is spatially infinite and contains infinite mass, while if $\rho > \rho_{\rm crit}$ the total mass is finite.

Exercises (29):

(i) Show for a dust-dominated universe with $K = 0$ that $a = (t/t_0)^{2/3}$. Hence estimate the age of the Universe if $\rho(t_0) = \rho_{\rm crit}(t_0)$.

(ii) Show for a radiation-dominated universe with $K = 0$ that $a = \sqrt{t/t_0}$.

(iii) Show that in Newton's theory the radial coordinate $a(t)$ of a particle embedded in a homogeneous spherical cloud of mutually gravitating particles which are initially receding from the origin with speeds proportional to radius, obeys (9.15). Identify the analogue of K in this case.

9.5 Inflation

When a thermodynamic system is rapidly expanded and therefore adiabatically cooled, it is liable to 'supercool' when it encounters the temperature at which a phase transition would occur if it were slowly cooled. A classic example of this phenomenon is water vapour in a Wilson cloud chamber: a sudden expansion supercools the vapour just before debris from a collision flies through, and water droplets rapidly condense along the tracks of the debris.

Since the vacuum is a complex, non-linear dynamical system, it is expected to exhibit phase transitions. In 1981 Alan Guth of M.I.T. pointed out[18] that supercooling at the temperature of a transition could have caused the vacuum to stumble temporarily into a false vacuum. Then the cosmic scale factor would obey the third option in equation (9.19) and we have

$$\ddot{a} = \frac{8\pi G\lambda}{3c^2} a \qquad \Rightarrow \qquad a(t) = a(0) \exp\left(\sqrt{\frac{8\pi G\lambda}{3c^2}}\, t\right). \tag{9.22}$$

Grand unified theories of the strong, weak and electromagnetic force suggest that the time constant associated with this exponential growth is $\approx 10^{-34}$ s.

Exercise (30):

Let the present age of the Universe be t_H and the distance over the *current* time-slice $t = t_H$ to the most distant fundamental observer it is in principle possible to see be D_H. Show that if the Universe had inflated from $t = 0$ to the present day we would have $D_H = ct_H$, while we would have $D_H = 2ct_H$ if the Universe had been always flat and radiation-dominated. The furthest fundamental observer we can see is said to be on the **particle horizon**. [Hint: use $0 = g_{rr}{\rm d}r^2 + g_{tt}{\rm d}t^2$.]

Guth's inflationary conjecture has two very seductive properties:

[18] *Phys. Rev.*, **D23**,347.

(i) It offers an explanation of why the Universe is so homogeneous on a large scale by suggesting that everything we see may have emerged from the explosive expansion of a single causally-connected fluctuation in the preinflationary Universe.

(ii) It offers an explanation of why $\rho(t_0)/\rho_{\rm crit}(t_0) \simeq 1$: with the definition (9.20) of $\rho_{\rm crit}$ the cosmic energy equation (9.15) can be written

$$\frac{\rho(t)}{\rho_{\rm crit}(t)} = 1 + \frac{Kc^2}{\dot{a}^2}. \tag{9.23}$$

Whatever the initial value of K, after a sufficient number of e-folding times $\dot{a}$ becomes enormous and the deviation of each side of (9.23) from unity becomes extremely small.

The inflationary period is supposed to have ended when the vacuum finally made the phase transition into the lower-energy configuration, releasing its former energy density as normal thermal radiation.

Extraordinarily, several astronomical phenomena are easier to explain if we live in a universe that is now mildly dominated by vacuum energy-density.[19] If vacuum energy-density is indeed significant now, it will soon become dominant and we must be at the start of a new inflationary episode. This proposition is harder to believe than that the long chain of astronomical inference upon which it rests is somewhere defective.

9.6 Cosmic Strings

It is thought that when the vacuum changed its phase from a symmetric high-temperature form to a less symmetrical low-temperature form, discontinuities may have arisen that would have persisted to the present day. The general idea is illustrated by what happens when a lump of iron cools in zero magnetic field through the Curie temperature T_c (at which iron becomes ferromagnetic). At T_c groups of atoms here and there in the lump decide to align their spins in some common direction. Since the direction is chosen at random, widely separated groups choose different directions. So long as the groups remain isolated they can all grow by convincing adjacent uncommitted atoms to align with them. But eventually the swelling groups touch each other – the lump has become a mass of interlocking domains. Between the domains are regions of high B and therefore of large magnetic energy. So it is energetically desirable for each domain boundary to shrink. But usually the boundary around one domain can shrink only if the boundaries of adjacent domains grow. So the domains are effectively locked into place.

When the Universe cools two-dimensional domain boundaries may form, but the most important discontinuities are one-dimensional – strings. The complex field ψ associated with charged particles such as electrons can give rise to a string like this.[20] Imagine that it is decided that the field shall everywhere have amplitude $|\psi| = 1$ and

[19] e.g., Efstathiou et al., *Mon. Not. R. Astr. Soc.*, 303 L47.

[20] The treatment here is a little oversimplified inasmuch as it neglects the fact that for electrons ψ is a Dirac spinor rather than a scalar.

you are told to specify its phase $0 \leq \arg(\psi) \leq 2\pi$ throughout space. You decide to set $\arg[\psi(\boldsymbol{x})] = \phi(\boldsymbol{x})$, where ϕ is the usual cylindrical-polar coordinate of the point $\boldsymbol{x}$. This assignment works fine everywhere except at your coordinate origin, $r = 0$. Here $\nabla \arg(\psi)$ diverges since any phase can be reached arbitrarily close to $r = 0$. It is not hard to persuade oneself that by adjusting the values of ψ in any finite volume you can move but not eliminate this singularity, which is associated with a line of energy-momentum. This is a cosmic string.

What does the energy momentum tensor $\mathbf{T}$ look like in the narrow tube around $r = 0$ in which $\mathbf{T} \neq 0$? We'd expect $\mathbf{T}$ to be Lorentz invariant with respect to boosts parallel to the string's line. So in the (t, z) plane $\mathbf{T}$ has to be proportional to the Minkowski metric. Also it's hard to see how the string could be carrying anything in the x or y directions. So

$$T_{\mu\nu} = -\rho c^2 \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{9.24}$$

where ρ is a constant.

Now consider the line element

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + r_0^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) + \mathrm{d}z^2, \tag{9.25}$$

where r_0 is a constant. This is almost the line element $\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + \mathrm{d}z^2$ of flat spacetime in cylindrical polars; $r_0\theta$ is a kind of radial variable. The only non-zero Christoffel symbols generated by (9.25) are

$$\Gamma^\theta_{\phi\phi} = -\tfrac{1}{2}\sin 2\theta \quad ; \quad \Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta.$$

The only non-zero components of the Ricci tensor are

$$R^\theta_\theta = R^\phi_\phi = -r_0^{-2}.$$

Thus $R = -2r_0^{-2}$ and the Einstein equations (6.17) read

$$\begin{aligned} R^\beta_\alpha - \tfrac{1}{2}\delta^\beta_\alpha R = \begin{pmatrix} r_0^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_0^{-2} \end{pmatrix} &= -\frac{8\pi G}{c^4}T^\beta_\alpha \\ &= \frac{8\pi G\rho}{c^2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{9.26}$$

Hence with $\rho > 0$ (which corresponds to a positive energy density and tension in the string) the metric (9.25) solves Einstein's equations inside the string.

What we really need is the metric outside the string, where we live. Let the outer surface of the string be $\theta = \theta_m$. Then the exterior metric is

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + r_0^2\Big(\frac{\cos^2\theta}{\cos^2\theta_m}\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\Big) + \mathrm{d}z^2. \tag{9.27}$$

This metric obviously joins smoothly to the interior metric (9.25) on $\theta = \theta_m$. To show that it is a vacuum solution of Einstein's equations, we transform to a new coordinate set (t, r', ϕ', z), where the t and z coordinates are the old ones and

$$r' \equiv r_0\frac{\sin\theta}{\cos\theta_m} \quad ; \quad \phi' \equiv \cos(\theta_m)\phi. \tag{9.28}$$

The metric (9.27) now becomes

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}r'^2 + r'^2\mathrm{d}\phi'^2 + \mathrm{d}z^2, \tag{9.29}$$

which is just the cylindrical-polar metric of flat spacetime. But on a large scale the spacetime outside the string is very odd because the range of ϕ' is $(0, 2\pi\cos\theta_m)$. [This follows from (9.28) and the fact that ϕ is in $(0, 2\pi)$]. Consider for example a large circle $r' = a \gg r_0$. The radius of this circle is

$$R = \int_0^a \sqrt{g_{r'r'}}\,\mathrm{d}r' \simeq a, \tag{9.30a}$$

while its circumference is

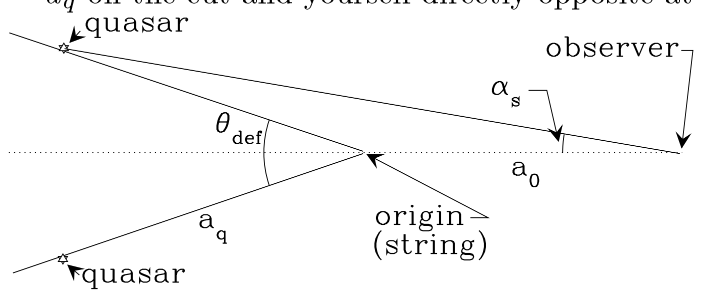
$$C = \int \sqrt{g_{\phi'\phi'}}\,\mathrm{d}\phi' = a2\pi\cos(\theta_m). \tag{9.30b}$$

So the usual flat-space relation $C = 2\pi R$ does not apply. Thinking about a cone may help to clarify this strange state of affairs. At each point a cone is flat in the sense that it can be made out of a piece of paper without stretching the paper (you can't make a paper sphere as easily), but circles distance a from the cone's apex have a circumference smaller than $2\pi a$.

How could we detect a cosmic string? Our best bet is to look for lines of gravitationally lensed objects. To understand how a string lenses an object, think of the exterior space as a piece of paper with a wedge of angle

$$\theta_{\mathrm{def}} \equiv 2\pi(1 - \cos\theta_m) \tag{9.31}$$

cut out and corresponding points along the cuts identified. Place the object to be lensed at radius $r' = a_q$ on the cut and yourself directly opposite at $r' = a_o$.



Rays travel over the paper in straight lines, so you can see the object along two lines of sight separated by $2\alpha_s$, where

$$\frac{\sin(\pi - \frac{1}{2}\theta_{\rm def})}{\sqrt{a_o^2 + a_q^2 + 2a_o a_q \cos(\pi - \frac{1}{2}\theta_{\rm def})}} = \frac{\sin\alpha_s}{a_q}.$$

The largest possible value of α_s is clearly $\frac{1}{2}\theta_{\rm def}$. It should be possible to detect a cosmic string by looking for a line in the sky either side of which lie members of pairs of similar objects.

The mass per unit length μ of the string would follow immediately from $\theta_{\rm def}$: from the interior metric (9.25) it follows that the string's cross-sectional area is

$$A = \int_0^{\theta_m} r_0 {\rm d}\theta \int_0^{2\pi} r_0 \sin\theta {\rm d}\phi = 2\pi r_0^2 (1 - \cos\theta_m).$$

Hence using (9.26) we have that the string's mass per unit length is $\mu = \rho A = c^2(1 - \cos\theta_m)/(4G) = c^2\theta_{\rm def}/(8\pi G)$ independently of the string's physical width r_0. There won't be room outside the string for the Universe as we know it unless $\mu < \frac{1}{4}c^2/G = 3.37 \times 10^{26}\,{\rm kg\,m^{-1}}$. Particle theorists think strings may exist with line densities of order a thousandth of this.

9.7 Summary

The cosmic microwave background defines a natural coordinate system for cosmology. On large scales the Universe appears to be strikingly homogeneous and isotropic. This implies that equal-time hypersurfaces must have the geometry of either (i) the 3-sphere, (ii) flat space, or (iii) hyperbolic space according as the mean cosmic density ρ is greater than, equal to, or less than $\rho_{\rm crit} \simeq 10^{-26}\,{\rm kg\,m^{-3}}$. It is widely believed that $\rho = \rho_{\rm crit}$ although measurements suggest a smaller value.

The cosmic scale when the light we detect from a distant object was emitted can be deduced from the redshift z of the object's spectrum: $1 + z = \omega_{\rm emit}/\omega_{\rm obs} = a(t_{\rm obs})/a(t_{\rm emit})$. The most distant objects are seen at an epoch when a was smaller than now by more than a factor 5.

The expansion of the Universe will cease only if $\rho > \rho_{\rm crit}$. At early times we always have $\rho \simeq \rho_{\rm crit}$ and the cosmic scale grows as $a \propto t^{2/3}$. If the wild speculations of high-energy physicists are to be believed, very early on there may have been an inflationary phase in which $a \propto {\rm e}^{\gamma t}$ and the entire observable Universe grew out of a single quantum fluctuation. If the calibrations of astronomers are to be believed, about two thirds of the energy density in the Universe is currently contributed by vacuum energy, and the Universe is just starting on a second inflationary episode.

A Appendices

A Matrix Manipulation

Many calculations in relativity are best performed by matrix multiplication. Conventionally the first index i on a matrix A_{ij} labels a row and the second, j a column. Then we form the product $\boldsymbol{A} \cdot \boldsymbol{B}$ by summing over adjacent indices:

$$\left(\boldsymbol{A} \cdot \boldsymbol{B}\right)_{ik} = \sum_j A_{ij} B_{jk}$$

Thus to evaluate $A^{\lambda}{}_{\nu} \equiv g_{\mu\nu} B^{\lambda\mu}$ we first rearrange to ensure that we are summing over adjacent indices:

$$\begin{aligned} A^{\lambda}{}_{\nu} &\equiv g_{\mu\nu} B^{\lambda\mu} \\ &= B^{\lambda\mu} g_{\mu\nu} = (\mathbf{B} \cdot \mathbf{g})^{\lambda}{}_{\nu}. \end{aligned}$$

We may have to transpose a tensor to do this:

$$\begin{aligned} A_{\nu}{}^{\lambda} &\equiv B_{\mu\nu} C^{\mu\lambda} \\ &= (B^T)_{\nu\mu} C^{\mu\lambda} = (\mathbf{B}^T \cdot \mathbf{C})_{\nu}{}^{\lambda}. \end{aligned}$$

In particular:

(i) *to raise/lower first index*, premultiply by $\mathbf{g}$ – in special relativity this just changes the sign of the top row;

(ii) *to raise/lower second index*, postmultiply by $\mathbf{g}$ – in special relativity this just changes the sign of the left column.

Doubly contracted 2$^{\text{nd}}$ rank tensors are just the trace of a product matrix:

$$\begin{aligned} A^{\mu\nu} B_{\mu\nu} &= \sum_{\mu} \left(\mathbf{A} \cdot \mathbf{B}^T\right)_{\mu\mu} \\ &= \text{trace}(\mathbf{A} \cdot \mathbf{B}^T). \end{aligned}$$

The epsilon symbol In *special* relativity we define the Levi-Civita symbol by

$$\epsilon_{\alpha\beta\gamma\delta} = \begin{cases} 0 & \text{if any two indices equal} \\ = +1 & \text{if } \alpha, \beta, \gamma, \delta \text{ cyclic permutation of 0,1,2,3} \\ = -1 & \text{if } \alpha, \beta, \gamma, \delta \text{ anticyclic permutation of 0,1,2,3} \end{cases} \qquad \text{(special rel. only)}.$$

It's easy to see that on raising each index with an $\boldsymbol{\eta}$ we get the same pattern for the up symbol.

The ϵ symbols are useful for taking determinants:

$$\begin{aligned} \epsilon_{\alpha\beta\gamma\delta} |\mathbf{A}| &= \epsilon_{\kappa\lambda\mu\nu} A^{\kappa}_{\alpha} A^{\lambda}_{\beta} A^{\mu}_{\gamma} A^{\nu}_{\delta} \\ \epsilon^{\alpha\beta\gamma\delta} |\mathbf{B}| &= \epsilon_{\kappa\lambda\mu\nu} B^{\kappa\alpha} B^{\lambda\beta} B^{\mu\gamma} B^{\nu\delta} \end{aligned}$$

etc.

Transforming to a curvilinear coordinate system we find

$$\begin{aligned}\epsilon'^{\alpha\beta\gamma\delta} &= \frac{\partial x'^\alpha}{\partial x^\kappa}\frac{\partial x'^\beta}{\partial x^\lambda}\frac{\partial x'^\gamma}{\partial x^\mu}\frac{\partial x'^\delta}{\partial x^\nu}\epsilon^{\lambda\kappa\mu\nu} \\ &= \frac{\partial(\mathbf{x}')}{\partial(\mathbf{x})}\epsilon^{\alpha\beta\gamma\delta}\end{aligned} \tag{9.1}$$

But

$$\begin{aligned}|g'^{\alpha\beta}| &= \left|\frac{\partial x'^\alpha}{\partial x^\kappa}\frac{\partial x'^\beta}{\partial x^\lambda}\eta^{\kappa\lambda}\right| \\ &= -\left(\frac{\partial(\mathbf{x}')}{\partial(\mathbf{x})}\right)^2.\end{aligned}$$

So we can write (9.1) as

$$\epsilon'^{\alpha\beta\gamma\delta} = \sqrt{-|g^{\mu\nu}|}\epsilon^{\alpha\beta\gamma\delta}. \tag{9.2}$$

Similarly, $\epsilon'_{0123} = \sqrt{-|g_{\mu\nu}|} = 1/\sqrt{-|g^{\mu\nu}|}$. Hence in g.r. the ϵ symbols are not made up of nought and one, but of nought and $\sqrt{-|g^{\mu\nu}|}$. Also in g.r. the up and down forms of ϵ are distinct.

In general ϵ has two jobs: (i) it extracts the totally antisymmetric parts of tensors; (ii) it maps one-to-one totally antisymmetric n^{th} rank tensors into totally antisymmetric tensors of rank $(4-n)$. The correspondence $\mathbf{F} \leftrightarrow \overline{\mathbf{F}}$ is an example of this map at work.

B Derivation of $R_\alpha{}^\beta{}_{;\beta} = \frac{1}{2}R_{;\alpha}$

We can calculate $R_\alpha{}^\beta{}_{;\beta}$ at a point $\mathbf{X}$ most cheaply as follows. We adopt a locally freely falling coordinate system at $\mathbf{X}$. In this system there are no pseudo-forces at $\mathbf{X}$, so $\mathbf{\Gamma}$ (but not its derivatives) vanishes there. Consequently, at $\mathbf{X}$ covariant derivatives are equivalent to partial derivatives, and we obtain from (6.13a,b) and (5.20)

$$\begin{aligned}R_\alpha{}^\beta{}_{;\beta} &= \frac{\partial}{\partial x^\beta}\left[g^{\beta\gamma}\left(\frac{\partial\Gamma^\mu_{\alpha\mu}}{\partial x^\gamma} - \frac{\partial\Gamma^\mu_{\alpha\gamma}}{\partial x^\mu}\right)\right] \\ &= \tfrac{1}{2}\frac{\partial}{\partial x^\beta}\left[g^{\beta\gamma}\frac{\partial}{\partial x^\gamma}\left(g^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)\right] \\ &\quad - \tfrac{1}{2}\frac{\partial}{\partial x^\beta}\left\{g^{\beta\gamma}\frac{\partial}{\partial x^\mu}\left[g^{\mu\nu}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\gamma} + \frac{\partial g_{\nu\gamma}}{\partial x^\alpha} - \frac{\partial g_{\gamma\alpha}}{\partial x^\nu}\right)\right]\right\}.\end{aligned} \tag{B.3}$$

Since the covariant derivative of $\mathbf{g}$ always vanishes, and $\mathbf{\Gamma} = 0$ at $\mathbf{X}$, all first derivatives of $\mathbf{g}$ must vanish at $\mathbf{X}$. Dropping from $R_\alpha{}^\beta{}_{;\beta}$ all terms which contain a first derivative of $\mathbf{g}$, we find

$$R_\alpha{}^\beta{}_{;\beta} = \tfrac{1}{2}g^{\beta\gamma}g^{\mu\nu}\left(\frac{\partial^3 g_{\mu\nu}}{\partial x^\alpha\partial x^\beta\partial x^\gamma} - \frac{\partial^3 g_{\alpha\nu}}{\partial x^\beta\partial x^\gamma\partial x^\mu} - \frac{\partial^3 g_{\nu\gamma}}{\partial x^\alpha\partial x^\beta\partial x^\mu} + \frac{\partial^3 g_{\gamma\alpha}}{\partial x^\beta\partial x^\mu\partial x^\nu}\right).$$

The second and fourth terms cancel because in each case two of the partial derivatives are contracted together and the third is contracted with an index of the component of **g** being differentiated. Hence

$$R_\alpha{}^\beta{}_{;\beta} = \tfrac{1}{2}\frac{\partial}{\partial x^\alpha}\Big[g^{\beta\gamma}g^{\mu\nu}\Big(\frac{\partial^2 g_{\mu\nu}}{\partial x^\beta \partial x^\gamma} - \frac{\partial^2 g_{\nu\gamma}}{\partial x^\beta \partial x^\mu}\Big)\Big]. \tag{B.4}$$

From the definition (6.15) of the Ricci scalar R we have in our special coordinate system

$$\begin{aligned} R &= g^{\beta\gamma}\Big(\frac{\partial\Gamma^\mu_{\beta\mu}}{\partial x^\gamma} - \frac{\partial\Gamma^\mu_{\beta\gamma}}{\partial x^\mu}\Big) \\ &= \tfrac{1}{2}g^{\beta\gamma}g^{\mu\nu}\Big(\frac{\partial^2 g_{\mu\nu}}{\partial x^\gamma \partial x^\beta} - \frac{\partial^2 g_{\nu\beta}}{\partial x^\gamma \partial x^\mu} - \frac{\partial^2 g_{\nu\gamma}}{\partial x^\mu \partial x^\beta} + \frac{\partial^2 g_{\gamma\beta}}{\partial x^\mu \partial x^\nu}\Big). \end{aligned} \tag{B.5}$$

The first and fourth terms are equal, as are the second and third. Comparing this expression with (9.4) we obtain the desired relation.