Conformal Field Theory and Statistical Mechanics*

John Cardy

July 2008

*Lectures given at the Summer School on *Exact methods in low-dimensional statistical physics and quantum computing*, les Houches, July 2008.

Contents

1 Introduction

It is twenty years ago almost to the day that the les Houches school on *Fields, Strings and Critical Phenomena* took place. It came on the heels of a frenzied period of five years or so following the seminal paper of Belavin, Polyakov and Zamolodchikov (BPZ) in which the foundations of conformal field theory and related topics had been laid down, and featured lectures on CFT by Paul Ginsparg, Ian Affleck, Jean-Bernard Zuber and myself, related courses and talks by Hubert Saleur, Robert Dijkgraaf, Bertrand Duplantier, Sasha Polyakov, Daniel Friedan, as well as lectures on several other topics. The list of young participants, many of whom have since gone on to make their own important contributions, is equally impressive.

Twenty years later, CFT is an essential item in the toolbox of many theoretical condensed matter physicists and string theorists. It has also had a marked impact in mathematics, in algebra, geometry and, more recently, probability theory. It is the purpose of these lectures to introduce some of these tools. In some ways they are an updated version of those I gave in 1988. However, there are some important topics there which, in order to include new material, I will omit here, and I would encourage the diligent student to read both versions. I should stress that the lectures will largely be about Conformal Field Theor<u>y</u>, rather than Conformal Field Theor<u>ies</u>, in the sense that I'll be describing rather generic properties of CFT rather than discussing particular examples. That said, I don't want the discussion to be too abstract, and in fact I will have a very specific idea about what are the kind of CFTs I will be discussing: the scaling limit (in a sense to be described) of critical lattice models (either classical, in two dimensions, or quantum in 1+1 dimensions.) This will allow us to have, I hope, more concrete notions of the mathematical objects which are being discussed in CFT. However there will be no attempt at mathematical rigour. Despite the fact that CFT can be developed axiomatically, I think that for this audience it is more important to understand the physical origin of the basic ideas.

2 Scale invariance and conformal invariance in critical behaviour

2.1 Scale invariance

The prototype lattice model we have at the back of our minds is the ferromagnetic Ising model. We take some finite domain $\mathcal{D}$ of d-dimensional euclidean space and impose a regular lattice, say hypercubic, with lattice constant a. With each node r of the lattice we associate a binary-valued spin $s(r) = \pm 1$. Each configuration $\{s\}$ carries a relative weight $W(\{s\}) \propto \exp(\sum_{rr'\in\mathcal{D}} J(r-r')s(r)s(r))$ where $J(r-r') > 0$ is some short-ranged interaction (i.e. it vanishes if $|r-r'|$ is larger than some fixed multiple of a.)

The usual kinds of local observable $\phi_j^{\rm lat}(r)$ considered in this lattice model are sums of products of nearby spins over some region of size $O(a)$, for example the local magnetisation $s(r)$ itself, or the energy density $\sum_{r'} J(r-r')s(r)s(r')$. However it will become clear later on that there are other observables, also labelled by a single point r, which are functions of the whole configuration $\{s\}$. For example, we shall show that this Ising model can be mapped onto a gas of non-intersecting loops. An observable then might depend on whether a particular loop passes through the given point r. From the point of view of CFT, these observables are equally valid objects.

Correlation functions, that is expectation values of products of local lattice observables, are then given by

$$\langle\phi_1^{\rm lat}(r_1)\phi_2^{\rm lat}(r_2)\ldots\phi_n^{\rm lat}(r_n)\rangle = Z^{-1}\sum_{\{s\}}\phi_1^{\rm lat}(r_1)\ldots\phi_n^{\rm lat}(r_n)W(\{s\})\,,$$

where $Z=\sum_{\{s\}}W(\{s\})$ is the partition function. In general, the connected pieces of these correlations functions fall off over the same distance scale as the interaction $J(r-r')$, but, close to a critical point, this correlation length ξ can become large, $\gg a$.

The *scaling limit* is obtained by taking $a\to 0$ while keeping ξ and the domain $\mathcal{D}$ fixed. In general the correlation functions as defined above do not possess a finite scaling limit. However, the theory of renormalisation (based on studies in exactly solved models, as well as perturbative analysis of cut-off quantum field theory) suggests that in general there are particular linear combinations of local lattice observables which are *multiplicatively renormalisable*. That is, the limit

$$\lim_{a\to 0} a^{-\sum_{j=1}^n x_j}\langle\phi_1^{\rm lat}(r_1)\phi_2^{\rm lat}(r_2)\ldots\phi_n^{\rm lat}(r_n)\rangle \tag{1}$$

exists for certain values of the $\{x_j\}$. We usually denote this by

$$\langle\phi_1(r_1)\phi_2(r_2)\ldots\phi_n(r_n)\rangle\,, \tag{2}$$

and we often think of it as the expectation value of the product the random variables $\phi_j(r_j)$, known as *scaling fields* (sometimes scaling operators, to be even more confusing) with respect to some 'path integral' measure. However it should be stressed that this is only an occasionally useful fiction, which ignores all the wonderful subtleties of renormalised field theory. The basic objects of QFT are the correlation functions. The numbers $\{x_j\}$ in (1) are called the *scaling dimensions*.

One important reason why this is not true in general is that the limit in (1) in fact only exists if the points $\{r_j\}$ are non-coincident. The correlation functions in (2) are singular in the limits when $r_i \to r_j$. However, the nature of these singularities is prescribed by the *operator product expansion* (OPE)

$$\langle\phi_i(r_i)\phi_j(r_j)\ldots\rangle = \sum_k C_{ijk}(r_i - r_j)\langle\phi_k((r_i + r_j)/2)\ldots\rangle\,. \tag{3}$$

The main point is that, in the limit when $|r_i - r_j|$ is much less than the separation between r_i and all the other arguments in $\ldots$, the coefficients C_{ijk} are independent of what is in the dots. For this reason, (3) is often written as

$$\phi_i(r_i)\cdot\phi_j(r_j) = \sum_k C_{ijk}(r_i - r_j)\phi_k((r_i + r_j)/2)\,, \tag{4}$$

although it should be stressed that this is merely a short-hand for (3).

So far we have been talking about how to get a continuum (euclidean) field theory as the scaling limit of a lattice model. In general this will be a massive QFT, with a mass scale given by the inverse correlation length ξ^{-1}. In general, the correlation functions will depend on this scale. However, at a (second-order) critical point the correlation length ξ diverges, that is the mass vanishes, and there is no length scale in the problem besides the overall size L of the domain $\mathcal{D}$.

The fact that the scaling limit of (1) exists then implies that, instead of starting with a lattice model with lattice constant a, we could equally well have started with one with some fraction a/b. This would, however, be identical with a lattice model with the original spacing a, in which all lengths (including the size of the domain $\mathcal{D}$) are multiplied by b. This implies that the correlation functions in (2) are *scale covariant*:

$$\langle\phi_1(br_1)\phi_2(br_2)\ldots\phi_n(br_n)\rangle_{b\mathcal{D}} = b^{-\sum_j x_j}\langle\phi_1(r_1)\phi_2(r_2)\ldots\phi_n(r_n)\rangle_{\mathcal{D}}\,. \tag{5}$$

Once again, we can write this in the suggestive form

$$\phi_j(br) = b^{-x_j}\phi_j(r)\,, \tag{6}$$

as long as what we really mean is (5).

In a massless QFT, the form of the OPE coefficients in (4) simplifies: by scale covariance

$$C_{ijk}(r_j - r_k) = \frac{c_{ijk}}{|r_i - r_j|^{x_i + x_j - x_k}}\,, \tag{7}$$

where the c_{ijk} are pure numbers, and *universal* if the 2-point functions are normalised so that $\langle\phi_j(r_1)\phi_j(r_2)\rangle = |r_1 - r_2|^{-2x_j}$. (This assumes that the scaling fields are all rotational scalars – otherwise it is somewhat more complicated, at least for general dimension.)

From scale covariance, it is a simple but powerful leap to *conformal covariance*: suppose that the scaling factor b in (5) is a slowly varying function of position r. Then we can try to write a generalisation of (5) as

$$\langle\phi_1(r_1')\phi_2(r_2')\ldots\phi_n(r_n')\rangle_{\mathcal{D}'} = \prod_{j=1}^n b(r_j)^{-x_j}\langle\phi_1(r_1)\phi_2(r_2)\ldots\phi_n(r_n)\rangle_{\mathcal{D}}\,, \tag{8}$$

where $b(r) = |\partial r'/\partial r|$ is the local jacobian of the transformation $r \to r'$.

For what transformations $r \to r'$ do we expect (8) to hold? The heuristic argument runs as follows: if the theory is local (that is the interactions in the lattice model are short-ranged), then as long as the transformations looks *locally* like a scale transformation (plus a possible rotation), then (8) may be expected to hold. (In Sec. 3 we will make this more precise, based on the assumed properties of the stress tensor, and argue that in fact it holds only for a special class of scaling fields $\{\phi_j\}$ called primary.)

It is most important that the underlying lattice does *not* transform (otherwise the statement is a tautology): (8) relates correlation functions in $\mathcal{D}$, defined in terms of the limit $a \to 0$ of a model on a regular lattice superimposed on $\mathcal{D}$, to correlation functions defined by a regular lattice superimposed on $\mathcal{D}'$.

Transformations which are locally equivalent to a scale transformation and rotation, that is, have no local components of shear, also locally preserve angles and are called *conformal*.

2.2 Conformal mappings in general

Consider a general infinitesimal transformation (in flat space) $r^\mu \to r'^\mu = r^\mu + \alpha^\mu(r)$ (we distinguish upper and lower indices in anticipation of using coordinates in which the metric is not diagonal.) The shear component is the traceless symmetric part

$$\alpha^{\mu,\nu} + \alpha^{\nu,\mu} - (2/d)\alpha^\lambda{}_{,\lambda} g^{\mu\nu} ,$$

all $\frac{1}{2}d(d+1) - 1$ components of which must vanish for the mapping to be conformal. For general d this is very restrictive, and in fact, apart from uniform translations, rotations and scale transformations, there is only one other type of solution

$$\alpha^\mu(r) = b^\mu r^2 - 2(b \cdot r) r^\mu ,$$

where b^μ is a constant vector. These are in fact the composition of the finite conformal mapping of inversion $r^\mu \to r^\mu/|r|^2$, followed by an infinitesimal translation b^μ, followed by a further inversion. They are called the special conformal transformations, and together with the others, they generate a group isomorphic to $\mathrm{SO}(d+1,1)$.

These special conformal transformations have enough freedom to fix the form of the 3-point functions in $\mathbf{R}^d$ (just as scale invariance and rotational invariance fixes the 2-point functions): for scalar operators[1]

$$\langle \phi_1(r_1)\phi_2(r_2)\phi_3(r_3)\rangle = \frac{c_{123}}{|r_1 - r_2|^{x_1+x_2-x_3}|r_2 - r_3|^{x_2+x_3-x_1}|r_3 - r_1|^{x_3+x_1-x_2}} . \qquad (9)$$

Comparing with the OPE (4,7), and assuming non-degeneracy of the scaling dimensions[2], we see that c_{123} is the same as the OPE coefficient defined earlier. This shows that the OPE coefficients c_{ijk} are symmetric in their indices.

[1] The easiest way to show this it to make an inversion with an origin very close to one of the points, say r_1, and then use the OPE, since its image is then very far from those of the other two points.

[2] This and other properties fail in so-called logarithmic CFTs.

In two dimensions, the condition that $\alpha^\mu(r)$ be conformal imposes only two differential conditions on two functions, and there is a much wider class of solutions. These are more easily seen using *complex coordinates*[3] $z \equiv r^1 + ir^2$, $\bar z \equiv r^1 - ir^2$, so that the line element is $ds^2 = dzd\bar z$, and the metric is

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} .$$

In this basis, two of the conditions are satisfied identically and the others become

$$\alpha^{z,z} = \alpha^{\bar z,\bar z} = 0 \,,$$

which means that $\partial\alpha^z/\partial\bar z = \partial\alpha^{\bar z}/\partial z = 0$, that is, α^z is a holomorphic function $\alpha(z)$ of z, and $\alpha^{\bar z}$ is an antiholomorphic function.

Generalising this to a finite transformation, it means that conformal mappings $r \to r'$ correspond to functions $z \to z' = f(z)$ which are analytic in $\mathcal{D}$. (Note that the only such functions on the whole Riemann sphere are the Möbius transformations $f(z) = (az + b)/(cz + d)$, which are the finite special conformal mappings.)

In passing, let us note that complex coordinates give us a nice way of discussing non-scalar fields: if, for example, under a rotation $z \to ze^{i\theta}$, $\phi_j(z, \bar z) \to e^{is_j\theta}\phi_j$, we say that ϕ_j has conformal spin s_j (not related to quantum mechanical spin), and under a combined transformation $z \to \lambda z$ where $\lambda = be^{i\theta}$ we can write (in the same spirit as (6))

$$\phi_j(\lambda z, \bar\lambda\bar z) = \lambda^{-\Delta_j}\bar\lambda^{-\overline{\Delta}_j}\phi_j(z, \bar z) \,,$$

where $x_j = \Delta_j + \overline{\Delta}_j$, $s_j = \Delta_j - \overline{\Delta}_j$. $(\Delta_j, \overline{\Delta}_j)$ are called the complex scaling dimensions of ϕ_j (although they are usually both real, and not necessarily complex conjugates of each other.)

3 The role of the stress tensor

Since we wish to explore the consequences of conformal invariance for correlation functions in a *fixed* domain $\mathcal{D}$ (usually the entire complex plane), it is necessary to consider transformations which are *not* conformal everywhere. This brings in the stress tensor $T_{\mu\nu}$ (also known as the stress-energy tensor or the (improved) energy-momentum tensor). It is the object appearing on the right hand side of Einstein's equations in curved space. In a classical field theory, it is defined in terms of the response of the action S to a general infinitesimal transformation $\alpha^\mu(r)$:

$$\delta S = -\frac{1}{2\pi}\int T_{\mu\nu}\alpha^{\mu,\nu}d^2r \qquad (10)$$

[3] For many CFT computations we may treat z and $\bar z$ as independent, imposing only at the end that they should be complex conjugates.

(the $(1/2\pi)$ avoids awkward such factors later on.) Invariance of the action under translations and rotations implies that $T_{\mu\nu}$ is conserved and symmetric. Moreover if S is scale invariant, $T_{\mu\nu}$ is also traceless. In complex coordinates, the first two conditions imply that $T_{z\bar{z}} + T_{\bar{z}z} = 0$ and $T_{z\bar{z}} = T_{\bar{z}z}$, so they both vanish, and the conservation equations then read $\partial^z T_{zz} = 2\partial T_{zz}/\partial\bar{z} = 0$ and $\partial T_{\bar{z}\bar{z}}/\partial z = 0$. Thus the non-zero components $T \equiv T_{zz}$ and $\overline{T} \equiv T_{\bar{z}\bar{z}}$ are respectively holomorphic and antiholomorphic fields. Now if we consider a more general transformation for which $\alpha^{\mu,\nu}$ is symmetric and traceless, that is a conformal transformation, we see that $\delta S = 0$ in this case also. Thus, at least classically, we see that scale invariance and rotational invariance imply conformal invariance of the action, at least if (10) holds. However if the theory contains long-range interactions, for example, this is no longer the case.

In a quantum 2d CFT, it is assumed that the above analyticity properties continue to hold at the level of correlation functions: those of $T(z)$ and $\overline{T}(\bar{z})$ are holomorphic and antiholomorphic functions of z respectively (except at coincident points.)

3.0.1 An example - free (gaussian) scalar field

The prototype CFT is the free, or gaussian, massless scalar field $h(r)$ (we use this notation for reasons that will emerge later). It will turn out that many other CFTs are basically variants of this. The classical action is

$$S[h] = (g/4\pi)\int(\partial_\mu h)(\partial^\mu h)d^2r\,.$$

Since $h(r)$ can take any real value, we could rescale it to eliminate the coefficient in front, but in later extensions this will have a meaning, so we keep it. In complex coordinates, $S \propto \int(\partial_z h)(\partial_{\bar{z}} h)d^2z$, and it is easy to see that this is conformally invariant under $z \to z' = f(z)$, since $\partial_z = f'(z)\partial_{z'}$, $\partial_{\bar{z}} = \overline{f'(z)}\partial_{\bar{z}'}$ and $d^2z = |f'(z)|^{-2}d^2z'$. This is confirmed by calculating $T_{\mu\nu}$ explicitly: we find $T_{z\bar{z}} = T_{\bar{z}z} = 0$, and

$$T = T_{zz} = -g(\partial_z h)^2, \qquad \overline{T} = T_{\bar{z}\bar{z}} = -g(\partial_{\bar{z}} h)^2\,.$$

These are holomorphic (resp. antiholomorphic) by virtue of the classical equation of motion $\partial_z\partial_{\bar{z}}h = 0$.

In the quantum field theory, a given configuration $\{h\}$ is weighted by $\exp(-S[h])$. The 2-point function is[4]

$$\langle h(z,\bar{z})h(0,0)\rangle = \frac{2\pi}{g}\int\frac{e^{ik\cdot r}}{k^2}\frac{d^2k}{(2\pi)^2} \sim -(1/2g)\log(z\bar{z}/L^2)\,.$$

This means that $\langle T\rangle$ is formally divergent. It can be made finite, for example, by point-splitting and subtracting off the divergent piece:

$$T(z) = -g\lim_{\delta\to 0}\left(\partial_z h(z+\tfrac{1}{2}\delta)\partial_z h(z-\tfrac{1}{2}\delta) - \frac{1}{2g\delta^2}\right)\,. \qquad (11)$$

[4]this is cut-off at small k by the assumed finite size L, but we are here also assuming that the points are far from the boundary.

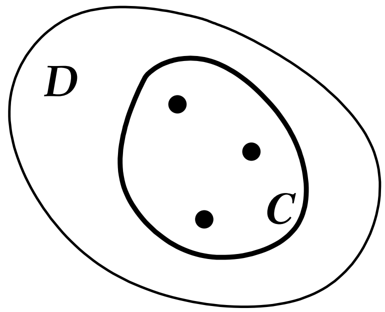

Figure 1: We consider an infinitesimal transformation which is conformal within C and the identity in the complement in $\mathcal{D}$.

This doesn't affect the essential properties of T.

3.1 Conformal Ward identity

Consider a general correlation function of scaling fields $\langle\phi_1(z_1,\bar z_1)\phi_2(z_2,\bar z_2)\ldots\rangle_{\mathcal{D}}$ in some domain $\mathcal{D}$. We want to make an infinitesimal conformal transformation $z\to z'=z+\alpha(z)$ on the points $\{z_j\}$, without modifying $\mathcal{D}$. This can be done by considering a contour C which encloses all the points $\{z_j\}$ but which lies wholly within $\mathcal{D}$, such that the transformation is conformal within C, and the identity $z'=z$ outside C (Fig. 1). This gives rise to an (infinitesimal) discontinuity on C, and, at least classically to a modification of the action S according to (10). Integrating by parts, we find $\delta S=(1/2\pi)\int_C T_{\mu\nu}\alpha^\mu n^\nu d\ell$, where n^ν is the outward-pointing normal and $d\ell$ is a line element of C. This is more easily expressed in complex coordinates, after some algebra, as

$$\delta S=\frac{1}{2\pi i}\int_C \alpha(z)T(z)dz+\text{complex conjugate}\,.$$

This extra factor can then be expanded, to first order in α, out of the weight $\exp(-S[h]-\delta S)\sim(1-\delta S)\exp(-S[h])$, and the extra piece δS considered as an insertion into the correlation function. This is balanced by the explicit change in the correlation function under the conformal transformation:

$$\delta\langle\phi_1(z_1,\bar z_1)\phi_2(z_2,\bar z_2)\ldots\rangle=\frac{1}{2\pi i}\int_C \alpha(z)\langle T(z)\phi_1(z_1,\bar z_1)\phi_2(z_2,\bar z_2)\ldots\rangle dz+\text{c.c.}\qquad(12)$$

Let us first consider the case when $\alpha(z)=\lambda(z-z_1)$, that is corresponds to a combined rotation and scale transformation. In that case $\delta\phi_1=(\Delta_1\lambda+\overline{\Delta}_1\bar\lambda)\phi_1$, and therefore, equating coefficients of λ and $\bar\lambda$,

$$\int_C (z-z_1)\langle T(z)\phi_1(z_1,\bar z_1)\phi_2(z_2,\bar z_2)\ldots\rangle\frac{dz}{2\pi i}=\Delta_1\langle\phi_1(z_1,\bar z_1)\phi_2(z_2,\bar z_2)\ldots\rangle+\cdots\,.$$

Similarly, if we take $\alpha =$ constant, corresponding to a translation, we have $\delta\phi_j \propto \partial_{z_j}\phi_j$, so

$$\int_C \langle T(z)\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\ldots\rangle \frac{dz}{2\pi i} = \sum_j \Delta_j \partial_{z_j}\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\ldots\rangle\,.$$

Using Cauchy's theorem, these two equations tell us about two of the singular terms in the OPE of $T(z)$ with a general scaling field $\phi_j(z_j,\bar{z}_j)$:

$$T(z)\cdot\phi_j(z_j,\bar{z}_j) = \cdots + \frac{\Delta_j}{(z-z_j)^2}\phi_j(z_j,\bar{z}_j) + \frac{1}{z-z_j}\partial_{z_j}\phi_j(z_j,\bar{z}_j) + \cdots\,. \qquad (13)$$

Note that only integer powers can occur, because the correlation function is a meromorphic function of z.

Example of the gaussian free field. If we take $\phi_q^{\rm lat}(r) = e^{iqh(r)}$ then

$$\langle \phi_q^{\rm lat}(r_1)\phi_{-q}^{\rm lat}(r_2)\rangle = \exp\big(-\tfrac{1}{2}q^2\langle (h(r_1)-h(r_2))^2\rangle\big) \sim \left(\frac{a}{|r_1-r_2|}\right)^{q^2/g}\,,$$

which means that the renormalised field $\phi_q \sim a^{-q^2/2g}\phi_q^{\rm lat}$ has scaling dimension $x_q = q^2/2g$. It is then a nice exercise in Wick's theorem to check that the OPE with the stress tensor (13) holds with $\Delta_q = x_q/2$. (Note that in this case the multiplicative renormalisation of ϕ_q is equivalent to ignoring all Wick contractions between fields $h(r)$ at the same point.)

Now suppose each ϕ_j is such that the terms $O((z-z_j)^{-2-n})$ with $n\geq 1$ in (13) are absent. Since a meromorphic function is determined entirely by its singularities, we then know the correlation function $\langle T(z)\ldots\rangle$ exactly:

$$\langle T(z)\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\ldots\rangle = \sum_j\left(\frac{\Delta_j}{z-z_j)^2} + \frac{1}{z-z_j}\partial_{z_j}\right)\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\ldots\rangle\,. \quad (14)$$

This (as well as a similar equation for an insertion of $\overline{T}$) is the *conformal Ward identity.* We derived it assuming that the quantum theory could be defined by a path integral and the change in the action δS follows the classical pattern. For a more general CFT, not necessarily 'defined' (however loosely) by a path integral, (14) is usually assumed as a property of T. In fact many basic introductions to CFT use this as a starting point.

Scaling fields $\phi_j(z_j,\bar{z}_j)$ such that the most singular term in their OPE with $T(z)$ is $O((z-z_j)^{-2})$ are called *primary.*[5] All the other fields like those appearing in the less singular terms in (13) are called *descendants*. Once one knows the correlation functions of all the primaries, those of the rest follow from (13).[6]

For correlations of such primary fields, we can now reverse the arguments leading to (13) for the case of a general infinitesimal conformal transformation $\alpha(z)$ and conclude that

$$\delta\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\ldots\rangle = \sum_j(\Delta_j\alpha'(z_j)+\alpha(z_j)\partial_{z_j})\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\ldots\rangle\,,$$

[5] If we assume that there is a lower bound to the scaling dimensions, such fields must exist.

[6] Since the scaling dimensions of the descendants differ from those of the corresponding primaries by positive integers, they are increasingly irrelevant in the sense of the renormalisation group.

which may be integrated up to get the result for a *finite* conformal mapping $z \to z' = f(z)$:

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\ldots\rangle_{\mathcal{D}} = \prod_j f'(z_j)^{\Delta_j}\overline{f'(z_j)}^{\bar{\Delta}_j}\langle \phi_1(z_1', \bar{z}_1')\phi_2(z_2', \bar{z}_2')\ldots\rangle_{\mathcal{D}'}\,.$$

This is just the result we wanted to postulate in (8), but now we see that it can hold only for correlation functions of *primary* fields.

It is important to realise that T itself is not in general primary. Indeed its OPE with itself must take the form[7]

$$T(z)\cdot T(z_1) = \frac{c/2}{(z-z_1)^4} + \frac{2}{(z-z_1)^2}T(z_1) + \frac{1}{z-z_1}\partial_{z_1}T(z_1)\cdots\,. \tag{15}$$

This is because (taking the expectation value of both sides) the 2-point function $\langle T(z)T(z_1)\rangle$ is generally non-zero. Its form is fixed by the fact that $\Delta_T = 2$, $\overline{\Delta}_T = 0$, but, since the normalisation of T is fixed by its definition (10), its coefficient $c/2$ is fixed. This introduces the *conformal anomaly* number c, which is part of the basic data of the CFT, along with the scaling dimensions $(\Delta_j, \overline{\Delta}_j)$ and the OPE coefficients c_{ijk}.[8]

This means that, under an infinitesimal transformation $\alpha(z)$, there is an additional term in the transformation law of T:

$$\delta T(z) = 2\alpha'(z)T(z) + \alpha(z)\partial_z T(z) + \tfrac{c}{12}\alpha'''(z)\,.$$

For a finite conformal transformation $z \to z' = f(z)$, this integrates up to

$$T(z) = f'(z)^2 T(z') + \tfrac{c}{12}\{z', z\}\,, \tag{16}$$

where the last term is the Schwarzian derivative

$$\{w(z), z\} = \frac{w'''(z)w'(z) - \frac{3}{2}w''(z)^2}{w'(z)^2}\,.$$

The form of the Schwarzian can be seen most easily in the example of a gaussian free field. In this case, the point-split terms in (11) transform properly and give rise to the first term in (16), but the fact that the subtraction has to be made separately in the origin frame and the transformed one leads to an anomalous term

$$\lim_{\delta\to 0} g\left(\frac{f'(z+\frac{1}{2}\delta)f'(z-\frac{1}{2}\delta)}{2g(f(z+\frac{1}{2}\delta)-f(z-\frac{1}{2}\delta))^2} - \frac{1}{2g\delta^2}\right)\,,$$

which, after some algebra, gives the second term in (16) with $c = 1$.[9]

[7]The $O((z-z_1)^{-3})$ term is absent by symmetry under exchange of z and z_1.

[8]When the theory is placed in curved background, the trace $\langle T^\mu_\mu\rangle \propto cR$, where R is the local scalar curvature.

[9]This is a classic example of an anomaly in QFT, which comes about because the regularisation procedure does not respect a symmetry.

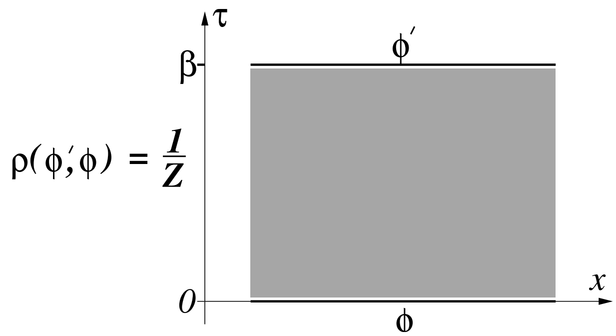

Figure 2: The density matrix is given by the path integral over a space-time region in which the rows and columns are labelled by the initial and final values of the fields.

3.2 An application - entanglement entropy

Let's take a break from the development of the general theory and discuss how the conformal anomaly number c arises in an interesting (and topical) physical context. Suppose that we have a massless relativistic quantum field theory in 1+1 dimensions, whose imaginary time behaviour therefore corresponds to a euclidean CFT. (There are many condensed matter systems whose large distance behaviour is described by such a theory.) The system is at zero temperature and therefore in its ground state $|0\rangle$, corresponding to a density matrix $\rho = |0\rangle\langle 0|$. Suppose an observer A has access to just part of the system, for example a segment of length ℓ inside the rest of the system, of total length $L \gg \ell$, observed by B. The measurements of A and B are entangled. It can be argued that a useful measure of the degree of entanglement is the *entropy* $S_A = -\mathrm{Tr}_A\,\rho_A \log \rho_A$, where $\rho_A = \mathrm{Tr}_B\,\rho$ is the reduced density matrix corresponding to A.

How can we calculate S_A using CFT methods? The first step is to realise that if we can compute $\mathrm{Tr}\,\rho_A^n$ for positive integer n, and then analytically continue in n, the derivative $\partial/\partial n|_{n=1}$ will give the required quantity. For any QFT, the density matrix at finite inverse temperature β is given by the path integral over some fundamental set of fields $h(x,t)$ (t is imaginary time)

$$\rho(\{h(x,0)\},\{h(x,\beta)\}) = Z^{-1}\int{}'[dh(x,t)]e^{-S[h]}\,,$$

where the rows and columns of ρ are labelled by the values of the fields at times 0 and β respectively, the the path integral is over all histories $h(x,t)$ consistent with these initial and final values (Fig. 2). Z is the partition function, obtained by sewing together the edges at these two times, that is setting $h(x,\beta) = h(x,0)$ and integrating $\int[dh(x,0)]$.

We are interested in the partial density matrix ρ_A, which is similarly found by sewing together the top and bottom edges for $x \in B$, that is, leaving open a slit along the interval A (Fig. 3). The rows and columns of ρ_A are labelled by the values of the fields on the edges of the slit. $\mathrm{Tr}\,\rho_A^n$ is then obtained by sewing together the edges of n copies of this slit cylinder in a cyclic fashion (Fig. 4). This gives an n-sheeted surface with branch

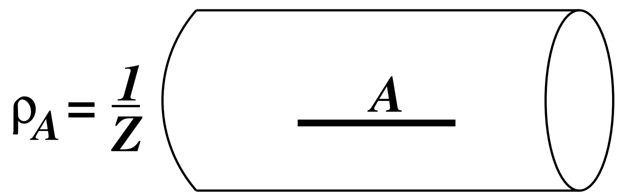

Figure 3: The reduced density matrix ρ_A is given by the path integral over a cylinder with a slit along the interval A.

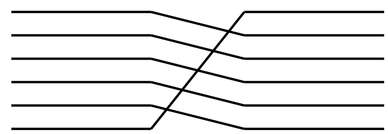

Figure 4: $\mathrm{Tr}\,\rho_A^n$ corresponds to sewing together n copies so that the edges are connected cyclically.

points, or conical singularities, at the ends of the interval A. If Z_n is the partition function on this surface, then

$$\mathrm{Tr}\,\rho_A^n = Z_n/Z_1^n\,.$$

Let us consider the case of zero temperature, $\beta \to \infty$, when the whole system is in the ground state $|0\rangle$. If the ends of the interval are at (x_1, x_2), the conformal mapping

$$z = \left(\frac{w - x_1}{w - x_2}\right)^{1/n}$$

maps the n-sheeted w-surface to the single-sheeted complex z-plane. We can use the transformation law (16) to compute $\langle T(w)\rangle$, given that $\langle T(z)\rangle = 0$ by translational and rotational invariance. This gives, after a little algebra,

$$\langle T(w)\rangle = \frac{(c/12)(1 - 1/n^2)(x_2 - x_1)^2}{(w - x_1)^2(w - x_2)^2}\,.$$

Now suppose we change the length $\ell = |x_2 - x_1|$ slightly, by making an infinitesimal non-conformal transformation $x \to x + \delta\ell\delta(x - x_0)$, where $x_1 < x_0 < x_2$. The response of the log of the partition function, by the definition of the stress tensor, is

$$\delta \log Z_n = -\frac{n\delta\ell}{2\pi}\int_{-\infty}^{\infty}\langle T_{xx}(x_0, t)\rangle dt$$

(the factor n occurs because it has to be inserted on each of the n sheets.) Writing $T_{xx} = T + \overline{T}$, the integration in each term can be carried out by wrapping the contour around either of the points x_1 or x_2. The result is

$$\frac{\partial \log Z_n}{\partial \ell} = -\frac{(c/6)(n - 1/n)}{\ell}\,,$$

so that $Z_n/Z_1^n \sim \ell^{-(c/6)(n-1/n)}$. Taking the derivative with respect to n at $n = 1$ we get the final result

$$S_A \sim (c/3)\log \ell\,.$$

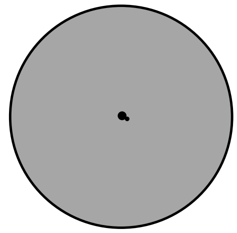

Figure 5: The state $|\phi_j\rangle$ is defined by weighting field configurations on the circle with the path integral inside, with an insertion of the field $\phi_j(0)$ at the origin.

4 Radial quantisation and the Virasoro algebra

4.1 Radial quantisation

Like any quantum theory CFT can be formulated in terms of local operators acting on a Hilbert space of states. However, as it is massless, the usual quantisation of the theory on the infinite line is not so useful since it is hard to disentangle the continuum of eigenstates of the hamiltonian, and we cannot define asymptotic states in the usual way. Instead it is useful to exploit the scale invariance, rather than the time-translation invariance, and quantise on a circle of fixed radius r_0. In the path integral formulation, heuristically the Hilbert space is the space of field configurations $|\{h(r_0,\theta)\}\rangle$ on this circle. The analogue of the hamiltonian is then the generator $\hat{D}$ of scale transformations. It will turn out that the spectrum of $\hat{D}$ is discrete. In the vacuum state each configuration is weighted by the path integral over the disc $|z|<r_0$, conditioned on taking the assigned values on r_0, see Fig. 5:

$$|0\rangle = \int [dh(r \leq r_0)] e^{-S[h]} |\{h(r_0,\theta)\}\rangle \,.$$

The scale invariance of the action means that this state is independent of r_0, up to a constant. If instead we insert a scaling field $\phi_j(0)$ into the above path integral, we get a different state $|\phi_j\rangle$. On the other hand, more general correlation functions of scaling fields are given in this operator formalism by[10]

$$\langle \phi_j(z_1,\bar{z}_1)\phi_j(z_2,\bar{z}_2)\rangle = \langle 0|\mathbf{R}\,\hat{\phi}_j(z_1,\bar{z}_1)\hat{\phi}_j(z_2,\bar{z}_2)|0\rangle \,,$$

where $\mathbf{R}$ means the r-ordered product (like the time-ordered product in the usual case), and $\langle 0|$ is defined similarly in terms of the path integral over $r>r_0$. Thus we can identify

$$|\phi_j\rangle = \lim_{z\to 0} \hat{\phi}_j(z,\bar{z})|0\rangle \,.$$

This is an example of the *operator-state correspondence* in CFT.

[10] We shall make an effort consistently to denote actual operators (as opposed to fields) with a hat.

Just as the hamiltonian in ordinary QFT can be written as an integral over the appropriate component of the stress tensor, we can write

$$\hat{D} = \hat{L}_0 + \hat{\bar{L}}_0 \equiv \frac{1}{2\pi i}\int_C z\hat{T}(z)dz + \text{c.c.}\,,$$

where C is any closed contour surrounding the origin. This suggests that we define more generally

$$\hat{L}_n \equiv \frac{1}{2\pi i}\int_C z^{n+1}\hat{T}(z)dz\,,$$

and similarly $\hat{\bar{L}}_n$.

If there are no operator insertions inside C it can be shrunk to zero for $n \geq -1$, thus

$$\hat{L}_n|0\rangle = 0 \qquad \text{for } n \geq -1\,.$$

On the other hand, if there is an operator ϕ_j inserted at the origin, we see from the OPE (13) that

$$\hat{L}_0|\phi_j\rangle = \Delta_j|\phi_j\rangle\,.$$

If ϕ_j is *primary*, we further see from the OPE that

$$\hat{L}_n|\phi_j\rangle = 0 \qquad \text{for } n \geq 1\,,$$

while for $n \leq -1$ we get states corresponding to descendants of ϕ_j.

4.2 The Virasoro algebra

Now consider $\hat{L}_m\hat{L}_n$ acting on some arbitrary state. In terms of correlation functions this involves the contour integrals

$$\int_{C_2}\frac{dz_2}{2\pi i}z_2^{m+1}\int_{C_1}\frac{dz_1}{2\pi i}z_1^{n+1}T(z_2)T(z_1)\,,$$

where C_2 lies *outside* C_1, because of the **R**-ordering. If instead we consider the operators in the reverse order, the contours will be reversed. However we can then always distort them to restore them to their original positions, as long as we leave a piece of the z_2 contour wrapped around z_1. This can be evaluated using the OPE (15) of T with itself:

$$\int_C\frac{dz_1}{2\pi i}z_1^{n+1}\oint\frac{dz_2}{2\pi i}z_2^{m+1}\left(\frac{2}{(z_2-z_1)^2}T(z_1)+\frac{1}{z_2-z_1}\partial_{z_1}T(z_1)+\frac{c/2}{)z_2-z_1)^4}\right)$$
$$=\int_C\frac{dz_1}{2\pi i}z_1^{n+1}\left(2(m+1)z_1^mT(z_1)+z_1^{m+1}\partial_{z_1}T(z_1)+\tfrac{c}{12}m(m^2-1)z_1^{m-2}\right)\,.$$

The integrals can then be re-expressed in terms of the $\hat{L}_n$. This gives the *Virasoro algebra*:

$$[\hat{L}_m,\hat{L}_n] = (m-n)\hat{L}_{m+n} + \tfrac{c}{12}m(m^2-1)\delta_{m+n,0}\,, \tag{17}$$

with an identical algebra satisfied by the $\hat{\bar{L}}_n$. It should be stressed that (17) is completely equivalent to the OPE (15), but of course algebraic methods are often more efficient in understanding the structure of QFT. The first term on the right hand side could have been foreseen if we think of $\hat{L}_n$ as being the generator of infinitesimal conformal transformations with $\alpha(z) \propto z^{n+1}$. Acting on functions of z this can therefore be represented by $\hat{\ell}_n = z^{n+1}\partial_z$, and it is easy to check that these satisfy (17) without the central term (called the Witt algebra.) However the $\hat{L}_n$ act on states of the CFT rather than functions, which allows for the existence of the second term, the central term. The form of this, apart from the undetermined constant c, is in fact dictated by consistency with the Jacobi identity. Note that there is a closed subalgebra generated by $(\hat{L}_1, \hat{L}_0, \hat{L}_{-1})$, which corresponds to special conformal transformations.

One consequence of (17) is

$$[\hat{L}_0, \hat{L}_{-n}] = n\hat{L}_{-n}\,,$$

so that $\hat{L}_0(\hat{L}_{-n}|\phi_j\rangle) = (\Delta_j + n)\hat{L}_{-n}|\phi_j\rangle$, which means that the $\hat{L}_n$ with $n < 0$ act as raising operators for the weight, or scaling dimension, Δ, and those with $n > 0$ act as lowering operators. The state $|\phi_j\rangle$ corresponding to a primary operator is annihilated by all the lowering operators. It is therefore a *lowest weight state.*[11] By acting with all possible raising operators we build up a *lowest weight representation* (called a Verma module) of the Virasoro algebra:

$$\begin{array}{l}
\vdots \\
\hat{L}_{-3}|\phi_j\rangle, \hat{L}_{-2}\hat{L}_{-1}|\phi_j\rangle, \hat{L}_{-1}^3|\phi_j\rangle; \\
\hat{L}_{-2}|\phi_j\rangle, \hat{L}_{-1}^2|\phi_j\rangle; \\
\hat{L}_{-1}|\phi_j\rangle; \\
|\phi_j\rangle\,.
\end{array}$$

4.3 Null states and the Kac formula

One of the most important issues in CFT is whether, for a given c and Δ_j, this representation is unitary, and whether it is reducible (more generally, decomposable). It turns out that these two are linked, as we shall see later. Decomposability implies the existence of null states in the Verma module, that is, some linear combination of states at a given level is itself a lowest state. The simplest example occurs at level 2, if

$$\hat{L}_n\left(\hat{L}_{-2}|\phi_j\rangle - (1/g)\hat{L}_{-1}^2|\phi_j\rangle\right) = 0\,,$$

for $n > 0$ (the notation with g is chosen to correspond to the Coulomb gas later on.) By taking $n = 1$ and $n = 2$ and using the Virasoro algebra and the fact that $|\phi_j\rangle$ is a lowest

[11] In the literature this is often called, confusingly, a highest weight state.

weight state, we get

$$\Delta_j = \frac{3g-2}{4}\,, \qquad c = \frac{(3g-2)(3-2g)}{g}\,.$$

This is special case $(r,s)=(2,1)$ of the *Kac formula*: with c parametrised as above, if[12]

$$\Delta_j = \Delta_{r,s}(g) \equiv \frac{(rg-s)^2-(g-1)^2}{4g}\,, \tag{18}$$

then $|\phi_j\rangle$ has a null state at level $r\cdot s$. We will not prove this here, but later will indicate how it is derived from Coulomb gas methods.

Removing all the null states from a Verma module gives an irreducible representation of the Virasoro algebra. Null states (and all their descendants) can consistently be set to zero in a given CFT. (This is no guarantee that they are in fact absent, however.)

One important consequence of the null state is that the correlation functions of $\phi_j(z,\bar z)$ satisfy linear differential equations in z (or $\bar z$) of order rs. The case $rs=2$ will be discussed as an example in the last lecture. This allows us in principle to calculate all the four-point functions and hence the OPE coefficients.

4.4 Fusion rules

Let us consider the 3-point function

$$\langle \phi_{2,1}(z_1)\phi_{r,s}(z_2)\phi_\Delta(z_3)\rangle\,,$$

where the first two fields sit at the indicated places in the Kac table, but the third is a general primary scaling field of dimension Δ. The form of this is given by (9). If we insert $\int_C (z-z_1)^{-1}T(z)dz$ into this correlation function, where C surrounds z_1 but not the other two points, this projects out $L_{-2}\phi_{2,1}\propto\partial_{z_1}^2\phi_{2,1}$. On the other hand, the full expression is given by the Ward identity (14). After some algebra, we find that this is consistent only if

$$\Delta = \Delta_{r\pm1,s}\,,$$

otherwise the 3-point function, and hence the OPE coefficient of ϕ_Δ in $\phi_{2,1}\cdot\phi_{r,s}$, vanishes.

This is an example of the *fusion rules* in action. It shows that Kac operators compose very much like irreducible representations of SU(2), with the r label playing the role of spin $\frac12(r-1)$. The s-labels compose in the same way. More generally the *fusion rule coefficients* N_{ij}^k tell us not only which OPEs can vanish, but which ones actually do appear in a particular CFT.[13] In this simplest case we have (suppressing the s-indices for clarity)

$$N_{rr'}^{r''} = \delta_{r'',r+r'-1} + \delta_{r'',r+r'-3} + \cdots + \delta_{r'',|r-r'|+1}\,.$$

[12] Note that g and $1/g$ given the same value of c, and that $\Delta_{r,s}(g)=\Delta_{s,r}(1/g)$. This has led to endless confusion in the literature.

[13] They can actually take values ≥ 2 if there are distinct primary fields with the same dimension.

A very important thing happens if g is rational $= p/p'$. Then we can write the Kac formula as

$$\Delta_{r,s} = \frac{(rp - sp')^2 - (p - p')^2}{4pp'} \,,$$

and we see that $\Delta_{r,s} = \Delta_{p'-r,p-s}$, that is, the same primary field sits at two different places in the rectangle $1 \leq r \leq p' - 1, 1 \leq s \leq p - 1$, called the Kac table. If we now apply the fusion rules to these fields we see that we get consistency between the different constraints only if the fusion algebra is *truncated*, that fields within the rectangle do not couple to those outside.

This shows that, at least at the level of fusion, we can have CFTs with a *finite number* of primary fields. These are called the *minimal models*[14]. However, it can be shown that, among these, the only ones admitting *unitary* representations of the Virasoro algebra, that is for which $\langle\psi|\psi\rangle \geq 0$ for all states $|\psi\rangle$ in the representation, are those with $|p - p'| = 1$ and $p, p' \geq 3$. Moreover these are the *only* unitary CFTs with $c < 1$. The physical significance of unitarity will be mentioned shortly.

5 CFT on the cylinder and torus

5.1 CFT on the cylinder

One of the most important conformal mappings is the logarithmic transformation $w = (L/2\pi)\log z$, which maps the z-plane (punctured at the origin) to an infinitely long cylinder of circumference L (equivalently, a strip with periodic boundary conditions.) It is useful to write $w = t + iu$, and to think of the coordinate t running along the cylinder as imaginary time, and u as space. CFT on the cylinder then corresponds to euclidean QFT on a circle.

The relation between the stress tensor on the cylinder and in the plane is given by (16):

$$T(w)_{\text{cyl}} = (dz/dw)^2 T(z) + \tfrac{c}{12}\{z, w\} = (2\pi/L)^2 \left(z^2 T(z)_{\text{plane}} - \tfrac{c}{24}\right) \,,$$

where the last term comes from the Schwarzian derivative.

The hamiltonian $\hat{H}$ on the cylinder, which generates infinitesimal translations in t, can be written in the usual way as an integral over the time-time component of the stress tensor

$$\hat{H} = \frac{1}{2\pi}\int_0^L \hat{T}_{tt}(u)du = \frac{1}{2\pi}\int_0^L (\hat{T}(u) + \hat{\overline{T}}(u))du \,,$$

which corresponds in the plane to

$$\hat{H} = \frac{2\pi}{L}\left(\hat{L}_0 + \hat{\overline{L}}_0\right) - \frac{\pi c}{6L} \,. \tag{19}$$

[14] The minimal models are examples of *rational* CFTs: those which have only a finite number of fields which are primary with respect to some algebra, more generally an extended one containing Virasoro as a sub-algebra.

Similarly the total momentum $\hat{P}$, which generates infinitesimal translations in u, is the integral of the T_{tu} component of the stress tensor, which can be written as $(2\pi/L)(\hat{L}_0 - \hat{\overline{L}}_0)$.

Eq. (19), although elementary, is one of the most important results of CFT in two dimensions. It relates the dimensions of all the scaling fields in the theory (which, recall, are the eigenvalues of $\hat{L}_0$ and $\hat{\overline{L}}_0$) to the spectra of $\hat{H}$ and $\hat{P}$ on the cylinder. If we have a lattice model on the cylinder whose scaling limit is described by a given CFT, we can therefore read off the scaling dimensions, up to finite-size corrections in (a/L), by diagonalising the transfer matrix $\hat{t} \approx 1 - a\hat{H}$. This can be done either numerically for small values of L, or, for integrable models, by solving the Bethe ansatz equations.

In particular, we see that the lowest eigenvalue of $\hat{H}$ (corresponding to the largest eigenvalue of the transfer matrix) is

$$E_0 = -\frac{\pi c}{6L} + \frac{2\pi}{L}(\Delta_0 + \overline{\Delta}_0)\,,$$

where $(\Delta_0, \overline{\Delta}_0)$ are the lowest possible scaling dimensions. In many CFTs, and all unitary ones, this corresponds to the identity field, so that $\Delta_0 = \overline{\Delta}_0 = 0$. This shows that c can be measured from finite-size behaviour of the ground state energy.

E_0 also gives the leading term in the partition function $Z = \mathrm{Tr}\, e^{-\ell\hat{H}}$ on a finite cylinder (a torus) of length $\ell \gg L$. Equivalently, the free energy (in units of k_BT) is

$$F = -\log Z \sim -\frac{\pi c\ell}{6L}\,.$$

In this equation F represents the scaling limit of free energy of a 2d *classical* lattice model[15]. However we can equally well think of t as being space and u imaginary time, in which case periodic boundary conditions imply finite inverse *temperature* $\beta = 1/k_BT = L$ in a 1d *quantum* field theory. For such a theory we then predict that

$$F \sim -\frac{\pi c\ell k_BT}{6}\,,$$

or, equivalently, that the low-temperature specific heat, at a quantum critical point described by a CFT (generally, with a linear dispersion relation $\omega \sim |q|$), has the form

$$C_v \sim \frac{\pi c k_B^2 T}{3}\,.$$

Note that the Virasoro generators can be written in terms of the stress tensor on the cylinder as

$$\hat{L}_n = \frac{L}{2\pi}\int_0^L e^{inu}\hat{T}(u,0)du\,.$$

[15] This treatment overlooks UV divergent terms in F of order $(\ell L/a^2)$, which are implicitly set to zero by the regularisation of the stress tensor.

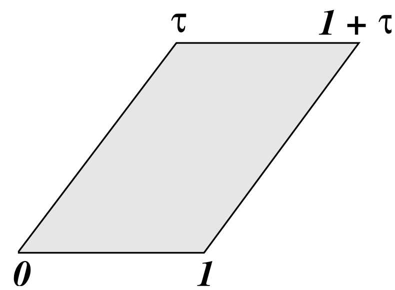

Figure 6: A general torus is obtained by identifying opposite sides of a parallelogram.

In a unitary theory, $\hat{T}$ is self-adjoint, and hence $\hat{L}_n^\dagger = \hat{L}_{-n}$. Unitarity of the QFT corresponds to *reflection positivity* of correlation functions: in general

$$\langle \phi_1(u_1, t_1)\phi_2(u_2, t_2) \ldots \phi_1(u_1, -t_1)\phi_2(u_2, -t_2)\rangle$$

is positive if the transfer matrix $\hat{T}$ can be made self-adjoint, which is generally true if the Boltzmann weights are positive. Note however, that a given lattice model (e.g. the Ising model) contains fields which are the scaling limit of lattice quantities in which the transfer matrix can be locally expressed (e.g. the local magnetisation and energy density) and for which one would expect reflection positivity to hold, and other scaling fields (e.g. the probability that a given edge lies on a cluster boundary) which are not locally expressible. Within such a CFT, then, one would expect to find a unitary *sector* – in fact in the Ising model this corresponds to the $p = 3, p' = 4$ minimal model – but also possible non-unitary sectors in addition.

5.2 Modular invariance on the torus

We have seen that unitarity (for $c < 1$), and, more generally, rationality, fix which scaling fields may appear in a given CFT, but they don't fix which ones actually appear. This is answered by considering another physical requirement: that of modular invariance on the torus.

We can make a general torus by imposing periodic boundary conditions on a parallelogram, whose vertices lie in the complex plane. Scale invariance allows us to fix the length of one of the sides to be unity: thus we can choose the vertices to be at $(0, 1, 1 + \tau, \tau)$, where τ is a complex number with positive imaginary part. In terms of the conventions of the previous section, we start with a finite cylinder of circumference $L = 1$ and length $\operatorname{Im}\tau$, twist one end by an amount $\operatorname{Re}\tau$, and sew the ends together – see Fig. 6. An important feature of this parametrisation of the torus is that it is not unique: the transformations $T : \tau \to \tau + 1$ and $S : \tau \to -1/\tau$ give the same torus (Fig. 7). Note that S interchanges space u and imaginary time t in the QFT. S and T generate an infinite discrete group of transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

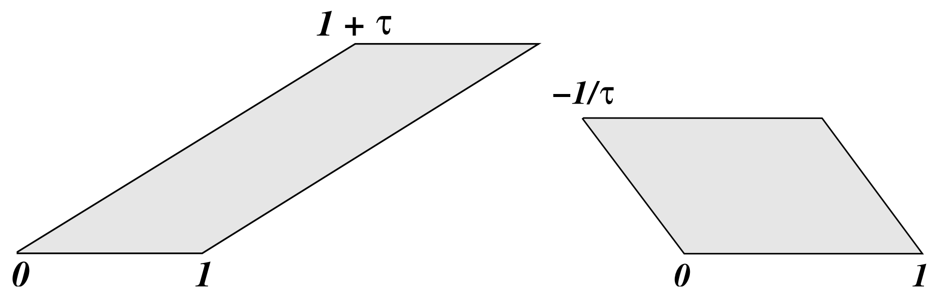

Figure 7: Two ways of viewing the same torus, corresponding to the modular transformations T and S.

with (a,b,c,d) all integers and $ad-bc=1$. This is called $\mathrm{SL}(2,\mathbf{Z})$ or the *modular group*. Note that $S^2=1$ and $(ST)^3=1$.

Consider the scaling limit of the partition function Z of a lattice model on this torus. Apart from the divergent term in $\log Z$, proportional to the area divided by a^2, which in CFT is set to zero by regularisation, the rest should depend only on the aspect ratio of the torus and thus be *modular invariant*. This would be an empty statement were it not that Z can be expressed in terms of the spectrum of scaling dimensions of the CFT in a manner which is not, manifestly, modular invariant.

Recall that the generators of infinitesimal translations along and around the cylinder can be written as

$$\hat{H} = 2\pi(\hat{L}_0 + \hat{\bar{L}}_0) - \tfrac{\pi c}{6} \qquad \hat{P} = 2\pi(\hat{L}_0 - \hat{\bar{L}}_0)\,.$$

The action of twisting the cylinder corresponds to a finite translation around its circumference, and sewing the ends together corresponds to taking the trace. Thus

$$\begin{aligned} Z &= \mathrm{Tr}\, e^{-(\mathrm{Im}\,\tau)\hat{H} + i(\mathrm{Re}\tau)\hat{P}} \\ &= e^{\pi c \mathrm{Im}\,\tau/6}\, \mathrm{Tr}\, e^{2\pi i\tau \hat{L}_0}\, e^{-2\pi i\bar{\tau}\hat{\bar{L}}_0} \\ &= (q\bar{q})^{-c/24}\, \mathrm{Tr}\, q^{\hat{L}_0}\, \bar{q}^{\hat{\bar{L}}_0}\,, \end{aligned}$$

where in the last line we have defined $q \equiv e^{2\pi i\tau}$.

The trace means that we sum over all eigenvalues of $\hat{L}_0$ and $\hat{\bar{L}}_0$, that is all scaling fields of the CFT. We know that these can be organised into irreducible representations of the Virasoro algebra, and therefore have the form $(\Delta + N, \overline{\Delta} + \bar{N})$, where Δ and $\overline{\Delta}$ correspond to primary fields and $(N,\bar{N})$ are non-negative integers labelling the levels of the descendants. Thus we can write

$$Z = \sum_{\Delta,\overline{\Delta}} n_{\Delta,\overline{\Delta}}\, \chi_\Delta(q)\chi_{\overline{\Delta}}(\bar{q})\,,$$

where $n_{\Delta,\overline{\Delta}}$ is the number of primary fields with lowest weights $(\Delta,\overline{\Delta})$, and

$$\chi_\Delta(q) = q^{-c/24+\Delta} \sum_{N=0}^{\infty} d_\Delta(N) q^N\,,$$

where $d_\Delta(N)$ is the degeneracy of the representation at level N. It is purely a property of the representation, not the particular CFT, and therefore so is $\chi_\Delta(q)$. This is called the *character* of the representation.

The requirement of modular invariance of Z under T is rather trivial: it says that all fields must have integer conformal spin[16]. However the invariance under S is highly non-trivial: it states that Z, which is a power series in q and $\bar q$, can equally well be expressed as an identical power series in $\tilde q \equiv e^{-2\pi i/\tau}$ and $\bar{\tilde q}$.

We can get some idea of the power of this requirement by considering the limit $q \to 1$, $\tilde q \to 0$, with q real. Suppose the density of scaling fields (including descendants) with dimension $x = \Delta + \overline{\Delta}$ in the range $(x, x+\delta x)$ (where $1 \gg \delta x \gg x$) is $\rho(x)\delta x$. Then, in this limit, when $q = 1-\epsilon$, $\epsilon \ll 1$,

$$Z \sim \int^\infty \rho(x) e^{x\log q} dx \sim \int^\infty \rho(x) e^{-\epsilon x} dx\,.$$

On the other hand, we know that as $\tilde q \to 0$, $Z \sim \tilde q^{-c/12+x_0} \sim e^{(2\pi)^2(c/12-x_0)/\epsilon}$ where $x_0 \leq 0$ is the lowest scaling dimension (usually 0). Taking the inverse Laplace transform,

$$\rho(x) \sim \int e^{\epsilon x + (2\pi)^2(c/12-x_0)/\epsilon} \frac{d\epsilon}{2\pi i}\,.$$

Using the method of steepest descents we then see that, as $x \to \infty$,

$$\rho(x) \sim \exp\left(4\pi(c/12-x_0)^{1/2}\, x^{1/2}\right)\,,$$

times a (calculable) prefactor. This relation is of importance in understanding black hole entropy in string theory.

5.2.1 Modular invariance for the minimal models

Let us apply this to the minimal models, where there is a finite number of primary fields, labelled by (r,s). We need the characters $\chi_{r,s}(q)$. If there were no null states, the degeneracy at level N would be the number of states of the form $\ldots \hat L_{-3}^{n_3} \hat L_{-2}^{n_2} \hat L_{-1}^{n_1} |\phi\rangle$ with $\sum_j j n_j = N$. This is just the number of distinct partitions of N into positive integers, and the generating function is $\prod_{k=1}^\infty (1-q^k)^{-1}$.

However, we know that the representation has a null state at level rs, and this, and all its descendants, should be subtracted off. Thus

$$\chi_{rs}(q) = q^{-c/24} \prod_{k=1}^\infty (1-q^k)^{-1} (1 - q^{rs} + \cdots)\,.$$

But, as can be seen from the Kac formula (18), $\Delta_{r,s} + rs = \Delta_{p'+r,p-s}$, and therefore the null state at level rs has itself null states in its Verma module, which should not have been

[16]It is interesting to impose other kinds of boundary conditions, e.g. antiperiodic, on the torus, when other values of the spin can occur.

subtracted off. Thus we must add these back in. However, it is slightly more complicated than this, because for a minimal model each primary field sits in two places in the Kac rectangle, $\Delta_{r,s} = \Delta_{p'-r,p-s}$. Therefore this primary field also has a null state at level $(p'-r)(p-s)$, and this has the dimension $\Delta_{2p'-r,s} = \Delta_{r,2p-s}$ and should therefore also be added back in if it has not be included already. A full analysis requires understanding how the various submodules sit inside each other, but fortunately the final result has a nice form

$$\chi_{rs}(q) = q^{-c/24} \prod_{k=1}^{\infty} (1-q^k)^{-1} \left(K_{r,s}(q) - K_{r,-s}(q) \right) , \tag{20}$$

where

$$K_{r,s}(q) = \sum_{n=-\infty}^{\infty} q^{(2npp'+rp-sp')^2/4pp'} . \tag{21}$$

The partition function can then be written as finite sum

$$Z = \sum_{r,s;\bar{r},\bar{s}} n_{r,s;\bar{r},\bar{s}} \chi_{rs}(q) \chi_{\bar{r}\bar{s}}(\bar{q}) = \sum_{r,s;\bar{r},\bar{s}} n_{r,s;\bar{r},\bar{s}} \chi_{rs}(\tilde{q}) \chi_{\bar{r}\bar{s}}(\bar{\tilde{q}}) .$$

The reason this can happen is that the characters themselves transform linearly under $S : q \to \tilde{q}$, as can be seen (after quite a bit of algebra, by applying the Poisson sum formula to (21) and Euler's identities to the infinite product):

$$\chi_{rs}(\tilde{q}) = \sum_{r',s'} S_{rs}^{r's'} \chi_{r's'}(q) ,$$

where $\mathbf{S}$ is a matrix whose rows and columns are labelled by (rs) and $(r's')$. Another way to state this is that the characters form a representation of the modular group. The form of $\mathbf{S}$ is not that important, but we give it anyway:

$$S_{rs}^{r's'} = \left(\frac{8}{pp'} \right)^{1/2} (-1)^{1+rs'+sr'} \sin \frac{\pi p r r'}{p'} \sin \frac{\pi p' s s'}{p} .$$

The important properties of $\mathbf{S}$ are that it is real and symmetric and $\mathbf{S}^2 = 1$.

This immediately implies that the *diagonal* combination, with $n_{r,s;\bar{r},\bar{s}} = \delta_{r\bar{r}} \delta_{s\bar{s}}$, is modular invariant:

$$\sum_{r,s} \chi_{rs}(\tilde{q}) \chi_{rs}(\bar{\tilde{q}}) = \sum_{r,s} \sum_{r',s'} \sum_{r'',s''} S_{rs}^{r's'} S_{rs}^{r''s''} \chi_{r's'}(q) \chi_{r''s''}(\bar{q}) = \sum_{r,s} \chi_{rs}(q) \chi_{rs}(\bar{q}) ,$$

where we have used $\mathbf{S}\mathbf{S}^T = 1$. This gives the diagonal series of CFTs, in which all possible scalar primary fields in the Kac rectangle occur just once. These are known as the A_n series.

It is possible to find other modular invariants by exploiting symmetries of $\mathbf{S}$. For example, if $p'/2$ is odd, the space spanned by $\chi_{rs} + \chi_{p'-r,s}$, with r odd, is an invariant subspace of **S**, and is multiplied only by a pure phase under T. Hence the diagonal combination within this subspace

$$Z = \sum_{r \text{ odd}, s} |\chi_{r,s} + \chi_{p'-r,s}|^2$$

is modular invariant. Similar invariants can be constructed if $p/2$ is odd. This gives the D_n series. Note that in this case some fields appear with degeneracy 2. Apart from these two infinite series, there are three special values of p and p' (12,18,30) denoted by $\mathrm{E}_{6,7,8}$. The reason for this classification will become more obvious in the next section when we construct explicit lattice models which have these CFTs as their scaling limits.[17]

6 Height models, loop models and Coulomb gas methods

6.1 Height models and loop models

Although the ADE classification of minimal CFTs with $c < 1$ through modular invariance was a great step forwards, one can ask whether there are in fact lattice models which have these CFTs as their scaling limit. The answer is yes – in the form of the ADE lattice models. These can be analysed non-rigorously by so-called Coulomb gas methods.

For simplicity we shall describe only the so-called dilute models, defined on a triangular lattice.[18] At each site r of the lattice is defined a 'height' $h(r)$ which takes values on the nodes of some connected graph $\mathcal{G}$. An example is the linear graph called A_m shown in Fig. 8, in which $h(r)$ can be thought of as an integer between 1 and m. There is a

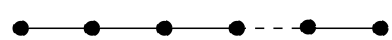

Figure 8: The graph A_m, with m nodes.

restriction in these models that the heights on neighbouring sites of the triangular lattice must either be the same, or be adjacent on $\mathcal{G}$. It is then easy to see that around a given triangle either all three heights are the same (which carries relative weight 1), or two of them are the same and the other is adjacent on $\mathcal{G}$.[19] In this case, if the heights are (h, h', h'), the weight is $x(S_h/S_{h'})^{1/6}$ where S_h is a function of the height h, to be made explicit later, and x is a positive temperature-like parameter.[20] (A simple example is A_2,

[17]This classification also arises in the finite subgroups of SU(2), of simply-laced Lie algebras, and in catastrophe theory.

[18]Similar models can be defined on the square lattice. They give rise to critical loop models in the dense phase.

[19]Apart from the pathological case when $\mathcal{G}$ itself has a 3-cycle, in which case we can enforce the restriction by hand.

[20]It is sometimes useful, e.g. for implementing a transfer matrix, to redistribute these weights around the loops.

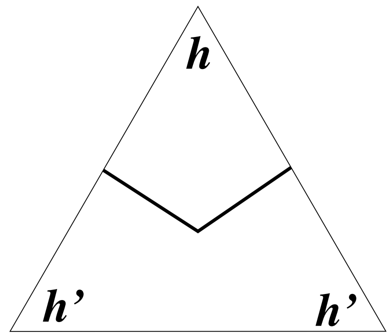

Figure 9: If the heights on the vertices are not all equal, we denote this by a segment of a curve on the dual lattice, as shown.

corresponding to the Ising model on the triangular lattice.)

The weight for a given configuration of the whole lattice is the product of the weights for each elementary triangle. Note that this model is local and has positive weights if S_h is a positive function of h. Its scaling limit at the critical point should correspond to a unitary CFT.

The height model can be mapped to a loop model as follows: every time the heights in a given triangle are not all equal, we draw a segment of a curve through it, as shown in Fig. 9. These segments all link up, and if we demand that all the heights on the boundary are the same, they form a set of nested, non-intersecting closed loops on the dual honeycomb lattice, separating regions of constant height on the original lattice. Consider a loop for which the heights just inside and outside are h and h' respectively. The loop has convex (outward-pointing) and concave (inward-pointing) corners. Each convex corner carries a factor $(S_h/S_{h'})^{1/6}$, and each concave corner the inverse factor. But each loop has exactly 6 more outward pointing corners than inward pointing ones, so it always carries an overall weight S_h/S'_h, times a factor x raised to the length of the loop. Let us now sum over the heights consistent with a fixed loop configuration, starting with innermost regions. Each sum has the form

$$\sum_{h:|h-h'|=1} (S_h/S_{h'})\,,$$

where $|h-h'|=1$ means that h and h' are adjacent on $\mathcal{G}$. The next stage in the summation will be simple only if this is a constant independent of h'. Thus we assume that the S_h satisfy

$$\sum_{h:|h-h'|=1} S_h = \Lambda S_{h'}\,,$$

that is, S_h is an eigenvector of the *adjacency matrix* of $\mathcal{G}$, with eigenvalue Λ. For A_m, for example, these have the form

$$S_h \propto \sin\frac{\pi k h}{m+1}\,, \tag{22}$$

where $1\leq k\leq m$, corresponding to $\Lambda = 2\cos\left(\pi k/(m+1)\right)$. Note that only the case $k=1$ gives all real positive weights.

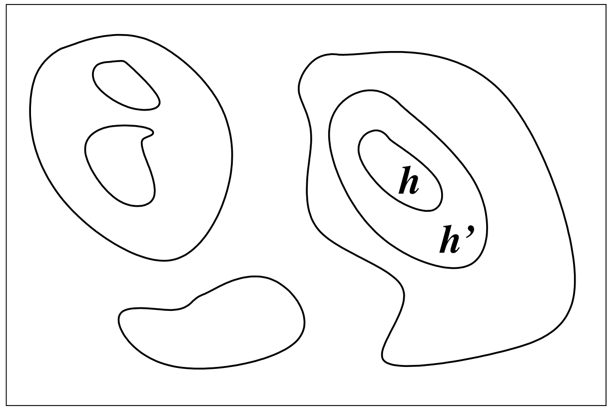

Figure 10: Nested set of loops, separating regions of constant height. We iteratively sum over the heights h, starting with the innermost.

Having chosen the S_h in this way, we can sum out all the heights consistent with a given loop configuration (Fig. 10), starting with the innermost and moving outwards, and thereby express the partition function as a sum over loop configurations:

$$Z = \sum_{\text{loop configs}} \Lambda^{\text{number of loops}} x^{\text{total length}} . \tag{23}$$

When x is small, the heights are nearly all equal (depending on the boundary condition,) and the typical loop length and number is small. At a critical point $x = x_c$ we expect this to diverge. Beyond this, we enter the *dense phase*, which is still critical in the loop sense, even though observables which are local in the original height variables may have a finite correlation length. For example, for $x > x_c$ in the Ising model, the Ising spins are disordered but the the cluster boundaries are the same, in the scaling limit, as those of critical percolation for site percolation on the triangular lattice.

However, we could have obtained the same expression for Z in several different ways. One is by introducing n-component spins $s_a(R)$ with $a = 1, \ldots, n$ on the sites of the dual lattice, and the partition function

$$Z_{O(n)} = \mathrm{Tr} \prod_{RR'} \left(1 + x \sum_{a=1}^{n} s_a(R) s_a(R') \right) ,$$

where the product is over edges of the honeycomb lattice, the trace of an odd power of $s_a(R)$ is zero, and $\mathrm{Tr} s_a(R) s_b(R) = \delta_{ab}$. Expanding in powers of x, and drawing in a curve segment each time the term proportional to x is chosen on a given edge, we get the same set of nested loop configurations, weighted as above with $\Lambda = n$. This is the O(n) model. Note that the final expression makes sense for all real positive values of n, but it can be expressed in terms of weights local in the original spins only for positive integer n. Only in the latter case do we therefore expect that correlations of the O(n) spins will satisfy reflection positivity and therefore correspond to a unitary CFT, even though the description in terms of heights is unitary. This shows how different sectors of the 'same' CFT can describe rather different physics.

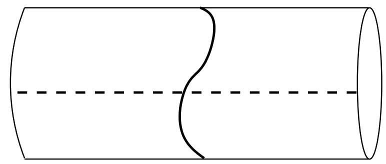

Figure 11: Non-contractible loops on the cylinder can be taken into account by the insertion of a suitable factor along a seam.

6.2 Coulomb gas methods

These loop models can be solved in various ways, for example by realising that their transfer matrix gives a representation of the Temperley-Lieb algebra, but a more powerful if less rigorous method is to use the so-called Coulomb gas approach. Recalling the arguments of the previous section, we see that yet another way of getting to (23) is by starting from a different height model, where now the heights $h(r)$ are defined on the integers (times π, for historical reasons) $\pi\mathbf{Z}$. As long as we choose the correct eigenvalue of the adjacency matrix, this will give the same loop gas weights. That is we take $\mathcal{G}$ to be A_∞. In this case the eigenvectors correspond to plane wave modes propagating along the graph, labelled by a quasi-momentum χ with $|\chi|<1$: $S_h \propto e^{i\chi h}$, whose eigenvalue is $\Lambda = 2\cos(\pi\chi)$. Because these modes are chiral, we have to orient the loops to distinguish between χ and $-\chi$. Each oriented loop then gets weighted with a factor $e^{\pm i\pi\chi/6}$ at each vertex of the honeycomb lattice it goes through, depending on whether it turns to the left or the right.

This version of the model, where the heights are unbounded, is much easier to analyse, at least non-rigorously. In particular, we might expect that in the scaling limit, after coarse-graining, we can treat $h(r)$ as taking all real values, and write down an effective field theory. This should have the property that it is local, invariant under $h(r) \to h(r)+$ constant, and with no terms irrelevant under the RG (that is, entering the effective action with positive powers of a.) The only possibility is a free gaussian field theory, with action

$$S = (g_0/4\pi)\int(\nabla h)^2 d^2r\,.$$

However, this cannot be the full answer, because we know this corresponds to a CFT with $c=1$. The resolution of this is most easily understood by considering the theory on a long cylinder of length ℓ and circumference $L \ll \ell$. Non-contractible loops which go around the cylinder have the same number of inside and outside corners, so they are incorrectly counted. This can be corrected by inserting a factor $\prod_t e^{i\chi h(t,0)}e^{-i\chi h(t+1,0)}$, which counts each loop passing between $(t,0)$ and $(t+1,0)$ with just the right factors $e^{\pm i\pi\chi}$. These factors accumulate to $e^{i\chi h(-\ell/2,0)}e^{-i\chi h(\ell/2,0)}$, corresponding to charges $\pm\chi$ at the ends of the cylinder. This means that the partition function is

$$Z \sim Z_{c=1}\langle e^{i\chi h(\ell/2)}e^{-i\chi h(-\ell/2)}\rangle\,.$$

But we know (Sec. 3.1) that this correlation function decays like r^{-2x_χ} in the plane, where $x_q = q^2/2g_0$, and therefore on the cylinder

$$Z \sim e^{\pi\ell/6L}\, e^{-2\pi(\chi^2/2g_0)\ell/L} ,$$

from which we see that the central charge is actually

$$c = 1 - \frac{6\chi^2}{g_0} .$$

However, we haven't yet determined g_0. This is fixed by the requirement that the *screening fields* $e^{\pm i2h(r)}$, which come from the the fact that originally $h(r) \in \pi\mathbf{Z}$, should be marginal, that is they do not affect the scaling behaviour so that we can add them to the action with impunity. This requires that they have scaling dimension $x_2 = 2$. However, now x_q should be calculated from the cylinder with the charges $e^{\pm i\chi h(\pm\ell/2)}$ at the ends:

$$x_q = \frac{(q \pm \chi)^2}{g_0} - \frac{\chi^2}{g_0} = \frac{q^2 \pm 2\chi q}{g_0} . \tag{24}$$

Setting $x_2 = 2$ we then find $g_0 = 1 \pm \chi$ and therefore

$$c = 1 - \frac{6(g_0 - 1)^2}{g_0} .$$

6.3 Identification with minimal models

The partition function for the height models (at least on cylinder) depends only on the eigenvalue Λ of the adjacency matrix and hence the Coulomb gas should work equally well for the models on A_m if we set $\chi = k/(m+1)$. The corresponding central charge is then

$$c = 1 - \frac{6k^2}{(m+1)(m+1 \pm k)} .$$

If we compare this with the formula for the minimal models

$$c = 1 - \frac{6(p - p')^2}{pp'} ,$$

we are tempted to identify $k = p - p'$ and $m + 1 = p'$. This implies $g_0 = p/p'$, which can therefore be identified with the parameter g introduced in the Kac formula.[21] Moreover, if we compute the scaling dimensions of local fields $\phi_r(R) = \cos\left((r-1)kh(R)/(m+1)\right)$ using (24) we find perfect agreement with the leading diagonal $\Delta_{r,r}$ of the Kac table.[22] We therefore have strong circumstantial evidence that the scaling limit of the dilute $A_{p'-1}$

[21] The other solution corresponds to interchanging p' and p.
[22] These are the relevant fields in the RG sense.

models (choosing the eigenvalue $\Lambda = 2\cos(\pi(p-p')/p'))$ is the (p,p') minimal model with $p > p'$. Note that only if $k=1$, that is $p = p'+1$, are these CFTs unitary, and this is precisely the case where the weights of the lattice model are real and positive.

For other graphs $\mathcal{G}$ we can try to make a similar identification. However, this is going to work only if the maximal eigenvalue of the adjacency matrix of $\mathcal{G}$ is strictly less than 2. A famous classification then shows that this restricts $\mathcal{G}$ to be either of the form A_m, D_m or one of three exceptional cases $E_{6,7,8}$.[23] These other graphs also have eigenvalues of the same form (refeigen) as A_m, but with $m+1$ now being the Coxeter number, and the allowed integers k being only a subset of those appearing in the A-series. These correspond to the Kac labels of the allowed scalar operators which appear in the appropriate modular invariant partition function.

7 Boundary conformal field theory

7.1 Conformal boundary conditions and Ward identities

So far we haven't considered what happens at the boundary of the domain $\mathcal{D}$. This is a subject with several important applications, for example to quantum impurity problems (see the lectures by Affleck) and to D-branes in string theory.

In any field theory in a domain with a boundary, one needs to consider how to impose a set of consistent boundary conditions. Since CFT is formulated independently of a particular set of fundamental fields and a lagrangian, this must be done in a more general manner. A natural requirement is that the off-diagonal component $T_{\parallel\perp}$ of the stress tensor parallel/perpendicular to the boundary should vanish. This is called the conformal boundary condition. If the boundary is parallel to the time axis, it implies that there is no momentum flow across the boundary. Moreover, it can be argued that, under the RG, any uniform boundary condition will flow into a conformally invariant one. For a given bulk CFT, however, there may be many possible distinct such boundary conditions, and it is one task of BCFT to classify these.

To begin with, take the domain to be the upper half plane, so that the boundary is the real axis. The conformal boundary condition then implies that $T(z) = \overline{T}(\bar z)$ when z is on the real axis. This has the immediate consequence that correlators of $\overline{T}$ are those of T analytically continued into the lower half plane. The conformal Ward identity now reads

$$\langle T(z)\prod_j \phi_j(z_j,\bar z_j)\rangle \quad = \quad \sum_j \left(\frac{\Delta_j}{(z-z_j)^2} + \frac{1}{z-z_j}\partial_{z_j}\right.$$

[23] $\Lambda = 2$ corresponds to the extended diagrams $\hat{A}_m$, etc., which give interesting rational CFTs with $c=1$. However models based on graphs with $\Lambda > 2$ probably have a different kind of a transition at which the mean loop length remains finite.

$$+\frac{\overline{\Delta}_j}{(z-\bar{z}_j)^2}+\frac{1}{z-\bar{z}_j}\partial_{\bar{z}_j}\Big)\,\langle\prod_j\phi_j(z_j,\bar{z}_j)\rangle\,. \tag{25}$$

In radial quantisation, in order that the Hilbert spaces defined on different hypersurfaces be equivalent, one must now choose semicircles centered on some point on the boundary, conventionally the origin. The dilatation operator is now

$$\hat{D}=\frac{1}{2\pi i}\int_S z\,\hat{T}(z)dz-\frac{1}{2\pi i}\int_S \bar{z}\,\hat{\overline{T}}(\bar{z})d\bar{z}\,, \tag{26}$$

where S is a semicircle. Using the conformal boundary condition, this can also be written as

$$\hat{D}=\hat{L}_0=\frac{1}{2\pi i}\int_C z\,\hat{T}(z)dz\,, \tag{27}$$

where C is a complete circle around the origin.

Note that there is now only one Virasoro algebra. This is related to the fact that conformal mappings which preserve the real axis correspond to real analytic functions. The eigenstates of $\hat{L}_0$ correspond to *boundary operators* $\hat{\phi}_j(0)$ acting on the vacuum state $|0\rangle$. It is well known that in a renormalizable QFT fields at the boundary require a different renormalization from those in the bulk, and this will in general lead to a different set of conformal weights. It is one of the tasks of BCFT to determine these, for a given allowed boundary condition.

However, there is one feature unique to boundary CFT in two dimensions. Radial quantization also makes sense, leading to the same form (27) for the dilation operator, if the boundary conditions on the negative and positive real axes are *different*. As far as the structure of BCFT goes, correlation functions with this mixed boundary condition behave as though a local scaling field were inserted at the origin. This has led to the term 'boundary condition changing (bcc) operator'.

7.2 CFT on the annulus and classification of boundary states

Just as consideration of the partition function on the torus illuminates the bulk operator content $n_{\Delta,\overline{\Delta}}$, it turns out that consistency on the annulus helps classify both the allowed boundary conditions, and the boundary operator content. To this end, consider a CFT in an annulus formed of a rectangle of unit width and height δ, with the top and bottom edges identified (see Fig. 12). The boundary conditions on the left and right edges, labelled by $a, b, \ldots$, may be different. The partition function with boundary conditions a and b on either edge is denoted by $Z_{ab}(\delta)$.

One way to compute this is by first considering the CFT on an infinitely long strip of unit width. This is conformally related to the upper half plane (with an insertion of boundary condition changing operators at 0 and ∞ if $a\neq b$) by the mapping $z\to(1/\pi)\ln z$. The generator of infinitesimal translations along the strip is

$$\hat{H}_{ab}=\pi\hat{D}-\pi c/24=\pi\hat{L}_0-c/24\,. \tag{28}$$

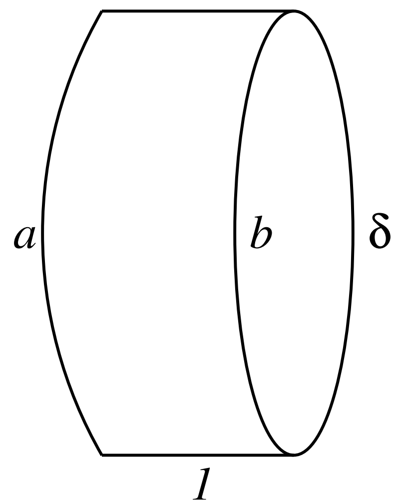

Figure 12: The annulus, with boundary conditions a and b on either boundary.

Thus for the annulus

$$Z_{ab}(\delta) = \mathrm{Tr}\, e^{-\delta\, \hat{H}_{ab}} = \mathrm{Tr}\, q^{\hat{L}_0 - \pi c/24}\,, \tag{29}$$

with $q \equiv e^{-\pi\delta}$. As before, this can be decomposed into characters

$$Z_{ab}(\delta) = \sum_{\Delta} n_{ab}^{\Delta}\, \chi_{\Delta}(q)\,, \tag{30}$$

but note that now the expression is linear. The non-negative integers n_{ab}^{Δ} give the operator content with the boundary conditions (ab): the lowest value of Δ with $n_{ab}^{\Delta} > 0$ gives the conformal weight of the bcc operator, and the others give conformal weights of the other allowed primary fields which may also sit at this point.

On the other hand, the annulus partition function may be viewed, up to an overall rescaling, as the path integral for a CFT on a circle of unit circumference, being propagated for (imaginary) time δ^{-1}. From this point of view, the partition function is no longer a trace, but rather the matrix element of $e^{-\hat{H}/\delta}$ between *boundary states*:

$$Z_{ab}(\delta) = \langle a | e^{-\hat{H}/\delta} | b \rangle\,. \tag{31}$$

Note that $\hat{H}$ is the same hamiltonian that appears on the cylinder, and the boundary states lie in the Hilbert space of states on the circle. They can be decomposed into linear combinations of states in the representation spaces of the two Virasoro algebras, labelled by their lowest weights $(\Delta, \overline{\Delta})$.

How are these boundary states to be characterized? Recalling that on the cylinder $\hat{L}_n \propto \int e^{inu}\hat{T}(u)du$, and $\hat{\overline{L}}_n \propto \int e^{-inu}\hat{\overline{T}}(u)du$, the conformal boundary condition implies that any boundary state $|B\rangle$ lies in the subspace satisfying

$$\hat{L}_n |B\rangle = \hat{\overline{L}}_{-n} |B\rangle\,. \tag{32}$$

This condition can be applied in each subspace. Taking $n = 0$ in (32) constrains $\overline{\Delta} = \Delta$. It can then be shown that the solution of (32) is unique within each subspace and has the following form. The subspace at level N of has dimension $d_\Delta(N)$. Denote an orthonormal basis by $|\Delta, N; j\rangle$, with $1 \le j \le d_\Delta(N)$, and the same basis for the representation space of $\overline{\mathrm{Vir}}$ by $\overline{|\Delta, N; j\rangle}$. The solution to (32) in this subspace is then

$$|\Delta\rangle\rangle \equiv \sum_{N=0}^{\infty} \sum_{j=1}^{d_\Delta(N)} |\Delta, N; j\rangle \otimes \overline{|\Delta, N; j\rangle}\,. \tag{33}$$

These are called Ishibashi states. One way to understand this is to note that (32) implies that

$$\langle B|\hat{L}_n|B\rangle = \langle B|\hat{\overline{L}}_{-n}|B\rangle = \langle B|\hat{\overline{L}}_n|B\rangle\,,$$

where we have used $\hat{\overline{L}}_{-n}^\dagger = \hat{\overline{L}}_n$ and assumed that the matrix elements are all real. This means that acting with the raising operators $\hat{\overline{L}}_n$ on $|B\rangle$ has exactly the same effect as the $\hat{L}_n$, so, starting with $N = 0$ we must build up exactly the same state in the two spaces.

Matrix elements of the translation operator along the cylinder between Ishibashi states are simple:

$$\langle\langle\Delta'|e^{-\hat{H}/\delta}|\Delta\rangle\rangle \tag{34}$$

$$= \sum_{N'=0}^{\infty} \sum_{j'=1}^{d_{\Delta'}(N')} \sum_{N=0}^{\infty} \sum_{j=1}^{d_\Delta(N)} \langle\Delta', N'; j'| \otimes \overline{\langle\Delta', N'; j'|} e^{-(2\pi/\delta)(\hat{L}_0 + \hat{\overline{L}}_0 - c/12)}$$

$$|\Delta, N; j\rangle \otimes \overline{|\Delta, N; j\rangle} \tag{35}$$

$$= \delta_{\Delta'\Delta} \sum_{N=0}^{\infty} \sum_{j=1}^{d_\Delta(N)} e^{-(4\pi/\delta)\big(\Delta + N - (c/24)\big)} = \delta_{\Delta'\Delta}\, \chi_\Delta(e^{-4\pi/\delta})\,. \tag{36}$$

Note that the characters which appear are related to those in (30) by the modular transformation S.

The *physical* boundary states satisfying (30) are linear combinations of these Ishibashi states:

$$|a\rangle = \sum_\Delta \langle\langle\Delta|a\rangle\, |\Delta\rangle\rangle\,. \tag{37}$$

Equating the two different expressions (30,31) for Z_{ab}, and using the modular transformation law for the characters and their linear independence, gives the (equivalent) conditions:

$$n_{ab}^\Delta = \sum_{\Delta'} S_{\Delta'}^\Delta \langle a|\Delta'\rangle\rangle\langle\langle\Delta'|b\rangle\,; \tag{38}$$

$$\langle a|\Delta'\rangle\rangle\langle\langle\Delta'|b\rangle = \sum_\Delta S_\Delta^{\Delta'} n_{ab}^\Delta\,. \tag{39}$$

The requirements that the right hand side of (38) should give a non-negative integer, and that the right hand side of (39) should factorize in a and b, give highly nontrivial constraints on the allowed boundary states and their operator content.

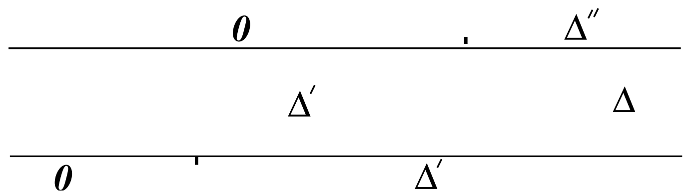

Figure 13: Argument illustrating the fusion rules.

For the diagonal CFTs considered here (and for the nondiagonal minimal models) a complete solution is possible. Since the elements S_0^Δ of $\mathbf{S}$ are all non-negative, one may choose $\langle\langle\Delta|\tilde{0}\rangle = (S_0^\Delta)^{1/2}$. This defines a boundary state

$$|\tilde{0}\rangle \equiv \sum_\Delta (S_0^\Delta)^{1/2} |\Delta\rangle\rangle\,, \tag{40}$$

and a corresponding boundary condition such that $n_{00}^\Delta = \delta_{\Delta 0}$. Then, for each $\Delta' \neq 0$, one may define a boundary state

$$\langle\langle\Delta|\tilde{\Delta}'\rangle \equiv S_{\Delta'}^\Delta / (S_0^\Delta)^{1/2}\,. \tag{41}$$

From (38), this gives $n_{\Delta' 0}^\Delta = \delta_{\Delta'\Delta}$. For each allowed Δ' in the torus partition function, there is therefore a boundary state $|\tilde{\Delta}'\rangle$ satisfying (38,39). However, there is a further requirement:

$$n_{\Delta'\Delta''}^\Delta = \sum_\ell \frac{S_\ell^\Delta S_{\Delta'}^\ell S_{\Delta''}^\ell}{S_0^\ell} \tag{42}$$

should be a non-negative integer. Remarkably, this combination of elements of $\mathbf{S}$ occurs in the *Verlinde formula*, which follows from considering consistency of the CFT on the torus. This states that the right hand side of (42) is equal to the fusion rule coefficient $N_{\Delta'\Delta''}^\Delta$. Since these are non-negative integers, the above ansatz for the boundary states is consistent. The appearance of the fusion rules in this context can be understood by the following argument, illustrated in Fig. 13. Consider a very long strip. At 'time' $t \to -\infty$ the boundary conditions on both sides are those corresponding to $\tilde{0}$, so that only states in the representation 0 propagate. At time t_1 we insert the bcc operator $(0|\Delta')$ on one edge: the states Δ' then propagate. This can be thought of as the fusion of 0 with Δ'. At some much later time we insert the bcc operator $(0|\Delta'')$ on the other edge: by the same argument this should correspond to the fusion of Δ' and Δ'', which gives all states Δ with $N_{\Delta',\Delta''}^\Delta = 1$. But by definition, these are those with $n_{\Delta',\Delta''}^\Delta = 1$.

We conclude that, at least for the diagonal models, there is a bijection between the allowed primary fields in the bulk CFT and the allowed conformally invariant boundary conditions. For the minimal models, with a finite number of such primary fields, this correspondence has been followed through explicitly.

7.2.1 Example

The simplest example is the diagonal $c=\frac{1}{2}$ unitary CFT corresponding to $p=4, p'=3$. The allowed values of the conformal weights are $h=0,\frac{1}{2}\frac{1}{16}$, and

$$\mathbf{S}=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \tag{43}$$

from which one finds the allowed boundary states

$$|\tilde{0}\rangle = \tfrac{1}{\sqrt{2}}|0\rangle\rangle + \tfrac{1}{\sqrt{2}}|\tfrac{1}{2}\rangle\rangle + \tfrac{1}{2^{1/4}}|\tfrac{1}{16}\rangle\rangle\,; \tag{44}$$

$$|\tilde{\tfrac{1}{2}}\rangle = \tfrac{1}{\sqrt{2}}|0\rangle\rangle + \tfrac{1}{\sqrt{2}}|\tfrac{1}{2}\rangle\rangle - \tfrac{1}{2^{1/4}}|\tfrac{1}{16}\rangle\rangle\,; \tag{45}$$

$$|\tilde{\tfrac{1}{16}}\rangle = |0\rangle\rangle - |\tfrac{1}{2}\rangle\rangle\,. \tag{46}$$

The nonzero fusion rule coefficients of this CFT are

$$N_{0,0}^{0} = N_{0,\frac{1}{16}}^{\frac{1}{16}} = N_{0,\frac{1}{2}}^{\frac{1}{2}} = N_{\frac{1}{16},\frac{1}{16}}^{0} = N_{\frac{1}{16},\frac{1}{16}}^{\frac{1}{2}} = N_{\frac{1}{2},\frac{1}{2}}^{0} = N_{\frac{1}{16},\frac{1}{2}}^{\frac{1}{16}} = 1\,.$$

The $c=\frac{1}{2}$ CFT is known to describe the continuum limit of the critical Ising model, in which spins $s=\pm1$ are localized on the sites of a regular lattice. The above boundary conditions may be interpreted as the continuum limit of the lattice boundary conditions $s=1$, $s=-1$ and free (f), respectively. Note there is a symmetry of the fusion rules which means that one could equally well have reversed the first two. This shows, for example, that the for (ff) boundary conditions the states with lowest weights 0 (corresponding to the identity operator) and $\frac{1}{2}$ (corresponding to the the magnetisation operator at the boundary) can propagate. Similarly, the scaling dimension of the $(f|\pm1)$ bcc operator is $\frac{1}{16}$.

7.3 Boundary operators and SLE

Let us now apply the above ideas to the A_m models. There should be a set of conformal boundary states corresponding to the entries of first row $(r,1)$ of the Kac table, with $1\le r\le m$. It is an educated guess (confirmed by exact calculations) that these in fact correspond to lattice boundary conditions where the heights on the boundary are fixed to be at a particular node r of the A_m graph. What about the boundary condition changing operators? These are given by the fusion rules. In particular, since (suppressing the index $s=1$)

$$N_{r,2}^{r'} = \delta_{|r-r'|,1}\,,$$

we see that the bcc operator between r and $r\pm1$, corresponding to a single cluster boundary intersecting the boundary of the domain, must be a $(2,1)$ operator in the Kac table.[24] This makes complete sense: if we want to go from r_1 to r_2 we must bring together at least $|r_1 - r_2|$ cluster boundaries, showing that the leading bcc operator in this case is at $(|r_1 - r_2|, 1)$, consistent once again with the fusion rules. If the bcc operators corresponding to a single curve are $(2,1)$ this means that the corresponding states satisfy

$$(2\hat{L}_{-2} - (2/g)\hat{L}_{-1}^2)|\phi_{2,1}\rangle = 0\,. \tag{47}$$

We are now going to argue that (47) is equivalent to the statement that the cluster boundary starting at this boundary point is described by SLE. In order to avoid being too abstract initially, we'll first show how the calculations of a particular observable agree in the two different formalisms.

Let ζ be a point in the upper half plane and let $P(\zeta)$ be the probability that the curve, starting at the origin, passes to the left of this point (of course it is not holomorphic). First we'll give the physicist's version of the SLE argument (assuming familiarity with Werner's lectures). We imagine making the exploration process for a small Loewner time δt, then continuing the process to infinity. Under the conformal mapping $f_{\delta t}(z)$ which removes the first part of the curve, we get a new curve with the same measure as the original one, but the point ζ is mapped to $f_{\delta t}(\zeta)$. But this will lie to the right of the new curve if and only if the original point lay to the right of the original curve. Also, by integrating the Loewner equation starting from $f_0(z) = z$, we have approximately

$$f_{\delta t}(z) \approx z + \frac{2\delta t}{z} + \sqrt{\kappa}\delta B_t\,,$$

at least for $z \gg \delta t$. Thus we can write down an equation[25]:

$$P(\zeta) = \mathbf{E}\left[P\left(\zeta + \frac{2\delta t}{\zeta} + \sqrt{\kappa}\delta B_t\right)\right]_{\delta B_t}\,,$$

where $\mathbf{E}[\ldots]_{\delta B_t}$ means an average over all realisations of the Brownian motion up to time δt. Expanding the right hand side to $O(\delta t)$, and remembering that $\mathbf{E}[\delta B_t] = 0$ and $\mathbf{E}[(\delta B_t)^2] = \delta t$, we find (with $\zeta = x + iy$), the linear PDE

$$\left(\frac{2x}{x^2+y^2}\frac{\partial}{\partial x} - \frac{2y}{x^2+y^2}\frac{\partial}{\partial y} + \frac{\kappa}{2}\frac{\partial^2}{\partial x^2}\right)P(x,y) = 0\,. \tag{48}$$

By scale invariance, $P(x,y)$ depends in fact only on the ratio y/x, and therefore this can be reduced to a second order ODE, whose solution, with appropriate boundary conditions, can be expressed in terms of hypergeometric functions (and is known as Schramm's formula.)

[24] If instead of the dilute lattice model we consider the dense phase, which corresponds, eg to the boundaries of the FK clusters in the Potts model, then r and s get interchanged for a given central charge c, and the bcc operator then lies at $(1,2)$.

[25] Some physicists will recognise this as the reverse Fokker-Planck equation.

Now let us give the CFT derivation. In terms of correlation functions, P can be expressed as

$$P = \frac{\langle\phi_{2,1}(0)\Phi(\zeta,\bar\zeta)\phi_{2,1}(\infty)\rangle}{\langle\phi_{2,1}(0)\phi_{2,1}(\infty)\rangle}\,.$$

The denominator is just the partition function restricted to there being a cluster boundary from 0 to infinity. Φ is an 'indicator operator' which takes the values 0 or 1 depending on whether the curve passes to the right (respectively left) of ζ. Since P is a probability it is dimensionless, so Φ has zero conformal dimensions and transforms trivially.

Now suppose we insert $\int_C(2T(z)/z)(dz/2\pi i) + \text{cc.}$ into the correlation function in the numerator, where C is a semicircular contour surrounding the origin but not ζ. Using the OPE of T with $\phi_{2,1}$, this gives

$$2L_{-2}\phi_{2,1}(0) = (2/g)\partial_x^2\phi_{1,2}(x)|_{x=0}\,.$$

Using translation invariance, this derivative can be made to act equivalently on the x-coordinate of ζ. On the other hand, we can also distort the contour to wrap around ζ, where it simply shifts the argument of Φ. The result is that we get exactly the same PDE as in (48), with the identification

$$g = 4/\kappa\,.$$

Of course this was just one example. Let us see how to proceed more generally. In radial quantisation, the insertion of the bcc field $\phi_{2,1}(0)$ gives a state $|\phi_{2,1}\rangle$. Under the infinitesimal mapping f_{dt} we get the state

$$\Big(1-(2\hat L_{-2}dt + \hat L_{-1}\sqrt{\kappa}dB_t)\Big)\,|\phi_{2,1}\rangle\,,$$

or, over a finite time, a time-ordered exponential

$$\mathbf{T}\exp\left(-\int_0^t(\hat L_{-2}dt' + \hat L_{-1}\sqrt{\kappa}dB_{t'})\right)|\phi_{2,1}\rangle\,. \tag{49}$$

The conformal invariance property of the measure on the curve then implies that, when averaged over $dB_{t'}$, this is in fact independent of t. Expanding to $O(t)$ we then again find (47) with $g = 4/\kappa$. Since this is a property of the state, it implies an equivalence between the two approaches for all correlation functions involving $\phi_{2,1}(0)$, not just the one considered earlier. Moreover if we replace $\sqrt{\kappa}dB_t$ by some more general random driving function dW_t, and expand (49) to any finite order in t using the Virasoro algebra and the null state condition, we can determine all moments of W_t and conclude that it must indeed be rescaled Brownian motion.

Of course the steps we used to arrive at this result in CFT are far less rigorous than the methods of SLE. However, CFT is more powerful in the sense that many other similar result can be conjectured which, at present, seem to be beyond the techniques of SLE. This is part of an ongoing symbiosis between the disciplines of theoretical physics and mathematics which, one hopes, will continue.

8 Further Reading

The basic reference for CFT is the 'big yellow book': *Conformal Field Theory* by P. di Francesco, P. Mathieu and D. Senechal (Springer-Verlag, 1996.) See also volume 2 of *Statistical Field Theory* by C. Itzykson and J.-M. Drouffe (Cambridge University Press, 1989.) A gentler introduction is provided in the 1988 les Houches lectures by P. Ginsparg and J. Cardy in *Fields, Strings and Critical Phenomena*, E. Brézin and J. Zinn-Justin, eds. (North-Holland, 1990.) Other specific pedagogical references are given below.

2.1 J. Cardy, *Scaling and Renormalization in Statistical Physics*, (Cambridge University Press, 1996.)

3.2 P. Calabrese and J. Cardy, *Entanglement entropy and quantum field theory – a non-technical introduction*, Int.J.Quant.Inf. **4**, 429 (2006); arXiv:quant-ph/0505193.

6.1 V. Pasquier, Nucl. Phys. B **285**, 162, 1986, and J. Phys. A **20**, L1229, 1987; I. Kostov, Nucl. Phys. B **376**, 539, 1992; arXiv:hep-th/9112059. The version in these lectures is discussed in J. Cardy, J.Phys. A **40**, 1427 (2007); arXiv:math-ph/0610030.

6.2 The basic reference is B. Nienhuis, in *Phase transitions and critical phenomena*, vol. 11, p. 1, C. Domb and J.L. Lebowitz, eds. (Academic, 1987.) A slightly different approach is explained in J. Kondev, Phys. Rev. Lett. **78**, 4320, 1997.

7.1,7.2 For reviews see V.B. Petkova and J.-B. Zuber, *Conformal boundary conditions and what they teach us*, lectures given at the Summer School and Conference on Non-perturbative Quantum Field Theoretic Methods and their Applications, August 2000, Budapest, Hungary, arXiv:hep-th/0103007, and J. Cardy, *Boundary Conformal Field Theory* in Encyclopedia of Mathematical Physics, (Elsevier, 2006); arXiv:hep-th/0411189.

7.3 For reviews on the connection between SLE and CFT, see J. Cardy, *SLE for Theoretical Physicists*, Ann. Phys. **318**, 81-118 (2005); arXiv:cond-mat/0503313, and M. Bauer and D. Bernard, *2D growth processes: SLE and Loewner chains*, Phys.Rept. **432**, 115-221 (2006); arXiv:math-ph/0602049.