Stability of the Standard Model ground state

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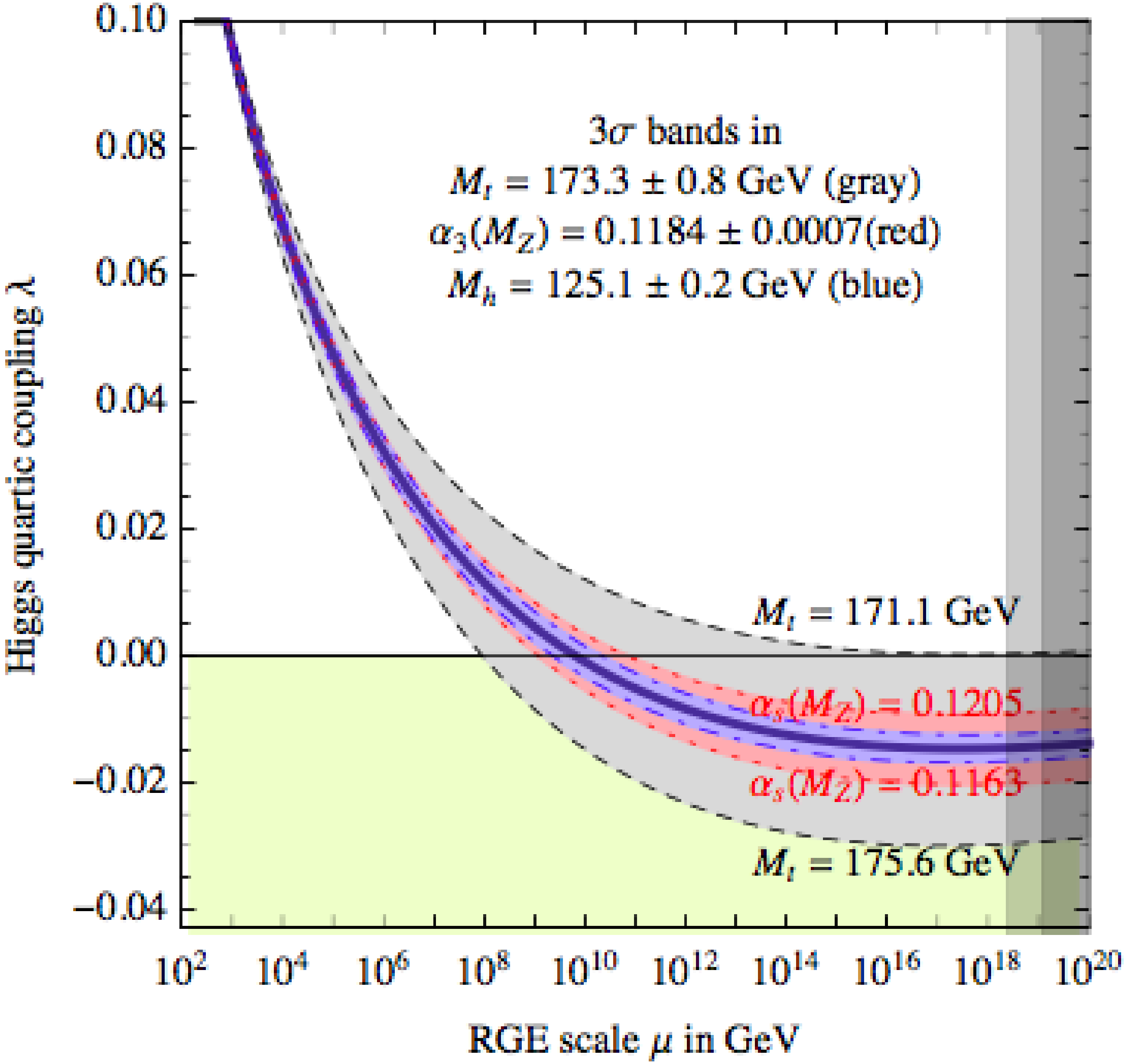
- **The standard model vacuum state $|0\rangle$ may not be stable:**

$$V_{\rm eff}(h) \sim \frac{1}{4}\lambda(h)h^4; \qquad \langle 0|h|0\rangle = v \simeq 246 \text{ GeV}$$

  $\lambda(\mu) < 0$ **for** μ **sufficiently large** [$+$ $m_{\rm Higgs}$ **sufficiently small**], **because of top quark loops.**

- **Before the Higgs discovery, this gives a lower bound to** $m_{\rm Higgs}$**, or an upper bound to the scale of non-standard physics (typical example: supersymmetric models, where** λ **is replaced by a positive-definite function of gauge couplings).**

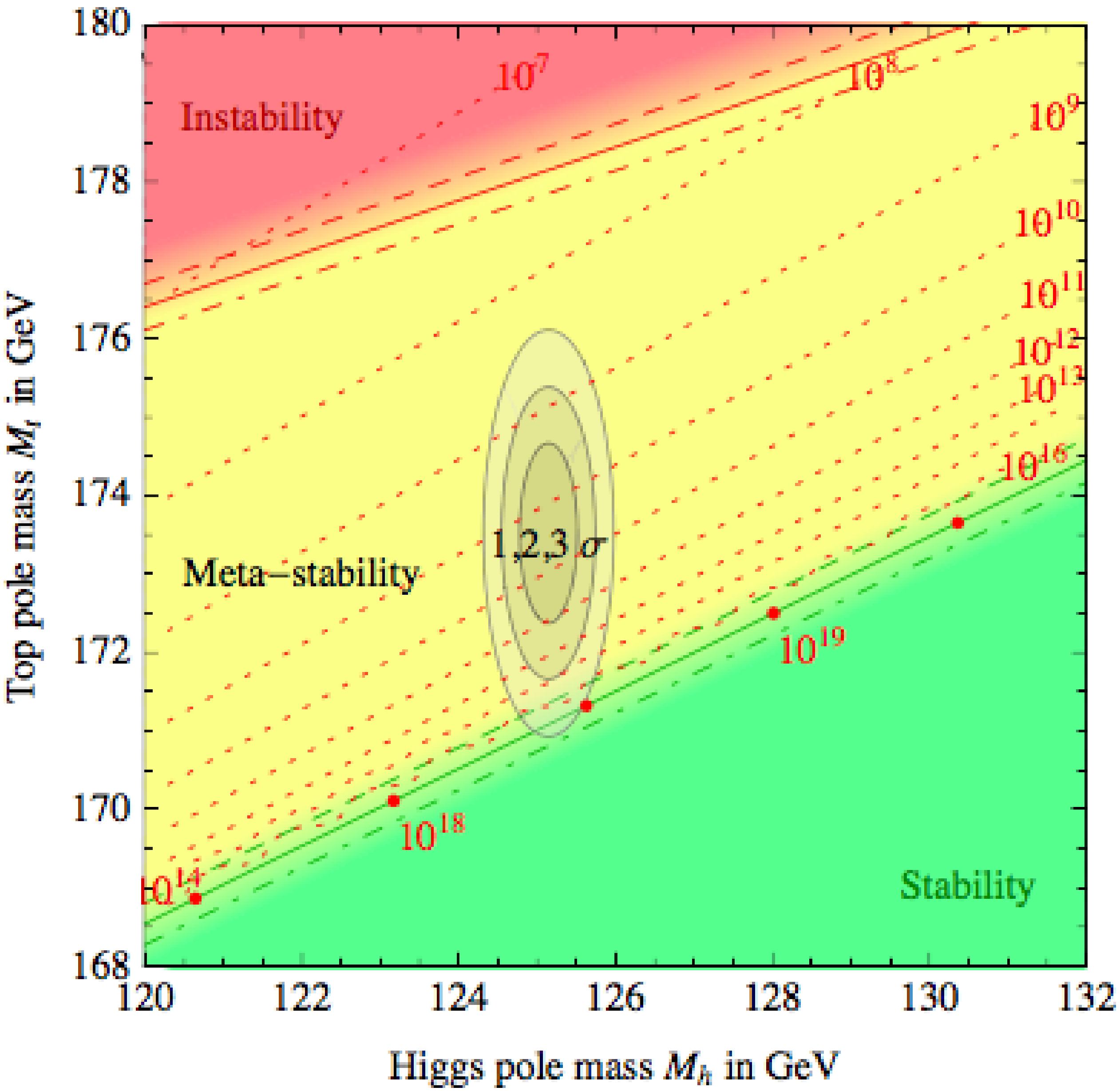
- **Not necessary to require absolute stability: an unstable vacuum with lifetime** $\tau > \tau_{\rm universe} \sim 1.3 \times 10^{10}$ **y also acceptable.**



[Buttazzo et al, JHEP 1312(2013)089]

- **The decay rate of the unstable vacuum state can be computed as a function of ew parameters within the pure standard model.**
- **With present values of the relevant parameters (m_{top} and m_{Higgs}) and no new physics the ground state turns out to be in the metastability region.**

No need for new physics to stabilize the ew ground state



[Buttazzo et al, JHEP 1312(2013)089]

Recent observation: Non-standard physics around the Planck mass may modify the ground state lifetime.

[Branchina, Messina, Platania, Sher, Zappalà, arXiv:1307.5193, 1407.4112, 1408.5302, 1507.08812, 1601.06963]

An opportunity to review the whole subject.

Results obtained in collaboration with Luca Di Luzio (Università di Genova) and Gino Isidori (University of Zürich).

Outline:

1. **A review of known facts**
2. **The role of scale invariance**
3. **Impact of new physics**
4. **Conclusions**

1. Known facts

In ordinary quantum mechanics in one dimension, the tunnelling probability through a potential barrier per unit time Γ is given by

$$\Gamma \simeq A e^{-B}$$

where

$$B = \frac{2}{\hbar} \int_a^b dx \, \sqrt{2m[V(x) - E]}$$

and $V(a) = V(b) = E$. This is standard semiclassical approximation.

We assume that $V(x)$ has a local minimum in $x = a$, and choose the zero of energy so that $E = V(a) = 0$.

Coefficient A not fixed in the semiclassical limit: first quantum fluctuations needed.

Generalization to the case of many variables not entirely trivial [Banks, Bender, Wu PRD 8(1973)3346; 3366].

Now $x \to \vec{x}$, $a \to \vec{a}$ and b is in general not a single point but a surface Σ in the coordinate space. Hence

$$B = \frac{2}{\hbar} \int_{\vec{a}}^{\vec{b}} ds \sqrt{2mV(x)}$$

The integral is performed along the path, and to the endpoint $\vec{b} \in \Sigma$, which minimize the integral:

$$\delta \int_{\vec{a}}^{\vec{b}} ds \sqrt{2mV(x)} = 0.$$

Problem reduced to finding the integration path.

For fixed endpoints $\vec{a}$ and $\vec{b}$, we know how to solve a similar variational problem: the path $\vec{x} = \vec{x}(t)$ which minimizes

$$\int_{\vec{a}}^{\vec{b}} dt\, \sqrt{2m[E - V(x)]}$$

is the solution of classical equations of motion

$$m\frac{d^2\vec{x}(t)}{dt^2} = -\vec{\nabla}V(\vec{x})$$

with

$$\frac{1}{2}m\frac{d\vec{x}(t)}{dt}\cdot\frac{d\vec{x}(t)}{dt} + V(\vec{x}) = E$$

and fixed endpoints $\vec{x}(t_0) = \vec{a}$, $\vec{x}(t_1) = \vec{b}$.

We have just the same problem, with $E = 0$ and $V \to -V$ (except that the upper bound $\vec{b}$ is not fixed: forget about it for the moment). We look for solutions of

$$m\frac{d^2\vec{x}(\tau)}{d\tau^2} = +\vec{\nabla}V(\vec{x})$$

with fixed energy

$$\frac{1}{2}m\frac{d\vec{x}(\tau)}{d\tau}\cdot\frac{d\vec{x}(\tau)}{d\tau} - V(\vec{x}) = 0$$

and boundary conditions $\vec{x}(\tau_0) = \vec{a}$, $\vec{x}(\tau_1) = \vec{b}$, some fixed point on the surface Σ.

A useful suggestion for the generalization to field theory:

view the same equations as classical equations of motion with imaginary time $\tau = it$; $\vec{x}(\tau)$ stationary point of the Euclidean action

$$S_E = \int d\tau \, L_E; \quad L_E = \frac{1}{2} m \frac{d\vec{x}(\tau)}{d\tau} \cdot \frac{d\vec{x}(\tau)}{d\tau} + V(\vec{x})$$

[S. Coleman, Phys. Rev. D15(1977)2929]

Features of the solution:

- **the solution starts at $\vec{x} = \vec{a}$ at $\tau_0 = -\infty$ and reaches $\vec{b}$ at an instant τ_1, which we may choose to be zero by translation invariance;**
- $\frac{d\vec{x}}{d\tau} = 0$ **for** $\tau = 0$**;**

Hence

$$\int_{\vec{a}}^{\vec{b}} ds \sqrt{2mV} = \int_{-\infty}^{0} d\tau \, L_E(\tau)$$

(which is insensitive to variations of $\vec{b}$ because $V(\vec{b}) = 0$), and

$$2\int_{\vec{a}}^{\vec{b}} ds \sqrt{2mV} = \int_{-\infty}^{+\infty} d\tau \, L_E(\tau)$$

The solution $\vec{x}(\tau)$ is called a bounce.

A proof (in one dimension): V has a local minimum in $x = a$, so for x close to a $V(x) \sim c^2(x-a)^2$. From energy conservation we have

$$\frac{m}{2}\left|\frac{dx}{d\tau}\right|^2 = V(x) = c^2(x-a)^2$$

Solution:

$$\tau = \frac{1}{c}\sqrt{\frac{m}{2}}\log(x-a) \to -\infty$$

Also,

$$V(b) = 0 \Rightarrow \left.\frac{dx}{d\tau}\right|_{\tau=0} = 0$$

and therefore

$$\frac{\delta}{\delta\vec{b}}\int_{\vec{a}}^{\vec{b}} ds\,\sqrt{2mV} = 0.$$

- **The solution must reach $\vec{b}$ (a point beyond the barrier) at $\tau = 0$: the trivial solution $\vec{x}(\tau) = \vec{a}$ not allowed.**
- **If more than one bounce is found, tunnelling probability dominated by the one with least classical action.**
- **If more than one bounce is found with the same action is found, their contributions to Γ must be summed (typically the case with symmetries). Integration only affects A.**

Third step: generalization to field theory (*the work of a moment*, to quote Sydney Coleman).

Scalar field theory with one local minimum of the potential at $h = v$: look for a solution of the euclidean field equation

$$\left(\frac{\partial^2}{\partial \tau^2} + \vec{\nabla}^2\right) h(\tau, \vec{r}) = V'(h)$$

such that

$$\lim_{\tau \to \pm\infty} h(\tau, \vec{r}) = v; \qquad \frac{\partial h(0, \vec{r})}{\partial \tau} = 0.$$

Finite action only if

$$\lim_{|\vec{r}| \to \infty} h(\tau, \vec{r}) = v$$

The decay probability per unit time Γ is given by

$$\Gamma \simeq A e^{-S[h]}$$

where

- $h(x)$ **(called "the bounce" by Sidney Coleman) a finite-action solution of the field equations**
- $S[h]$ **the corresponding euclidean action**

$$S[h] = \int d^4x \left[\frac{1}{2} \partial_\mu h \partial_\mu h + V(h) \right]$$

- $A \sim \frac{V_U}{R^4} = \frac{\tau_U^3}{R^4}$ **(**τ_U **the age of the Universe,** R **a length scale called the bubble radius) not fixed at the semiclassical level.**

Problem reduced to finding the bounce

$$\partial^2 h(x) = V'(h)$$

[S. Coleman, Phys. Rev. D15(1977)2929]

Considerably simplified by some considerations:

- $h(x) \to h(r)$, $r^2 = |\vec{x}|^2 + \tau^2$ **to minimize the action (a theorem)**
- **Since** $\lambda(\mu)$ **becomes negative around** $\mu = \Lambda_{\rm inst} \sim 10^{10}$ **GeV, electroweak mass effects neglected:** $v \to 0$ **(more on this later)**
- **running of** λ **neglected (higher order in** $\hbar$**);**

$$V(h) = -\frac{|\lambda|}{4} h^4$$

In this case the field equation

$$\frac{\partial^2 h(r)}{\partial r^2} + \frac{3}{r}\frac{\partial h(r)}{\partial r} = -|\lambda| h^3(r)$$

can be integrated analytically:

$$h(r) = \sqrt{\frac{8}{|\lambda|}} \frac{R}{r^2 + R^2}; \qquad S[h] = \frac{8\pi^2}{3|\lambda|}$$

[Often useful to view the field equation

$$\frac{\partial^2 h(r)}{\partial r^2} = V'(h) - \frac{3}{r}\frac{\partial h(r)}{\partial r}$$

as the classical equation of motion of a particle of unit mass and position h**, (with** r **playing the role of time) under a potential** $U(h) = -V(h)$**, and in the presence of a damping force with a time-dependent coefficient** $3/r$**.]**

The standard model bounce:

$$h(r) = \sqrt{\frac{8}{|\lambda|}} \frac{R}{r^2 + R^2}; \qquad S[h] = \frac{8\pi^2}{3|\lambda|}$$

Comments:

- **A bubble of size R:** $h(0) = \sqrt{\frac{8}{|\lambda|}} \frac{1}{R}$**,** $h(R) = \frac{h(0)}{2}$**,** $h(+\infty) = 0$**.**
- **Actually, an infinite family of bounces, parametrized by** R
- **The action** $S[h]$ **is (obviously)** R**-independent.**

A consequence of scale invariance of the field equation:

$$\frac{\partial^2 h(r)}{\partial r^2} + \frac{3}{r}\frac{\partial h(r)}{\partial r} = -|\lambda| h^3(r)$$

If $h(r)$ is a solution, then

$$h_a(r) = a h(ar)$$

is a solution too. Special of h^4 coupling: no dimensionful parameters.

Scale transformation amounts to replacing $R \to \frac{R}{a}$.

No way to fix R at the semiclassical level: one-loop corrections needed. **[Isidori, Strumia, GR, NPB 609(2001)387]**

Main effect of quantum fluctuations: breaking of scale invariance of the tree-level potential.

Bounces with different R turn out to have a one-loop action roughly given by

$$S[h] \sim \frac{8\pi^2}{3|\lambda(1/R)|}$$

For these configurations the dimensional factor due to the zero eigenvalues turns out to be of $\mathcal{O}(R^{-4})$.

Both scale ambiguities of the semi-classical result resolved:

$$P = \Gamma \tau_U = \max_R \frac{\tau_U^4}{R^4} \exp\left[-\frac{8\pi^2}{3\,|\lambda(1/R)|} - \Delta S \right]$$

Maximum at $R = R_{\rm SM} \approx 10^{-17}$ GeV^{-1}; decay rate dominated by the bounce with $\lambda = \lambda(1/R_{\rm SM})$.

A number of questions:

1. **Does it make sense to compute a tunnelling rate through a potential with no barrier?**
2. **How good is the fixed-λ approximation?**
3. **How good is the $v \ll \Lambda_{\mathrm{inst}}$ approximation?**
4. **Are we really allowed to ignore physics at scales $M \gg \Lambda_{\mathrm{inst}}$, e.g. $M \sim M_{\mathrm{Planck}}$?**

Answers to questions #1 and #2 are long known:

1. **Yes [K. Lee, E. Weinberg, NPB 267(1986)181]**
2. **Very good [IRS 2001]**

Naïve answer to question #3:

3. **Very good:** $\frac{v}{\Lambda_{\rm inst}} \sim 10^{-8}$

A more accurate answer closely related to question #4.

2. The role of scale invariance

SM euclidean action with a mass term and $\lambda < 0$:

$$S[h] = \int d^4x \left[\frac{1}{2}\partial_\mu h \partial_\mu h + \frac{1}{2} m^2 h^2 + \frac{1}{4}\lambda h^4 \right]$$

Upon scale transformations $h(r) \to h_a(r) = a h(ar)$

$$S[h] \to S[h_a] = S[h] + \frac{m^2}{2a^2} \int d^4x \, h^2(x),$$

which cannot be stationary unless $h = 0$:

$$\left. \frac{\partial S[h_a]}{\partial a} \right|_{a=1} = -m^2 \int d^4x \, h^2(x).$$

No scale invariance $\to$ no bounce!

Way out: constrained instantons: look for a bounce characterized by a definite length scale R much smaller than $\frac{1}{m}$ [Affleck 1981]. Constraint imposed by a Lagrange multiplier. Typically adopted in the context of gauge instantons.

Not directly relevant here:

In the context of the standard model scalar potential, including radiative corrections and with mass and cubic terms taken into account, a bounce exists, and differs from $h(r)$ by powers of $v^2 R_{\mathrm{SM}}^2$. Naïve expectation confirmed.

3. New physics at scale M

A possible parametrization of new physics around the scale M:

$$V_{\rm NP}(h) = -\frac{|\lambda|}{4}h^4 + \frac{\lambda_6}{6}\frac{h^6}{M^2} + \frac{\lambda_8}{8}\frac{h^8}{M^4}$$

A special case in a wider class:

$$V_{\rm NP}(h) = -\frac{|\lambda|}{4}h^4 + M^4\sum_{n=3}^{\infty}\frac{\lambda_{2n}}{2n}\left(\frac{h}{M}\right)^{2n-4}$$

(in principle, even derivative terms might be included.)

Also in this case, scale invariance is broken: e.g. keeping only the h^6 term,

$$S[h] \to S[h_a] = S[h] + \frac{\lambda_6 a^2}{6M^2} \int d^4x \, h^6(x),$$

which implies

$$\left. \frac{\partial S[h_a]}{\partial a} \right|_{a=1} = \frac{\lambda_6}{3M^2} \int d^4x \, h^6(x).$$

Again, no bounce, unless

- **either a constraint is imposed, to select bounces with $\frac{1}{R} \ll M$ (essentially equivalent to removing the effect of new couplings)**
- **or cancellations take place among different terms in $V_{\rm NP}(h)$, and a bounce with $R \sim \frac{1}{M}$ appear.**

Does a bounce exist in this case?

Unfortunately, the field equation

$$\frac{\partial^2 h(r)}{\partial r^2} + \frac{3}{r}\frac{\partial h(r)}{\partial r} = V'_{\rm NP}(h)$$

cannot be integrated analytically.

A numerical approach gives exact answers, but poor insight.

However:

Some insight by adopting a rough approximation: assume that the bounce (if it exists) is well approximated by a step function

$$h(r) = \begin{cases} h(0) & 0 \le r \le R \\ 0 & r > R \end{cases} = h(0)\theta(R - r)$$

A criterion to estimate $h(0)$ in the general case will be given shortly.

It is convenient to rewrite

$$V_{\rm NP}(h) = \frac{1}{4}\lambda_{\rm eff}(h)h^4$$

with

$$\lambda_{\rm eff}(h) = \lambda + 4\sum_{n=3}^{\infty}\frac{\lambda_{2n}}{2n}\left(\frac{h}{M}\right)^{2n-4}$$

Scale invariance broken by the new physics parameter M**:**

$$\left.\frac{\partial S[h_a]}{\partial a}\right|_{a=1} = \int d^4x\,\frac{1}{4}\frac{\partial\lambda_{\rm eff}(h)}{\partial h}h^5(x) \propto h(0)\lambda'_{\rm eff}(h(0))$$

Hence, in order for the bounce action to be stationary under scale transformations, $h(0)$ **should be chosen so that**

$$\lambda'_{\rm eff}(h(0)) = 0$$

If such a value of $h(0)$ exists, then scale invariance is locally restored, and the action of the bounce can be computed by analogy with the standard model:

$$S[h] \simeq \frac{8\pi^2}{3|\lambda_{\rm eff}(h(0))|}$$

The value of $h(0)$ (the position of the minimum of $\lambda_{\rm eff}$) is a function of λ_i and M. Can we relate it to the bubble radius R?

In the standard model (where the bounce is exactly known) we have

$$S_{\rm SM}[h] = \int d^4x \left[\frac{1}{2}(\partial_\mu h)^2 - \frac{|\lambda|}{4}h^4\right] = \frac{1}{|\lambda|}\int d^4x \left[\frac{1}{2}(\partial_\mu H)^2 - \frac{1}{4}H^4\right]$$

with

$$H = \sqrt{|\lambda|}h$$

independent of λ. Hence, on dimensional grounds,

$$h(0) = \frac{a}{R\sqrt{|\lambda|}}$$

with a a coefficient of order 1.

We may fix a by requiring that the action has the correct value:

$$S_{\mathrm{SM}}[h] = \frac{a^4\pi^2}{8|\lambda|} = \frac{8\pi^2}{3|\lambda|}$$

which gives $a = \sqrt{\frac{8}{\sqrt{3}}} \simeq 2.15$.

Hence h_0 is related to M by

$$h(0) = \frac{a}{R\sqrt{|\lambda_{\mathrm{eff}}(h(0))|}}; \qquad a = \sqrt{\frac{8}{\sqrt{3}}}$$

An example: the case considered by V. Branchina and collaborators.

$$V_{\rm NP}(h) = -\frac{|\lambda|}{4}h^4 + \frac{\lambda_6}{6M^2}h^6 + \frac{\lambda_8}{8M^4}h^8$$

$$\lambda_{\rm eff}(h) = \lambda + \frac{2}{3}\lambda_6\frac{h^2}{M^2} + \frac{1}{2}\lambda_8\frac{h^4}{M^4}$$

$$\lambda'_{\rm eff}(h) = \frac{2}{3}\frac{h}{M^2}\left(2\lambda_6 + 3\lambda_8\frac{h^2}{M^2}\right)$$

I will only consider the case $\lambda_8 > 0$, so that the scalar potential is bounded below (although the case $\lambda_6 > 0, \lambda_8 < 0$ may be conceptually interesting.)

Two possibilities:

i) $\lambda_6 > 0$ **(and** λ **a negative constant). Then** $\lambda'_{\rm eff}(h)$ **is always different from zero:**

$$\Downarrow$$

no bounce, because of the scale-invariance argument.

In the real word, with λ **running, there is an extra compensating contribution; a bounce of finite size with a SM-like bounce action can be found numerically, provided** $M \gg 1/R_{\rm SM} \approx 10^{17}$ **GeV, where** λ **has a minimum.**

Essentially no difference with respect to the pure standard model

ii) $\lambda_6 < 0$. **In this case, the two non-renormalizable terms compensate each other and locally restore scale invariance. Indeed,** $\lambda'_{\text{eff}}(h)$ **has a zero at**

$$h = h(0) = M\sqrt{\frac{2|\lambda_6|}{3\lambda_8}},$$

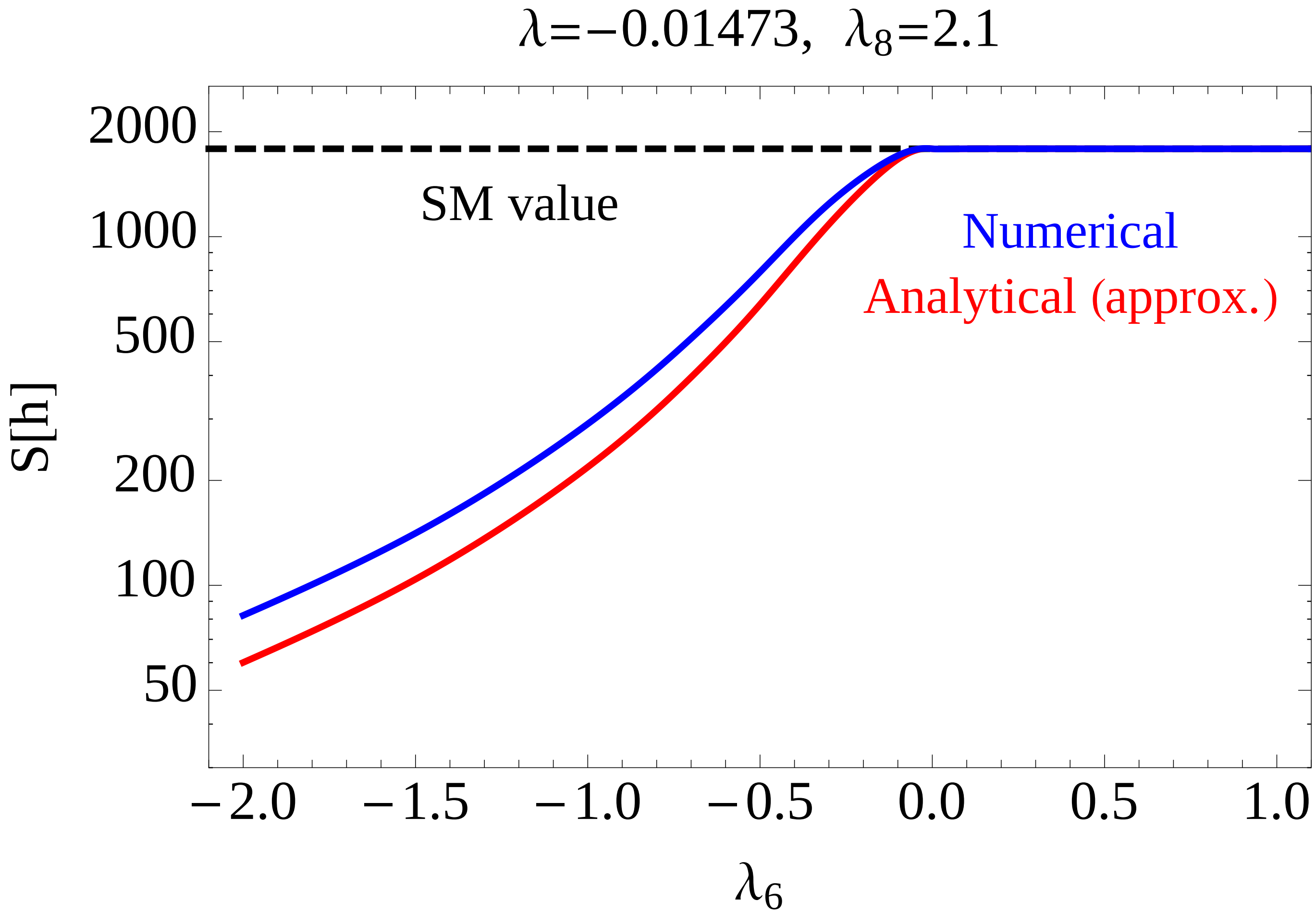
and

$$\lambda_{\text{eff}}(h(0)) = -|\lambda| - \frac{2|\lambda_6|^2}{9\lambda_8}.$$

The value of $h(0)$ **fixes the bounce size:**

$$R = \frac{a}{h(0)\sqrt{|\lambda_{\text{eff}}(h(0))|}} = \frac{a}{M}\frac{1}{\sqrt{\frac{2|\lambda_6|}{3\lambda_8}\left(|\lambda| + \frac{2|\lambda_6|^2}{9\lambda_8}\right)}} \sim \frac{1}{M}.$$

How good is this approximation?



Approximate vs. exact (numerical) bounce action: not too bad!

[Di Luzio, Isidori, GR, PLB753(2016)150]

Two important points:

1. We have

$$|\lambda_{\rm eff}(h(0))| = |\lambda| + \frac{2|\lambda_6|^2}{9\lambda_8},$$

always larger than $|\lambda|$; e.g. with $\lambda_6 = -2$ and $\lambda_8 = 2.1$,

$$|\lambda_{\rm eff}(h(0))| \simeq 0.015 + 0.423 = 0.438$$

and therefore

$$S[h] \ll S_{\rm SM}[h_{\rm SM}].$$

The decay probability of the false vacuum can only be increased with respect to the pure SM.

2. Modification of the tunnelling rate roughly independent of the scale M (not exactly):

$$\Gamma = A e^{-S[h]}$$

The bounce action (i.e. the dominant contribution to the tunnelling rate) is independent of M:

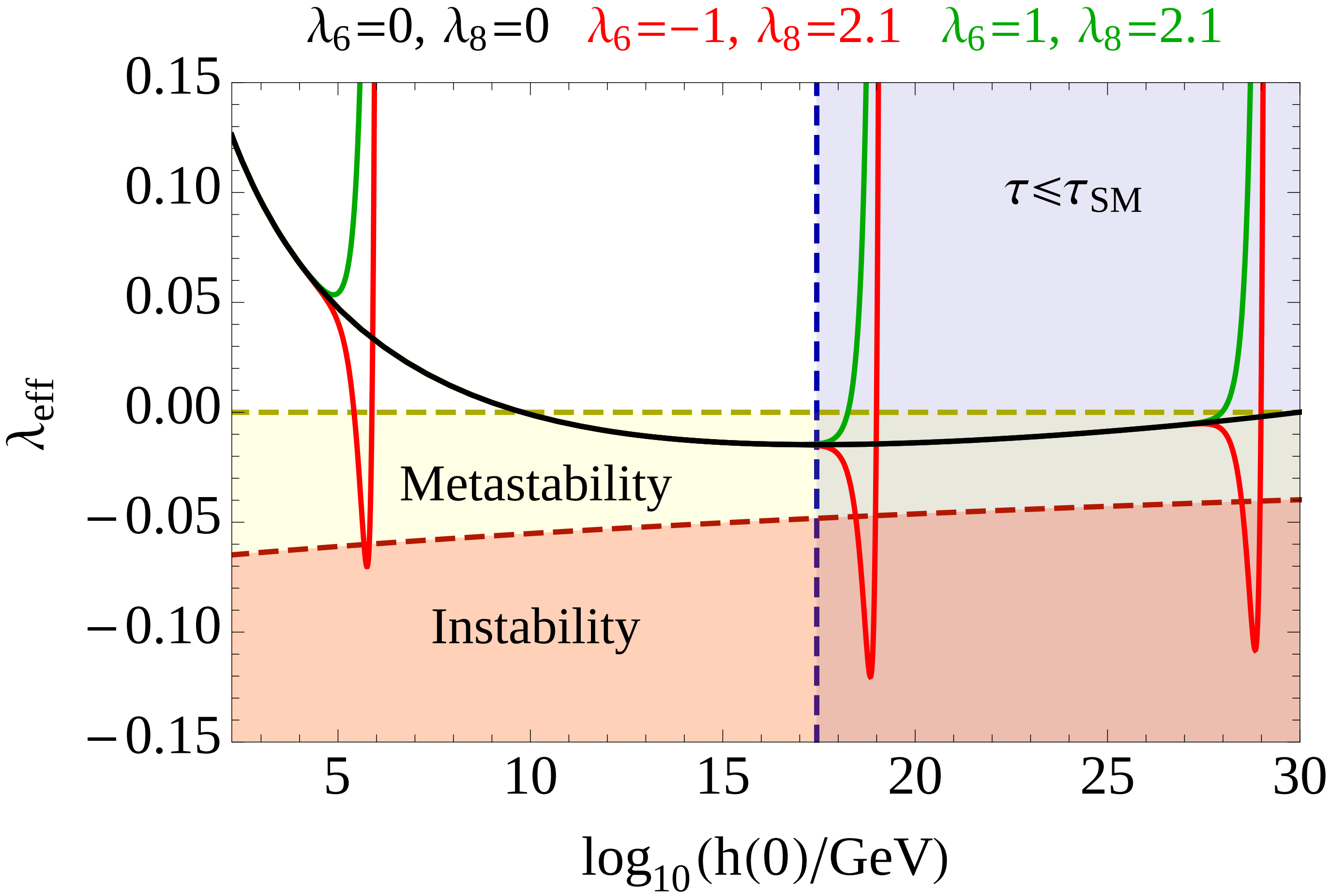
$$S[h] = 2\pi^2 \int_0^\infty dr\, r^3 \left[V(h) - \frac{1}{2} h(r) V'(h) \right]$$
$$= 2\pi^2 \int_0^\infty dx\, x^3 \left[\tilde{V}(\tilde{h}) - \frac{1}{2} \tilde{h}(x) \tilde{V}'(\tilde{h}) \right]$$

where

$$x = Mr; \qquad \tilde{h}(x) = \frac{h(r)}{M}; \qquad \tilde{V}(\tilde{h}) = \frac{V(h)}{M^4} = \frac{1}{4}\lambda \tilde{h}^4 + \sum_{n=3}^{\infty} \frac{\lambda_{2n}}{2n} \tilde{h}^{2n}$$

Not exactly independent:

$$A \sim \frac{\tau_U^3}{R^4} \sim M^4 \tau_U^3$$



$M = 10^6$ GeV $\quad M = M_P \quad M = 10^{10} M_P$

[Di Luzio, Isidori, GR]

4. Conclusions

- **The decay probability of the SM electroweak vacuum under quantum tunneling is related to the breaking of scale invariance in the Higgs effective potential.**
- **In the absence of NP, breaking occurs dominantly via radiative corrections, which selects a leading bounce with $1/R_{\rm SM} \approx 10^{17}$ GeV (the position of the minimum of $\lambda(\mu)$); present values of $m_{\rm top}, m_{\rm Higgs}$ give $\tau > \tau_U$.**
- **New degrees of freedom at high energies ($\sim M$) in general introduce an extra explicit breaking of scale invariance.**

- **If $M > 1/R_{\rm SM}$, two cases:**
  1. **NP simply stabilizes the SM potential**
  2. **NP introduce new decay channels for the electroweak vacuum.**

  In case 2. NP can only increase the vacuum decay probability with respect to the pure SM case. No longer a stabilized version of the SM?

- **The question** ***Does the extrapolation of the SM up to the Planck scale require NP below such scale?*** **can be answered (and the answer is** ***no*****) within the pure standard model.**

  e.g., with a top mass of 180 GeV, NP would be implied at scales smaller than $1/R_{\rm SM}$, regardless of physics in the deep UV which ***cannot*** **improve on stability.**

- **This result does not imply the absence of NP up to the Planck scale, or that any UV completion of the theory at the Planck scale is compatible with the stability of the electroweak vacuum.**

More questions:

- **Is $V_{\rm NP}(h)$ a realistic parametrization of new physics?**
- **What is the role of gravity?**

Back-up slides

1. Tunnelling without barriers

Tunnelling process in field theory:

$$\lim_{\tau_i \to -\infty} h(\vec{x}, \tau_i) = v; \qquad \lim_{\tau_f \to +\infty} h(\vec{x}, \tau_f) = v,$$

We have

$$S[h] = \int_{\tau_i}^{\tau_f} d\tau \, K(\tau) \sqrt{2U[h]},$$

where

$$K(\tau) = \left[\int d^3x \left(\frac{\partial h}{\partial \tau} \right)^2 \right]^{\frac{1}{2}}$$

yields the correct normalization of the path length, and

$$U[h] = \int d^3x \left[\frac{1}{2} \left(\vec{\nabla} h(\vec{x}, \tau) \right)^2 + V(h(\vec{x}, \tau)) \right]$$

plays the role of the potential energy as in ordinary quantum mechanics.

Using the SM bounce,

$$T[h] \equiv \int d^3x \, \frac{1}{2} \left(\vec{\nabla} h(\vec{x}, \tau) \right)^2 = \frac{2\pi^2}{|\lambda| \, R} \left(\frac{1}{1 + \frac{\tau^2}{R^2}} \right)^{\frac{3}{2}},$$

$$V[h] \equiv \int d^3x \, V(h(\vec{x}, \tau)) = -\frac{2\pi^2}{|\lambda| \, R} \left(\frac{1}{1 + \frac{\tau^2}{R^2}} \right)^{\frac{5}{2}},$$

and

$$K(\tau) = \frac{2\pi}{\sqrt{|\lambda| \, R}} \frac{\frac{\tau}{R}}{\left(1 + \frac{\tau^2}{R^2}\right)^{\frac{5}{4}}}.$$

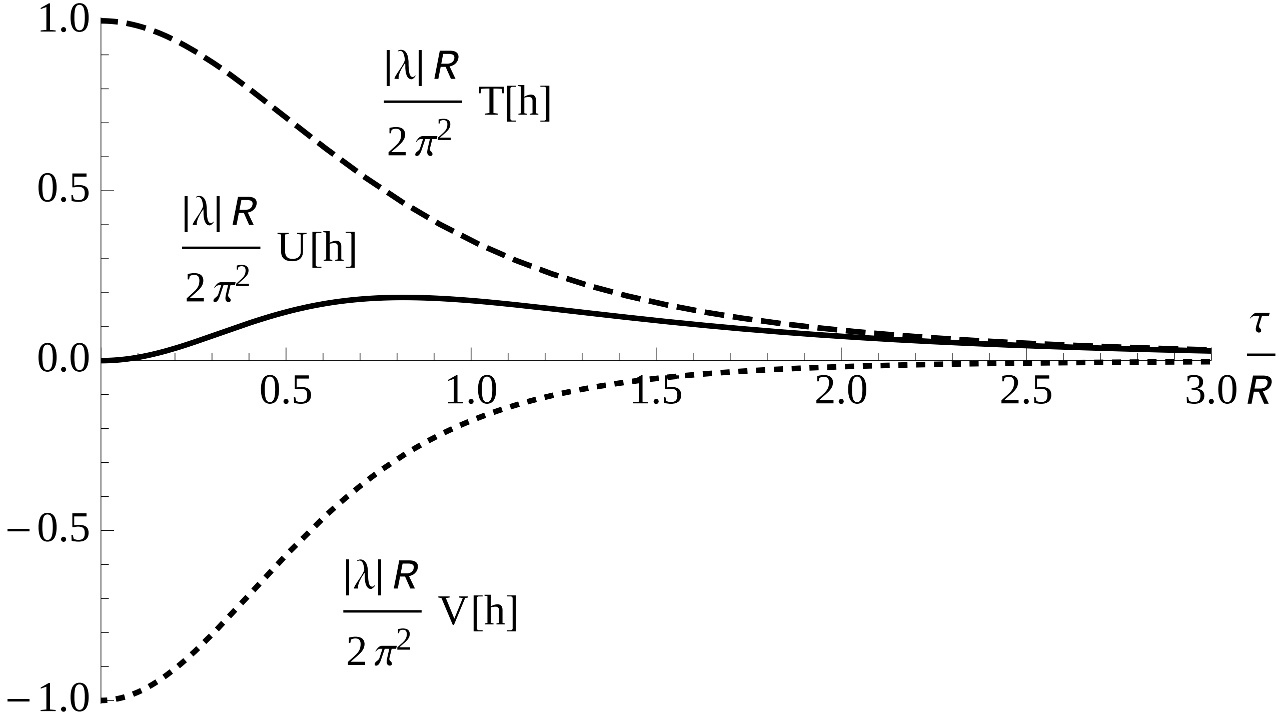
we finally get

$$U[h] = \frac{2\pi^2}{|\lambda| \, R} \left(\frac{1}{1 + \frac{\tau^2}{R^2}} \right)^{\frac{3}{2}} \left[1 - \frac{1}{1 + \frac{\tau^2}{R^2}} \right],$$

and

$$S[h] = 2 \int_0^\infty d\tau K(\tau) \sqrt{2U[h]} = \frac{8\pi^2}{3 \, |\lambda|},$$

which reproduces the correct result for the SM bounce action.



$U[h]$ as a function of τ can be interpreted as the potential energy along the path which minimizes the euclidean action. The positive gradient contribution $T[h]$ generates a barrier even for λ constant and negative.

2. Inclusion of the first quantum corrections

Including quantum fluctuations one gets

$$\frac{\Gamma}{\tau_U^4} = \frac{e^{-S_1[h]}}{\tau^4} = \frac{S_0^2[h]}{4\pi^2} \left| \frac{\mathrm{SDet}'\, S_0''[h]}{\mathrm{SDet}\, S_0''[0]} \right|^{-1/2} e^{-S_0[h]}$$

where

- h **the tree-level bounce,** S_0 **(**S_1**) the tree-level (one-loop) action functional;**
- $h = 0$ **the false (electroweak) vacuum, and** $S_1[0] = 0$**;**
- S_0'' **double functional differentiation of** S_0 **with respect to the various fields;**
- Det **the functional determinant,** $\mathrm{SDet} \equiv \mathrm{Det}$ **(bosons) or** $\mathrm{SDet} \equiv 1/\mathrm{Det}^2$ **(fermions).**

For constant h , S_1 is just the one-loop effective potential.

Here, computing $S_1[h]$ is a much harder task because *i)* the bounce is not a constant, *ii)* the bubble can appear everywhere in space-time.

The 'prime' on $\mathrm{SDet}'\, S_0''[h]$ indicates that these fluctuations, corresponding to zero modes, have been explicitly removed from the functional determinant. In this way the result acquires a dimensional factor that will be compensated by the integration over the volume of the universe.

Fortunately, in order to compute the one-loop corrections to the tunnelling rate we do not need to find the field configuration h_1 that extremizes the full one-loop action: we only need to compute $S_1[h]$, where h is the field configuration that extremizes S_0. The difference between $S_1[h]$ and $S_1[h_1]$ is a two-loop correction.

3. Constrained instantons

It is reasonable to think that, even in the presence of a mass term, instanton configurations of the scalar fields should exist, provided they are characterized by a length scale R such that $m \ll 1/R$.

Hence, the characteristic scale of h may be identified by the spacetime integral of a local operator, function of h, e.g.

$$\int d^4x\, h^n(x) \sim h^n(0) \int_0^R r^3 dr \sim R^{4-n}.$$

Affleck's suggestion: add to the action a term

$$S_c[h] = \sigma \left[\int d^4x\, O(h) - cR^{4-n} \right],$$

where $O(h)$ is a local operator of mass dimension $n \neq 4$, for example $O(h) = h^6$, and c a constant.

The constraint, with $\sigma > 0$, has the effect of generating an absolute minimum (the true vacuum) of the scalar potential, which would be unbounded from below with $\sigma = 0$ and $\lambda < 0$. Explicitly, the new scalar potential

$$V_c(h) = \frac{1}{2}m^2h^2 + \frac{\lambda}{4}h^4 + \sigma h^6$$

with $\lambda < 0$, $m^2 > 0$ has a local minimum at $h = 0$, with $V(0) = 0$, and an absolute minimum at $h \simeq \sqrt{\frac{|\lambda|}{6\sigma}}$ (for $m^2\sigma \ll 1$). The presence of the constraining term restores scale invariance of the action:

$$\begin{aligned} \frac{\partial}{\partial a}\left(S[h_a] + S_c[h_a]\right)\Big|_{a=1} &= -m^2\int d^4x\, h^2(x) + 2\sigma\int d^4x\, h^6(x) \\ &= -m^2\int d^4x\, h^2(x) + \frac{2c\sigma}{R^2}, \end{aligned}$$

which is zero for

$$\sigma = \frac{m^2R^2}{2c}\int d^4x\, h^2(x) \sim (mR)^2R^2 \ll R^2.$$