Scrambling & Chaos in Quantum Networks

Or Things To Think About ~~When You Can't Fall Asleep On Planes~~ During a Global Pandemic

J.M., J-G Hartmann, D. Rosa & J. Shock

based on: **1901.04561 and 2006.xxxxx**

Oxford Particle Theory Seminar, 2020

QGaSlab
The laboratory for Quantum Gravity & Strings

UNIVERSITY OF CAPE TOWN
IYUNIVESITHI YASEKAPA • UNIVERSITEIT VAN KAAPSTAD

The Small-World Phenomenon

[Milgram '67, Watts '99]

In 1967 **Stanley Milgram** attempted to quantify social connectivity in the US by:

- Identifying random individuals in Iowa
- Identifying a target individual in Boston
- Having the Iowans send a letter to the target by mailing only someone they knew and thought would be better placed to connect with the target

Milgram's team tracked the progress of the letter by having participants send a postcard directly to the researchers when they mailed the letter. This determined the links and nodes in the **social network**

How connected are we really?

What they found was that:

- With a small variance, the letter reached the target by 6 mailings
- The social network exhibited a **small world phenomenon** in that complete strangers were connected by a short acquaintance chain.
- This connectivity was mediated by a number of highly connected study participants

The Small-World Phenomenon

[Milgram '67, Watts '99]

In 1967 **Stanley Milgram** attempted to quantify social connectivity in the US by:

- Identifying random individuals in Nebraska
- Identifying a target individual in Boston
- Having the Nebraskans send a letter to the target by mailing only someone they knew and thought would be better placed to connect with the target

Milgram's team tracked the progress of the letter by having participants send a postcard directly to the researchers when they mailed the letter. This determined the links and nodes in the **social network**

How connected are we really?

What they found was that:

- With a small variance, the letter reached the target by 6 mailings
- The social network exhibited a **small world phenomenon** in that complete strangers were connected by a short acquaintance chain.
- This connectivity was mediated by a number of highly connected study participants

The Small-World Phenomenon

[Milgram '67, Watts '99]

In 1967 **Stanley Milgram** attempted to quantify social connectivity in the US by:

- Identifying random individuals in Nebraska
- Identifying a target individual in Boston
- Having the Nebraskans send a letter to the target by mailing only someone they knew and thought would be better placed to connect with the target

Milgram's team tracked the progress of the letter by having participants send a postcard directly to the researchers when they mailed the letter. This determined the links and nodes in the **social network**

How connected are we really?

What they found was that:

- With a small variance, the letter reached the target by 6 mailings
- The social network exhibited a **small world phenomenon** in that complete strangers were connected by a short acquaintance chain.
- This connectivity was mediated by a number of highly connected study participants

The Small-World Phenomenon

[Milgram '67, Watts '99]

In 1967 **Stanley Milgram** attempted to quantify social connectivity in the US by:

- Identifying random individuals in Nebraska
- Identifying a target individual in Boston
- Having the Nebraskans send a letter to the target by mailing only someone they knew and thought would be better placed to connect with the target

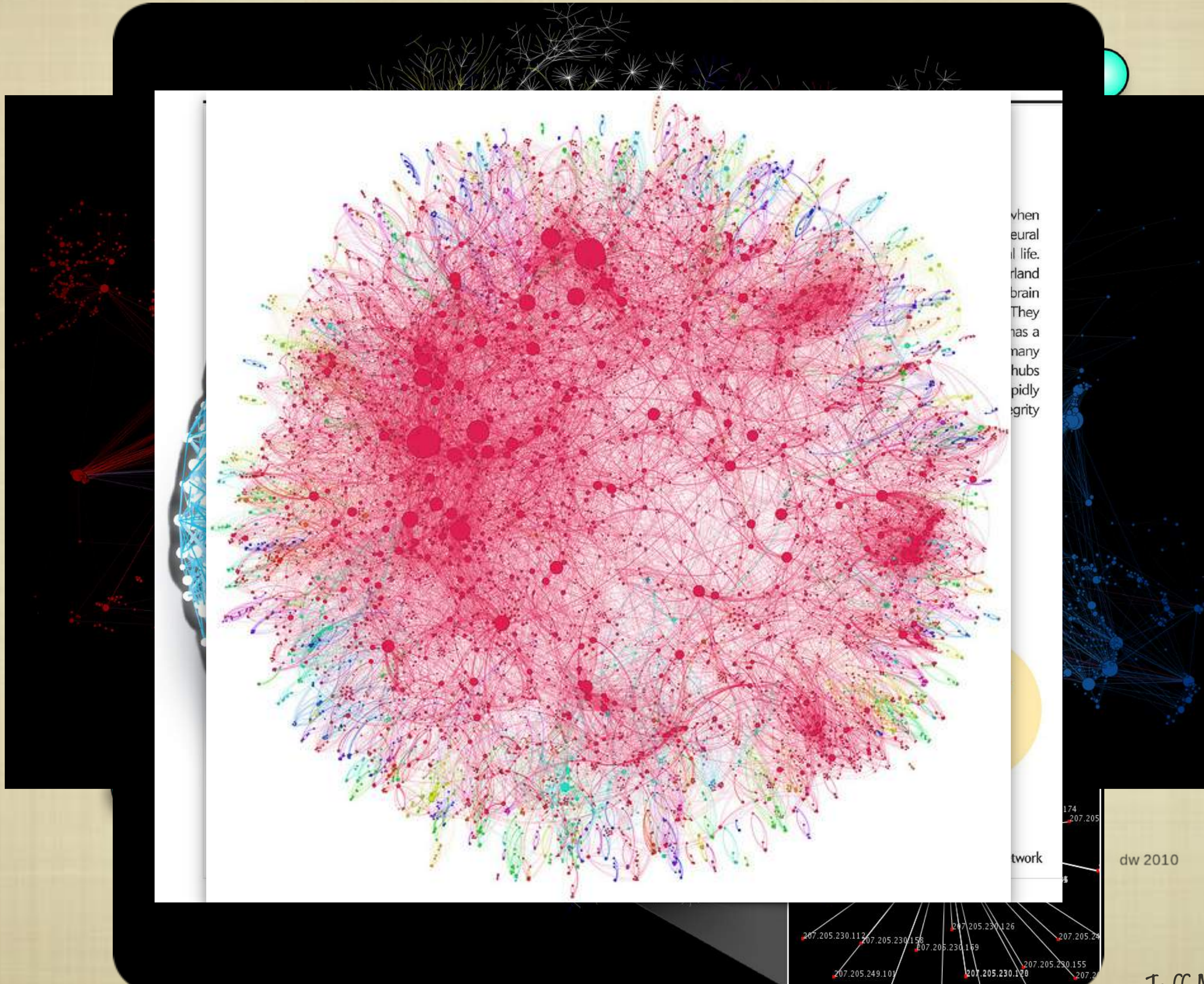
Milgram's team tracked the progress of the letter by having participants send a postcard directly to the researchers when they mailed the letter. This determined the links and nodes in the **social network**

How connected are we really?

What they found was that:

- With a small variance, the letter reached the target by 6 mailings
- The social network exhibited a **small world phenomenon** in that complete strangers were connected by a short acquaintance chain.
- This connectivity was mediated by a number of highly connected study participants

Six Degrees of Separation



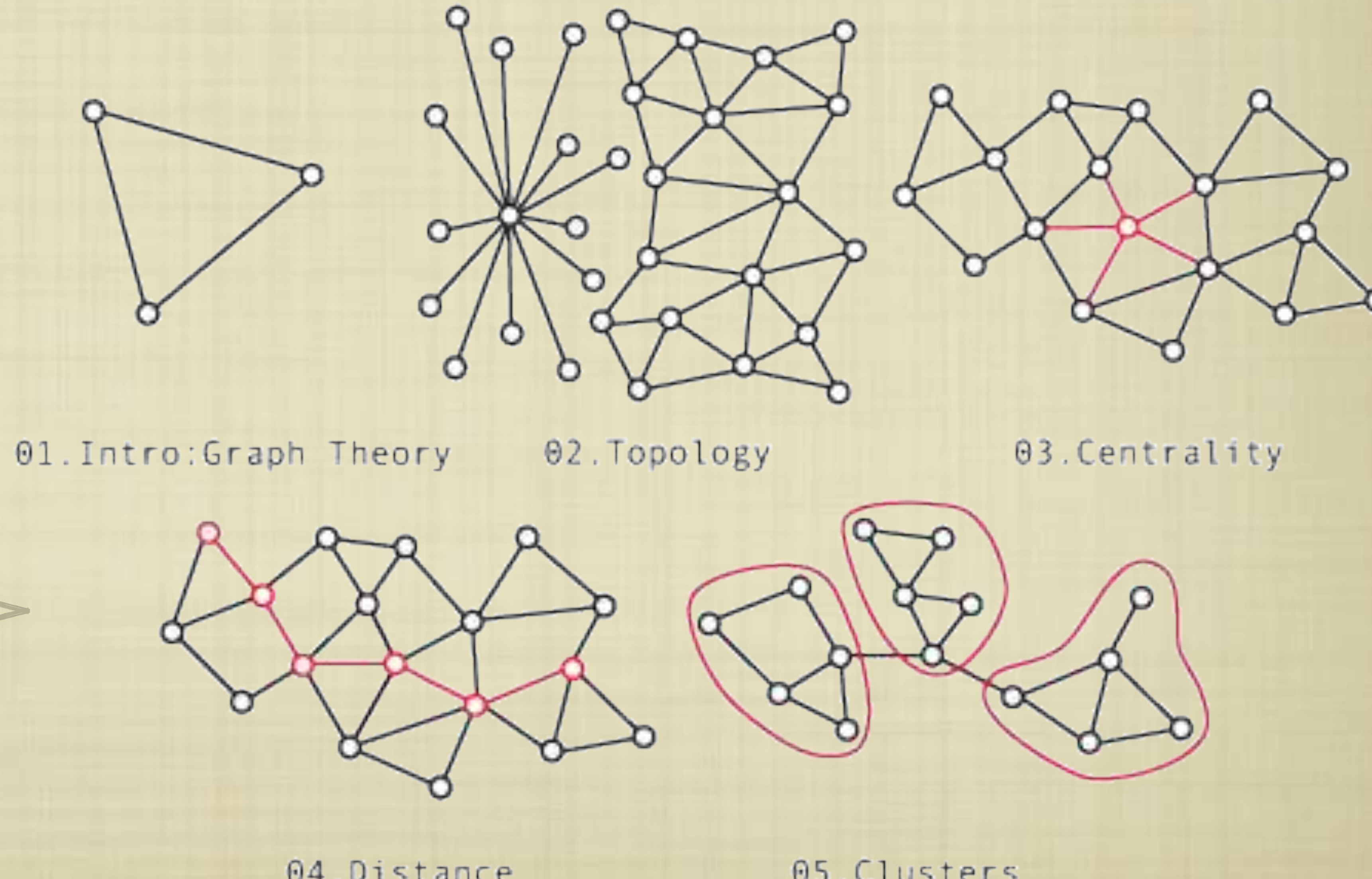
Jeff Murugan (UCT)

A Network Theory Primer

[Watts '99]

- A **graph** or **network** $G(V,E)$ is a collection of vertices $V=\{v_I, I=1\ldots N\}$ and **edges** $E\subseteq V\otimes V$ organised in a particular way.

- The organisation can be **regular** like a lattice or **random** like an Erdös-Rényi graph
- The organisation can be **sparse** or **dense.**
- If every node is connected to every other node, the graph is called **complete.**

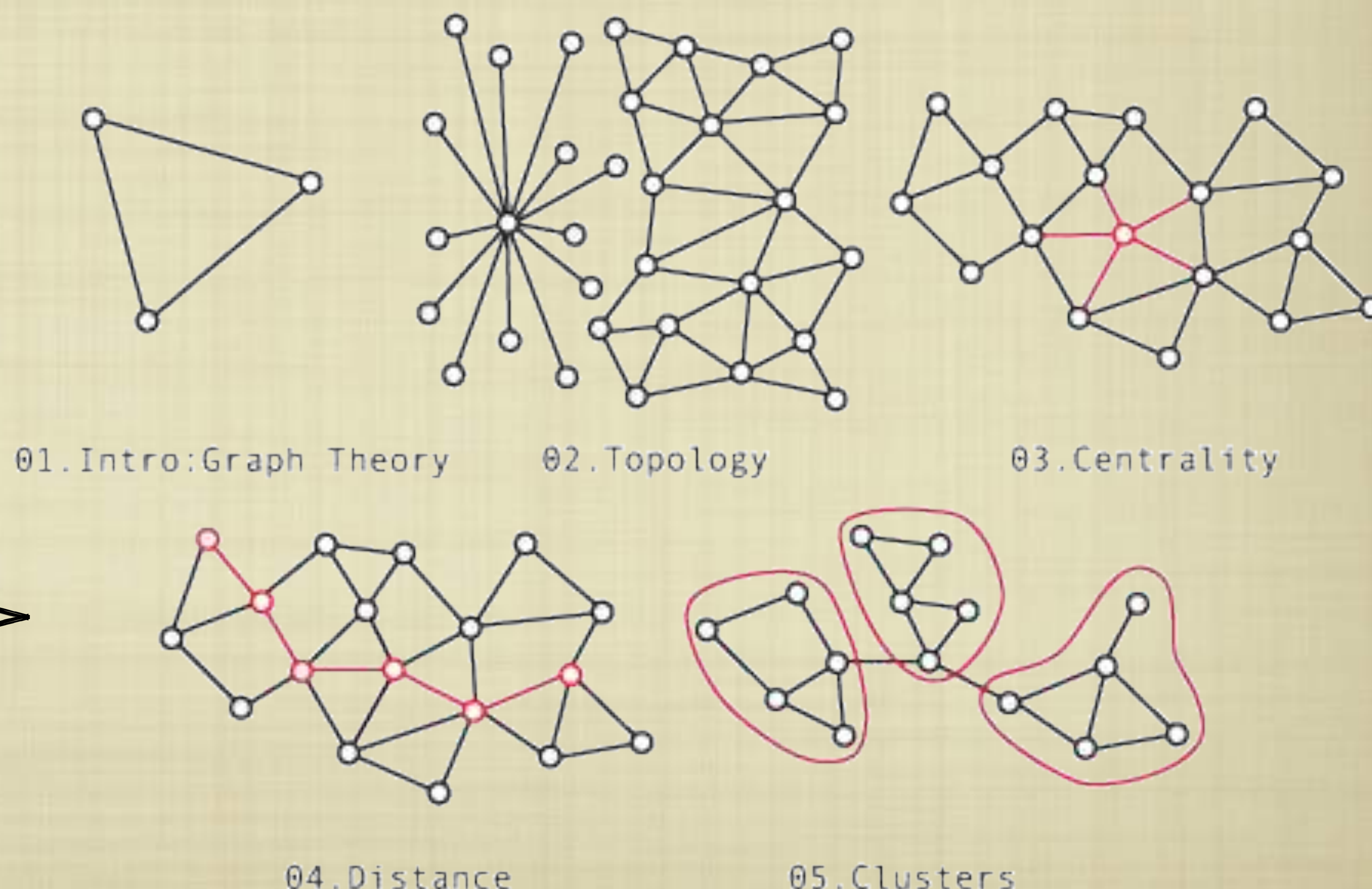


A Network Theory Primer

[Watts '99]

- A **graph** or **network** $G(V,E)$ is a collection of vertices $V=\{v_I, I=1\dots N\}$ and **edges** $E\subseteq V\otimes V$ organised in a particular way.

- The organisation can be **regular** like a lattice or **random** like an Erdös-Rényi graph
- The organisation can be **sparse** or **dense.**
- If every node is connected to every other node, the graph is called **complete.**



Network Properties

[Newman '18]

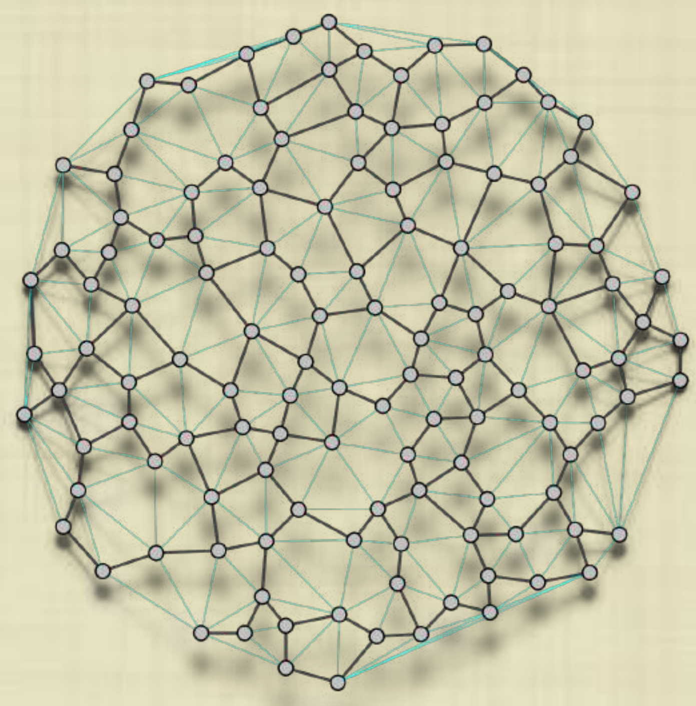
- The **degree of a node** is the number of edges connected to it and can be computed from the adjacency metric as $k_i = \sum_j A_{ij}$

- The **adjacency matrix**

$$A_{ij} = \begin{cases} 1 & (i,j) \in E \\ 0 & (i,j) \notin E \end{cases}$$

gives an unambiguous representation of any simple network

- Allows for a formulation of network properties in terms of matrix algebra.



- The **network Laplacian** generalises the idea of the usual Laplacian to a network as

$$L_{ij} = \begin{cases} k_i & i = j \\ -1 & i \neq j, (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- It can be related to the adjacency matrix through

$$L_{ij} = k_i \delta_{ij} - A_{ij}$$

- Useful for the **partitioning** of the network.

- The **spectrum** $\sigma(G)$ of the adjacency matrix is a network invariant that solves the characteristic polynomial

$$\det(\lambda I - A) = 0$$

Network Properties

[Newman '18]

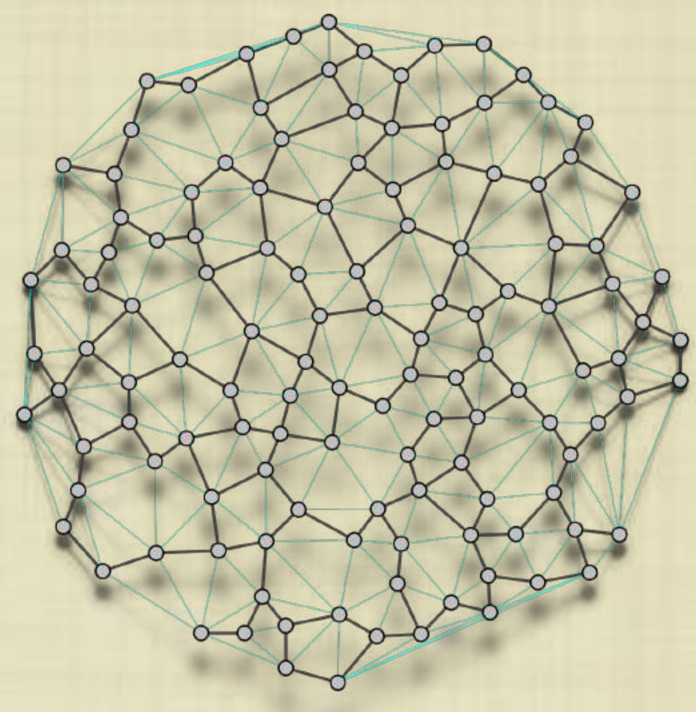
- The **degree of a node** is the number of edges connected to it and can be computed from the adjacency metric as $k_i = \sum_j A_{ij}$

- The **adjacency matrix**

$$A_{ij} = \begin{cases} 1 & (i,j) \in E \\ 0 & (i,j) \notin E \end{cases}$$

gives an unambiguous representation of any simple network

- Allows for a formulation of network properties in terms of matrix algebra.



- The **network Laplacian** generalises the idea of the usual Laplacian to a network as

$$L_{ij} = \begin{cases} k_i & i = j \\ -1 & i \neq j, (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- It can be related to the adjacency matrix through

$$L_{ij} = k_i \delta_{ij} - A_{ij}$$

- Useful for the **partitioning** of the network.

- The **spectrum** $\sigma(G)$ of the adjacency matrix is a network invariant that solves the characteristic polynomial

$$\det(\lambda I - A) = 0$$

Network Properties

[Newman '18]

- The **degree of a node** is the number of edges connected to it and can be computed from the adjacency metric as $k_i = \sum_j A_{ij}$

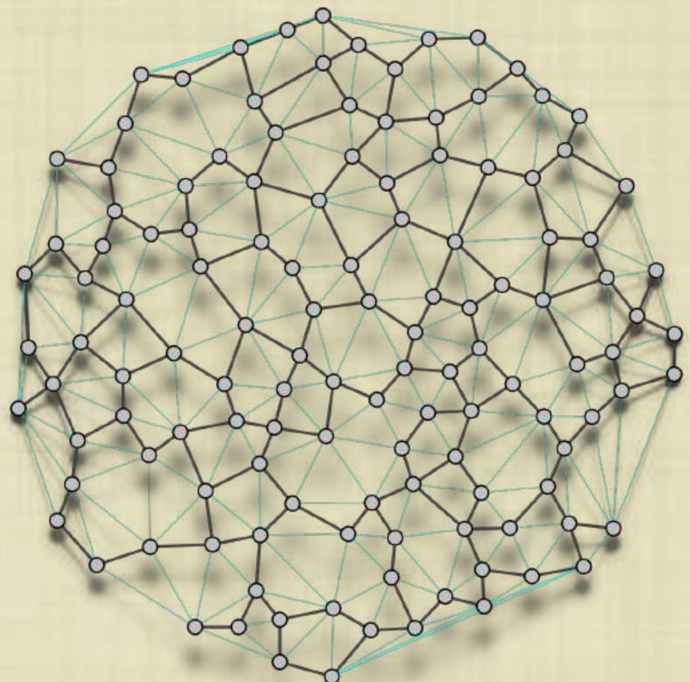
- The **network Laplacian** generalises the idea of the usual Laplacian to a network as

$$L_{ij} = \begin{cases} k_i & i = j \\ -1 & i \neq j, (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- It can be related to the adjacency matrix through

$$L_{ij} = k_i \delta_{ij} - A_{ij}$$

- Useful for the **partitioning** of the network.



- The **adjacency matrix**

$$A_{ij} = \begin{cases} 1 & (i,j) \in E \\ 0 & (i,j) \notin E \end{cases}$$

gives an unambiguous representation of any simple network

- Allows for a formulation of network properties in terms of matrix algebra.

- The **spectrum** $\sigma(G)$ of the adjacency matrix is a network invariant that solves the characteristic polynomial

$$\det(\lambda I - A) = 0$$

Jeff Murugan (UCT)

Network Properties

[Newman '18]

- The **degree of a node** is the number of edges connected to it and can be computed from the adjacency metric as $k_i = \sum_j A_{ij}$

- The **adjacency matrix**

$$A_{ij} = \begin{cases} 1 & (i,j) \in E \\ 0 & (i,j) \notin E \end{cases}$$

gives an unambiguous representation of any simple network

- Allows for a formulation of network properties in terms of matrix algebra.

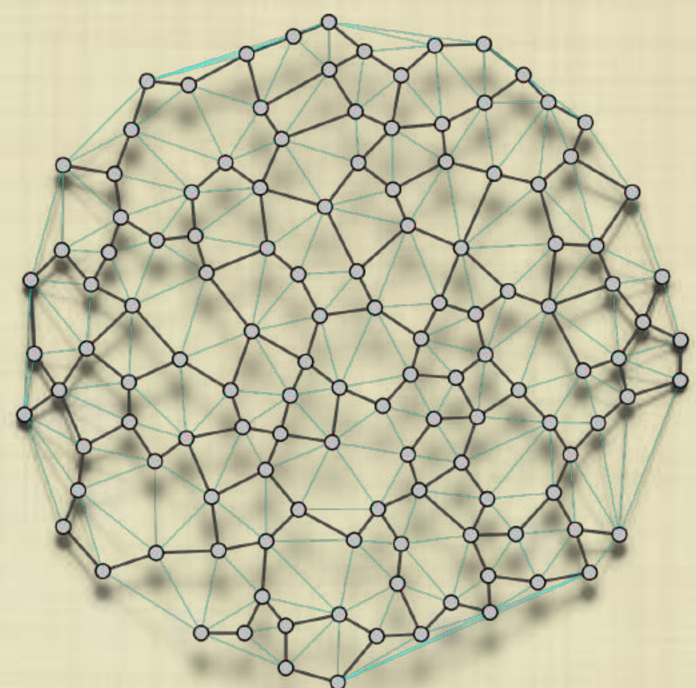
- The **network Laplacian** generalises the idea of the usual Laplacian to a network as

$$L_{ij} = \begin{cases} k_i & i = j \\ -1 & i \neq j, (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- It can be related to the adjacency matrix through

$$L_{ij} = k_i \delta_{ij} - A_{ij}$$

- Useful for the **partitioning** of the network.

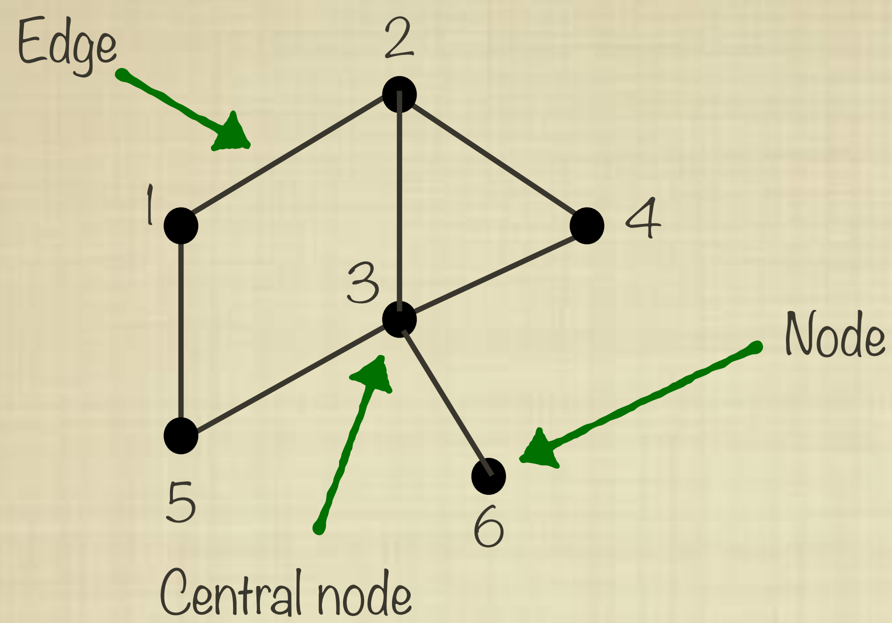


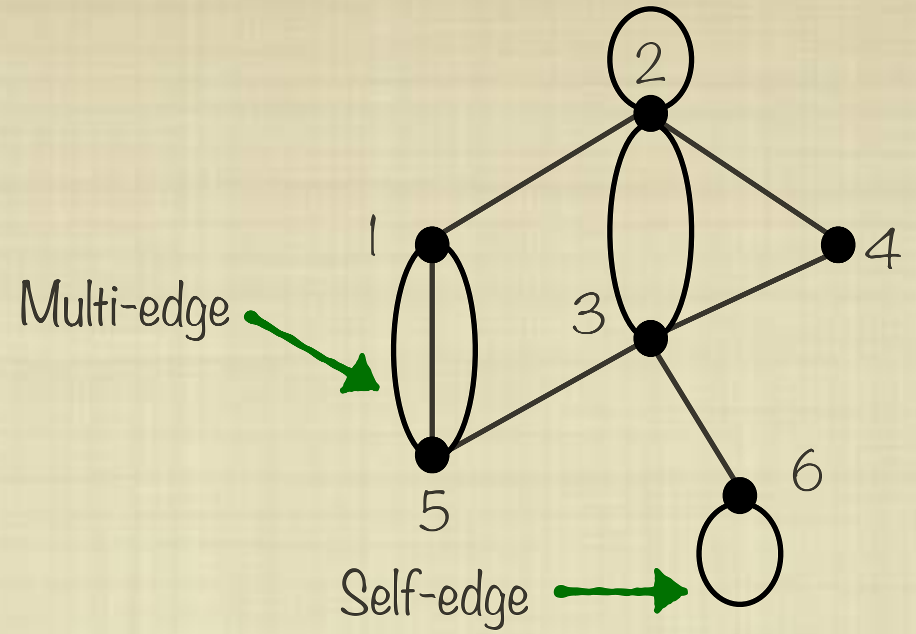
- The **spectrum** $\sigma(G)$ of the adjacency matrix is a network invariant that solves the characteristic polynomial

$$\det(\lambda I - A) = 0$$

Some Examples

[Newman '18]



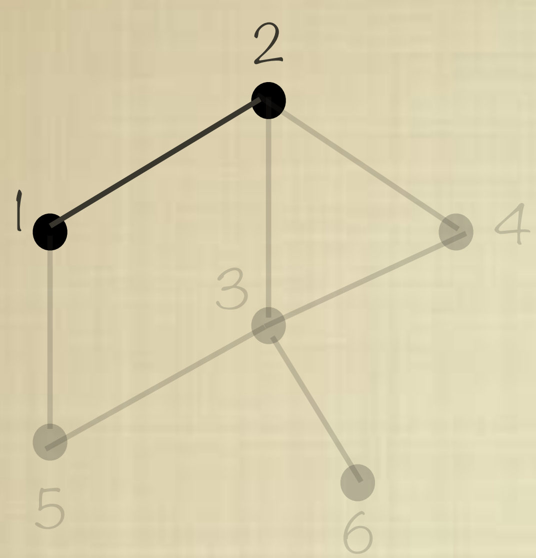


$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

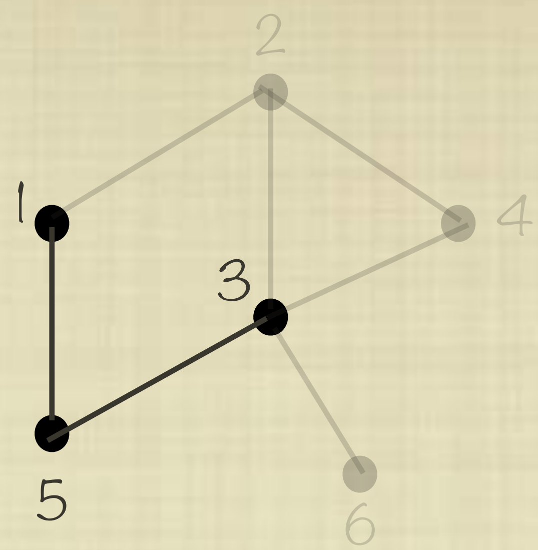
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 3 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$$L = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & -1 & 2 & 0 & 0 \\ -1 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

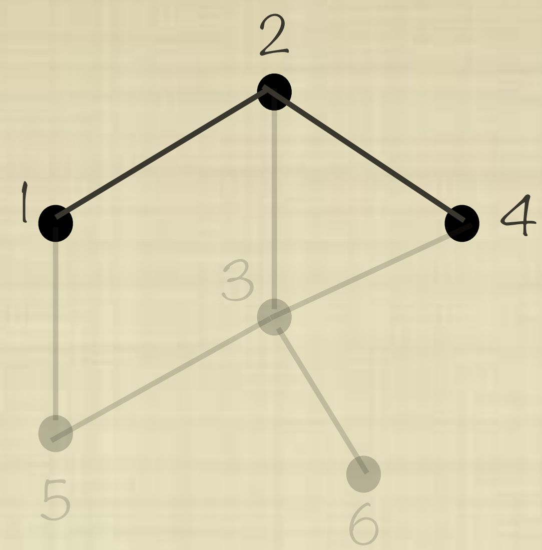
Path length



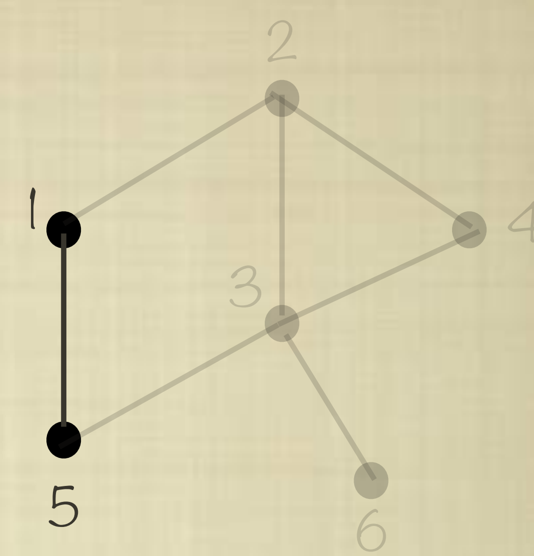
$d_{12} = 1$



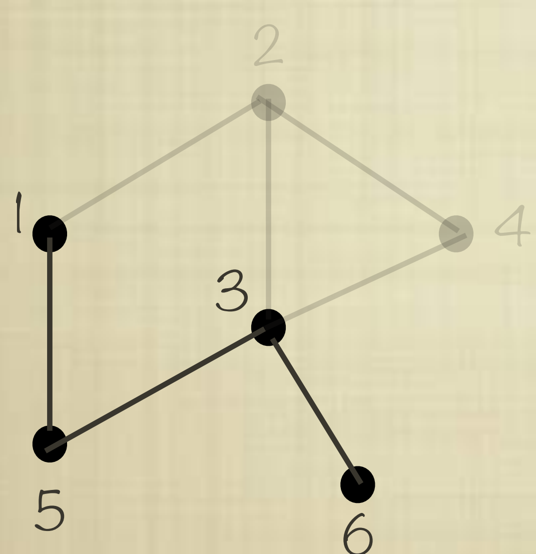
$d_{13} = 2$



$d_{14} = 2$



$d_{15} = 1$



$d_{16} = 3$

- Mean distance between node 1 and all other nodes:

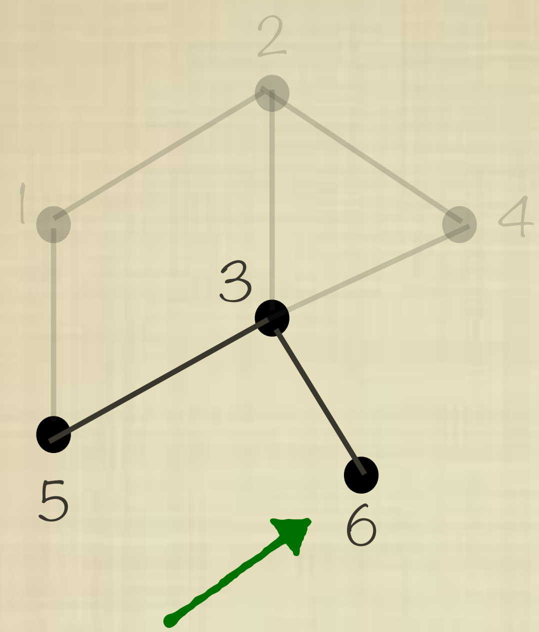
$$l_1 = \frac{1}{6}\sum_{j=1}^{6} d_{1j} = \frac{3}{2}$$

- **Mean distance** between nodes for the whole network:

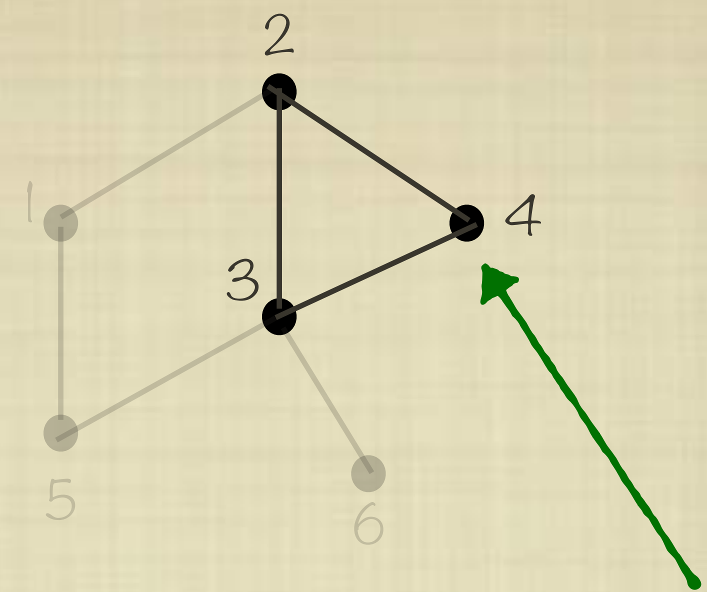
$$\langle l \rangle = \frac{1}{n}\sum_{i=1}^{n} l_i = \frac{4}{3}$$

Jeff Murugan (UCT)

Transitivity & Clustering



The friend of my friend is not necessarily my friend



The friend of my friend is likely to be my friend

- Paths of length 2 and are closed form a loop. Networks with many loops are highly clustered.
- We quantify the clustering of a network by the **clustering coefficient**

$$C = \frac{\text{\# of closed paths of length 2}}{\text{\# of paths of length 2}} = \frac{6 \times (\text{\# of triangles})}{\text{\# of paths of length 2}} = \frac{3 \times (\text{\# of triangles})}{\text{\# of connected triples}}$$

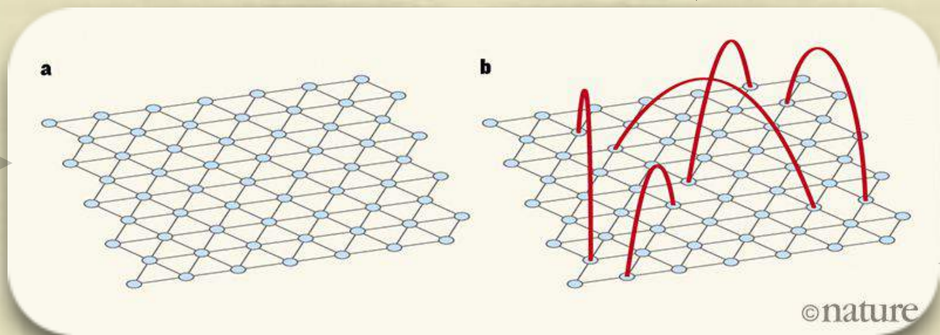
Small-World Networks

[Watts-Strogatz '98, Watts '99]

An N-node **small world network** is a graph in which:

- the **typical distance** between two randomly selected nodes in the network $L = \sum_{i \neq j} d_{ij}/(N^2 - N) \sim \log N$
- there is a large degree of **clustering**.

Small world networks interpolate between the clustering (**localising**) properties of regular graphs and the **rapid spreading** of information in random networks.



Scale-free networks are a special class of small word graphs that proliferate a large number of **hubs.** As a result, the mean path length are significantly shorter and scale like $L \sim \log \log N$

Small worldness of a graph can be measured by:

- The smallness coefficient $\sigma = (C/C_r)/(L/L_r)$ which is >1 for a small world network but very dependent on the network size.
- The small world parameter $\omega = 1 - |(L_r/L - C/C_l)|$ which ranges between 0 (regular) and 1 (small world)

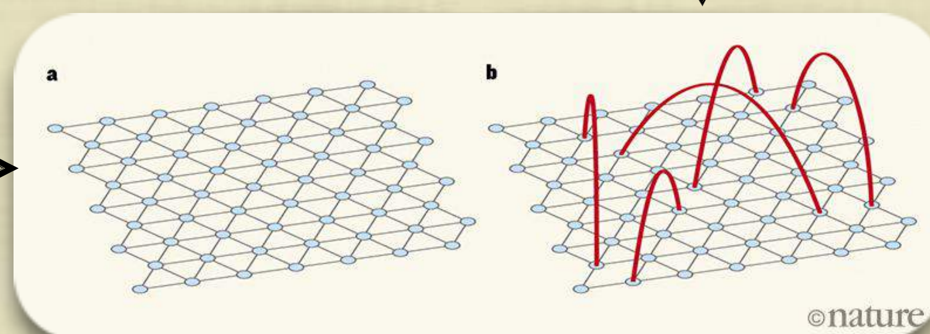
Small-World Networks

[Watts-Strogatz '98, Watts '99]

An N-node **small world network** is a graph in which:

- the **typical distance** between two randomly selected nodes in the network $L = \sum_{i \neq j} d_{ij}/(N^2 - N) \sim \log N$
- there is a large degree of **clustering**.

Small world networks interpolate between the clustering (**localising**) properties of regular graphs and the **rapid spreading** of information in random networks.



Scale-free networks are a special class of small word graphs that proliferate a large number of **hubs.** As a result, the mean path length are significantly shorter and scale like $L \sim \log \log N$

Small worldness of a graph can be measured by:

- The smallness coefficient $\sigma = (C/C_r)/(L/L_r)$ which is >1 for a small world network but very dependent on the network size.
- The small world parameter $\omega = 1 - |(L_r/L - C/C_l)|$ which ranges between 0 (regular) and 1 (small world)

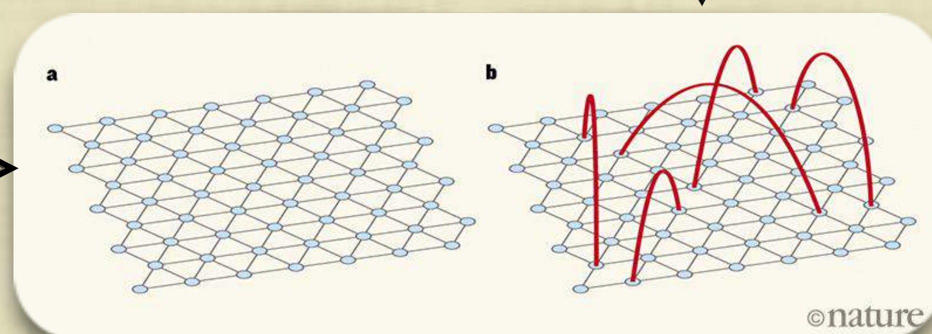
Small-World Networks

[Watts-Strogatz '98, Watts '99]

Small world networks interpolate between the clustering (**localising**) properties of regular graphs and the **rapid spreading** of information in random networks.

An N-node **small world network** is a graph in which:

- the **typical distance** between two randomly selected nodes in the network $L = \sum_{i \neq j} d_{ij}/(N^2 - N) \sim \log N$
- there is a large degree of **clustering**.



Scale-free networks are a special class of small word graphs that proliferate a large number of **hubs.** As a result, the mean path length are significantly shorter and scale like $L \sim \log \log N$

Small worldness of a graph can be measured by:

- The smallness coefficient $\sigma = (C/C_r)/(L/L_r)$ which is >1 for a small world network but very dependent on the network size.
- The small world parameter $\omega = 1 - |(L_r/L - C/C_l)|$ which ranges between 0 (regular) and 1 (small world)

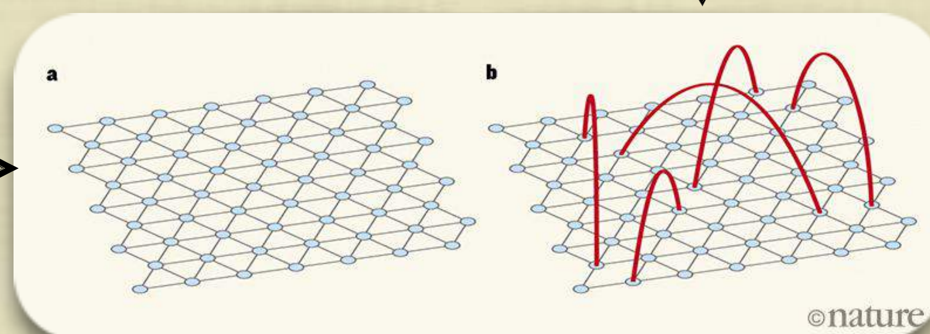
Small-World Networks

[Watts-Strogatz '98, Watts '99]

An N-node **small world network** is a graph in which:

- the **typical distance** between two randomly selected nodes in the network $L = \sum_{i \neq j} d_{ij}/(N^2 - N) \sim \log N$
- there is a large degree of **clustering**.

Small world networks interpolate between the clustering (**localising**) properties of regular graphs and the **rapid spreading** of information in random networks.



Scale-free networks are a special class of small word graphs that proliferate a large number of **hubs.** As a result, the mean path length are significantly shorter and scale like $L \sim \log \log N$

Small worldness of a graph can be measured by:

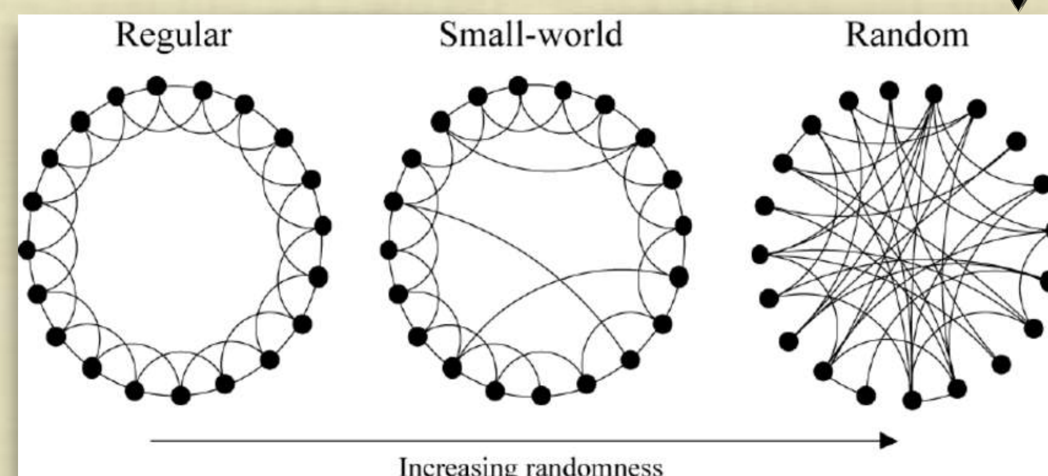
- The smallness coefficient $\sigma = (C/C_r)/(L/L_r)$ which is >1 for a small world network but very dependent on the network size.
- The small world parameter $\omega = 1 - |(L_r/L - C/C_l)|$ which ranges between 0 (regular) and 1 (small world)

The Watts-Strogatz Protocol

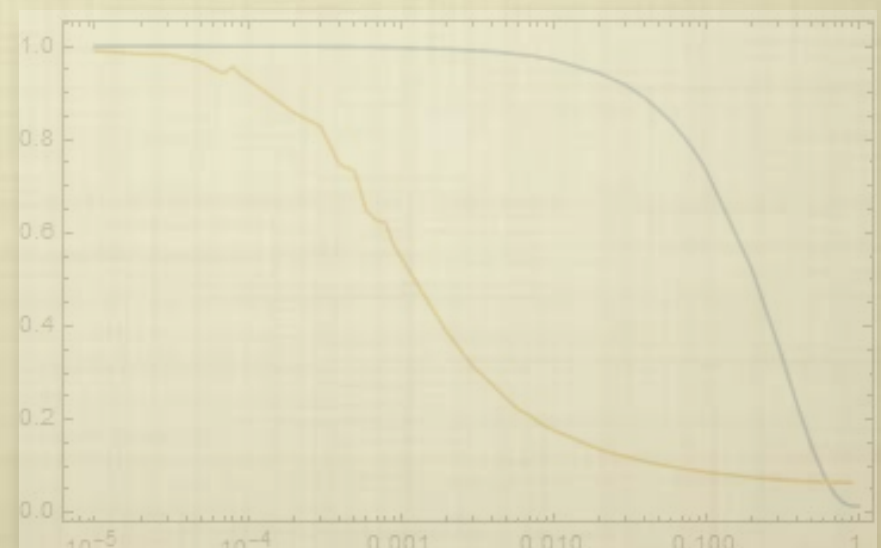
[Watts-Strogatz '98]

- Start with a regular **N-node lattice** with k/2-nearest-neighbour edges.
- At each node n_i :
  - Iterate through each edge (i,j) connecting n_i to $n_j \neq n_i$
  - With probability p, rewire the edge by replacing (i,j) with a random (i,k)

The resulting small world network inherits its **clustering properties** from the underlying lattice and its short path length from the random **long-range connections.**



- The **clustering coefficient** $C_i = 2E_i/(k_i(k_i-1))$ measures how cliquey the graph is.
- The **path length** $L = \sum_{i\neq j} d_{ij}/(N(N-1))$ of the network is the average of the shortest geodesic between any two nodes.

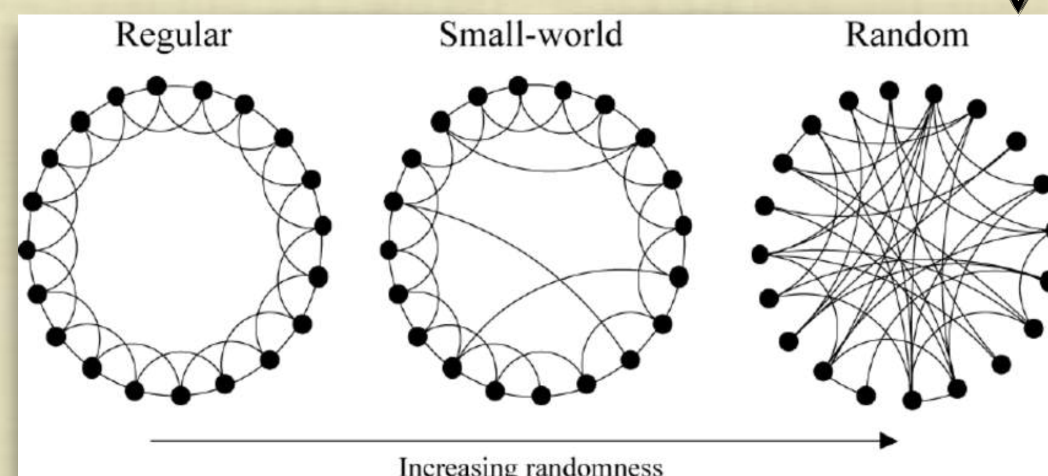


The Watts-Strogatz Protocol

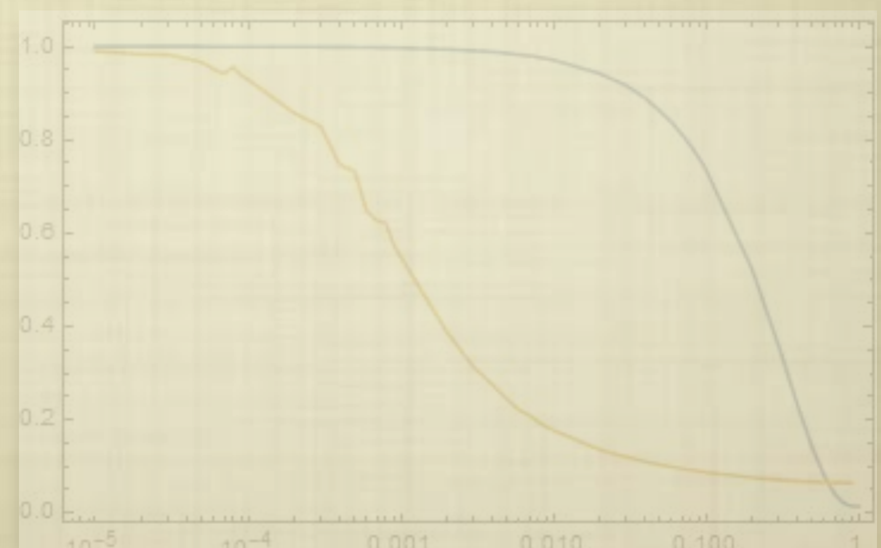
[Watts-Strogatz '98]

The resulting small world network inherits its **clustering properties** from the underlying lattice and its short path length from the random **long-range connections.**

- Start with a regular **N-node lattice** with k/2-nearest-neighbour edges.
- At each node n_i :
  - Iterate through each edge (i,j) connecting n_i to $n_j \neq n_i$
  - With probability p, rewire the edge by replacing (i,j) with a random (i,k)

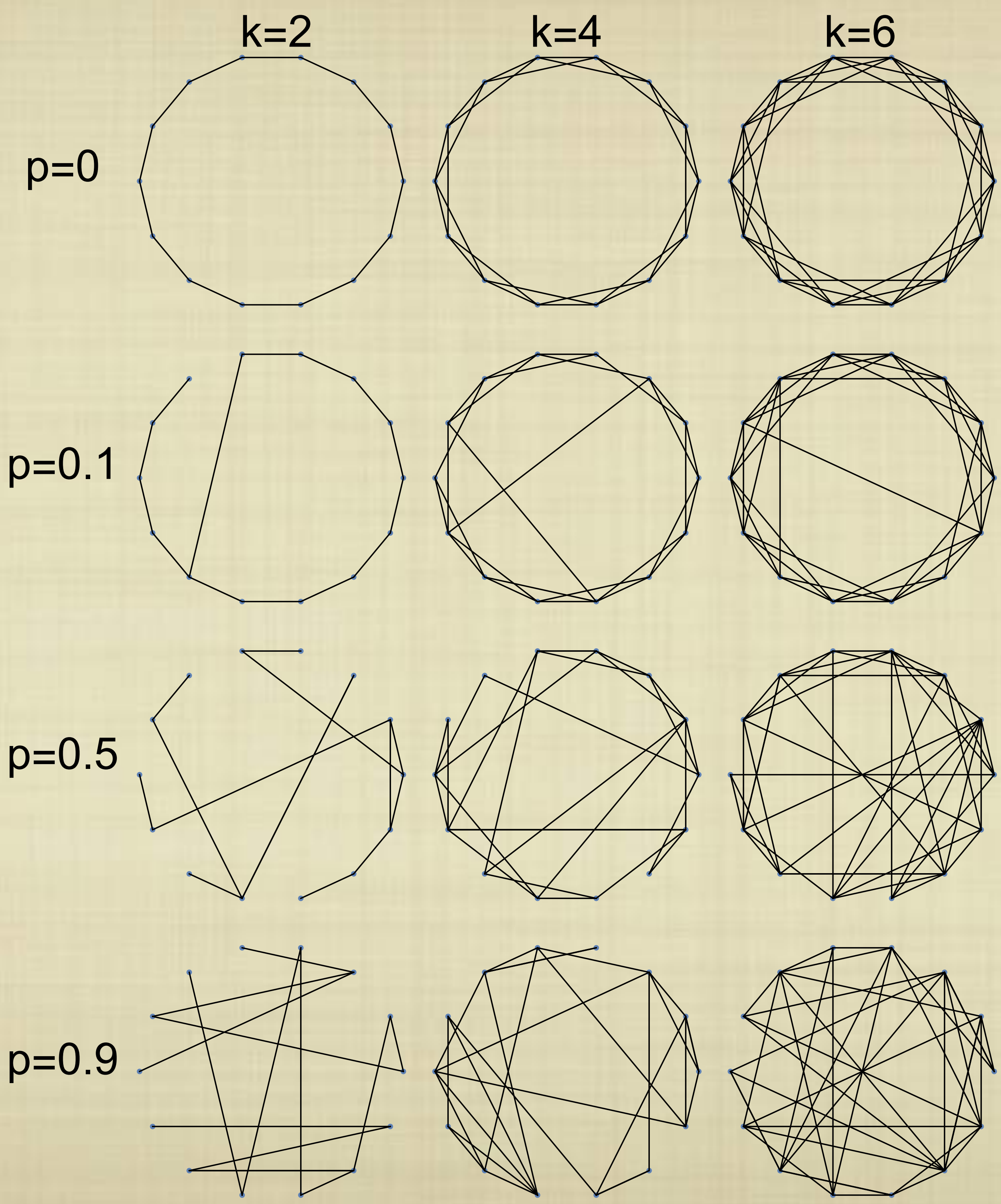


- The **clustering coefficient** $C_i = 2E_i/(k_i(k_i-1))$ measures how cliquey the graph is.
- The **path length** $L = \sum_{i\neq j} d_{ij}/(N(N-1))$ of the network is the average of the shortest geodesic between any two nodes.



The Watts-Strogatz Protocol

[Watts-Strogatz '98]

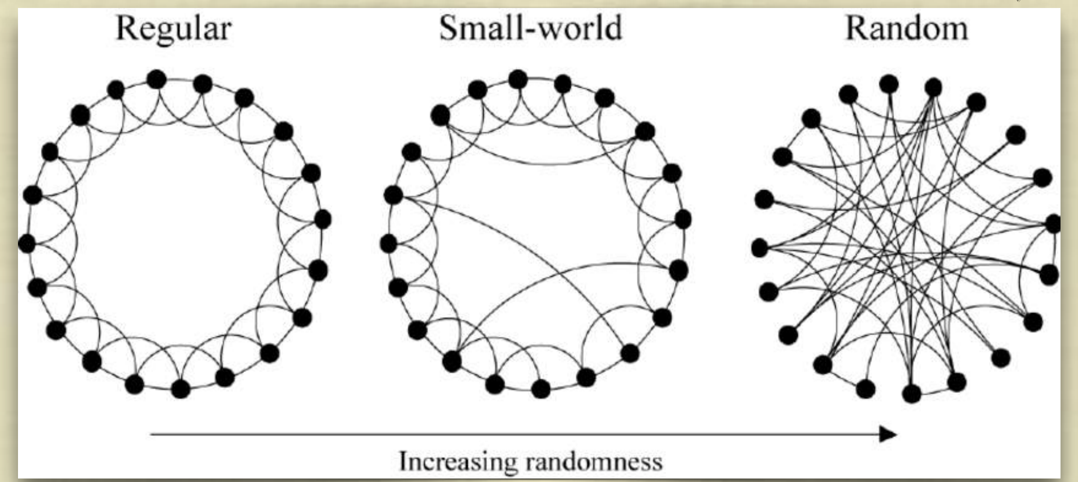


The Watts-Strogatz Protocol

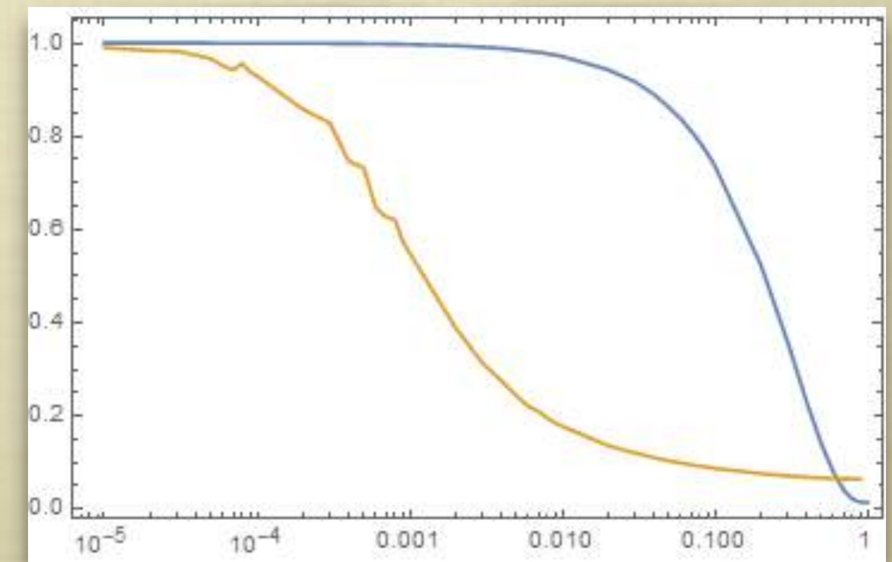
[Watts-Strogatz '98]

The resulting small world network inherits its **clustering properties** from the underlying lattice and its short path length from the random **long-range connections.**

- Start with a regular **N-node lattice** with k/2-nearest-neighbour edges.
- At each node n_i :
  - Iterate through each edge (i,j) connecting n_i to $n_j \neq n_i$
  - With probability p, rewire the edge by replacing (i,j) with a random (i,k)



- The **clustering coefficient** $C_i = 2E_i/(k_i(k_i-1))$ measures how cliquey the graph is.
- The **path length** $L = \sum_{i\neq j} d_{ij}/(N(N-1))$ of the network is the average of the shortest geodesic between any two nodes.



Jeff Murugan (UCT)

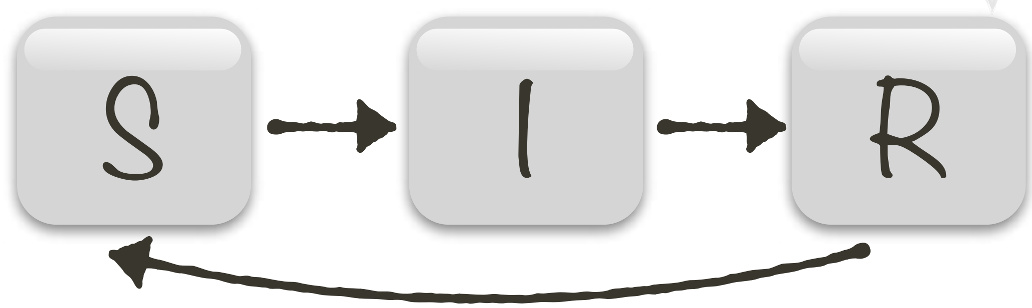
Epidemic Spreading in a Small World

[Moore-Newman '00]

- Traditional epidemic models **compartmentalise** the population into **S**usceptible, **I**nfectious and **R**emoved interacting in a fully mixed way:

$$\dot{s} = -\beta s x \qquad \dot{x} = \beta s x - \gamma x \qquad \dot{r} = \gamma x$$

subject to $s + x + r = 1$



- The SIR model can be solved as a quadrature

$$t = \frac{1}{\gamma}\int_0^r \frac{du}{1 - u - s_0 e^{-\beta u/\gamma}}$$

- At late times

$$r \to 1 - e^{-\beta r/\gamma}$$

- The critical point $\beta = \gamma$ defines the **epidemic threshold** below which there is no epidemic.
- In the SIR model, the **basic reproduction number**

$$R_0 = \beta\gamma \int_0^\infty d\tau\, \tau e^{-\gamma\tau} = \frac{\beta}{\gamma}$$

- Setting $\gamma = 0$ in the SIR model reduces to an SI model with **logistic growth**

$$x(t) = \frac{x_0 e^{\beta t}}{1 - x_0 + x_0 e^{\beta t}}$$

of infected individuals which **always** results in an epidemic.

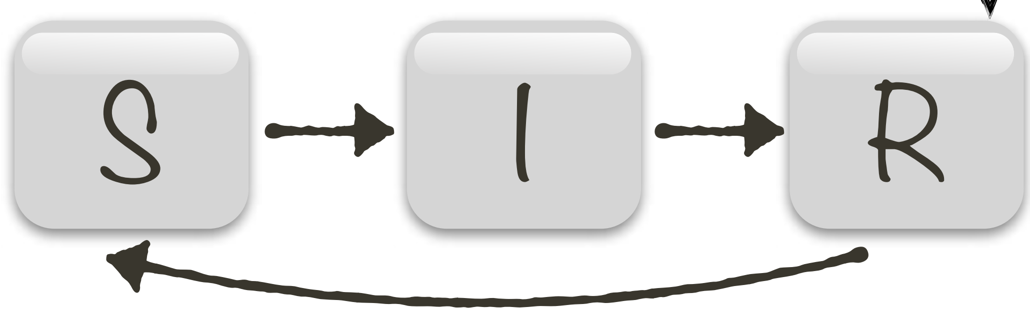
Epidemic Spreading in a Small World

[Moore-Newman '00]

- Traditional epidemic models **compartmentalise** the population into **S**usceptible, **I**nfectious and **R**emoved interacting in a fully mixed way:

$$\dot{s} = -\beta s x \qquad \dot{x} = \beta s x - \gamma x \qquad \dot{r} = \gamma x$$

subject to $s + x + r = 1$



- The SIR model can be solved as a quadrature

$$t = \frac{1}{\gamma}\int_0^r \frac{du}{1 - u - s_0 e^{-\beta u/\gamma}}$$

- At late times

$$r \to 1 - e^{-\beta r/\gamma}$$

- The critical point $\beta = \gamma$ defines the **epidemic threshold** below which there is no epidemic.
- In the SIR model, the **basic reproduction number**

$$R_0 = \beta\gamma \int_0^\infty d\tau\, \tau e^{-\gamma\tau} = \frac{\beta}{\gamma}$$

- Setting $\gamma = 0$ in the SIR model reduces to an SI model with **logistic growth**

$$x(t) = \frac{x_0 e^{\beta t}}{1 - x_0 + x_0 e^{\beta t}}$$

of infected individuals which **always** results in an epidemic.

Epidemic Spreading in a Small World

[Moore-Newman '00]

- The SIR model can be solved as a quadrature

$$t = \frac{1}{\gamma}\int_0^r \frac{du}{1-u-s_0 e^{-\beta u/\gamma}}$$

- At late times

$$r \to 1 - e^{-\beta r/\gamma}$$

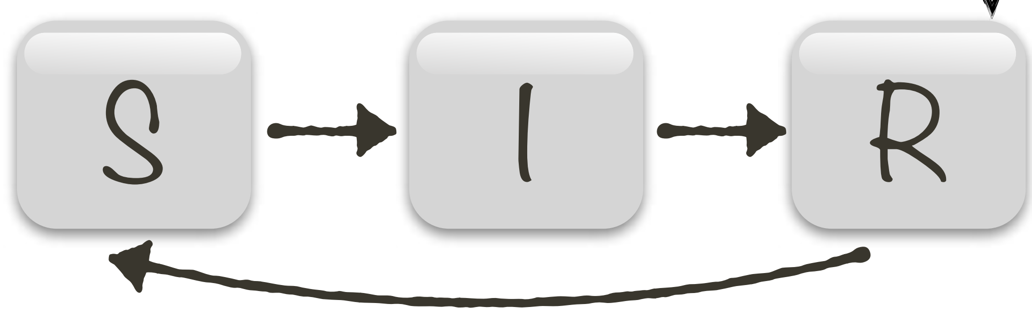
- The critical point $\beta = \gamma$ defines the **epidemic threshold** below which there is no epidemic.
- In the SIR model, the **basic reproduction number**

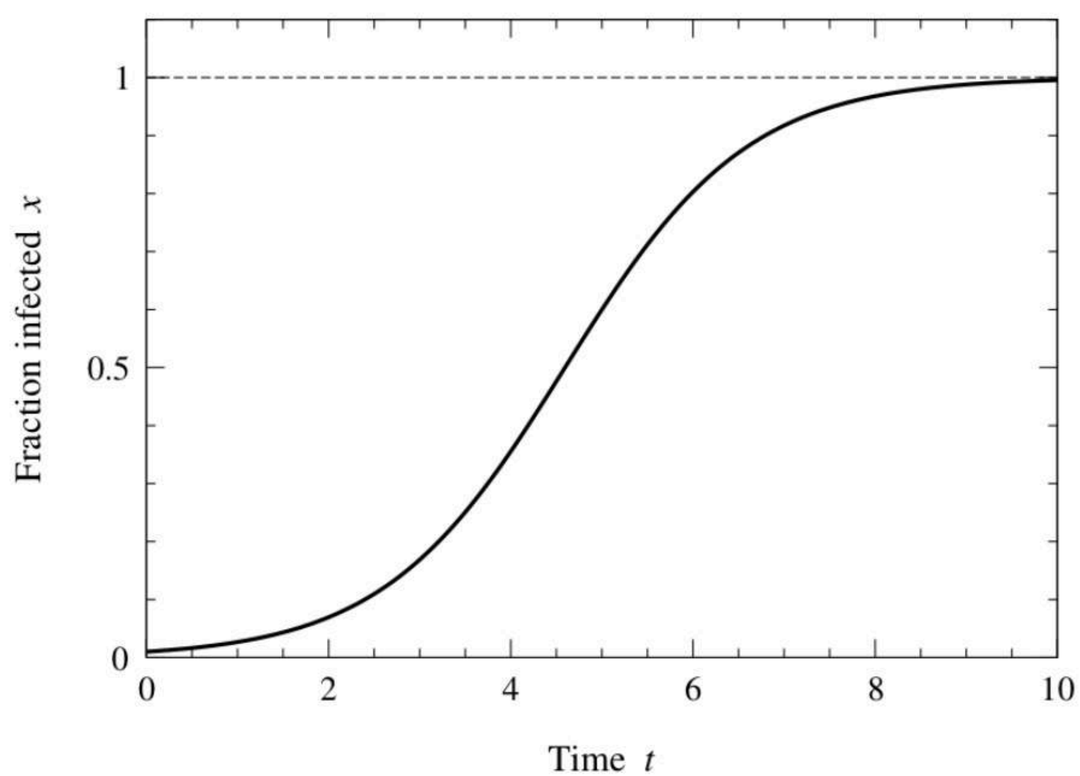
$$R_0 = \beta\gamma\int_0^\infty d\tau\, \tau e^{-\gamma\tau} = \frac{\beta}{\gamma}$$

- Traditional epidemic models **compartmentalise** the population into **S**usceptible, **I**nfectious and **R**emoved interacting in a fully mixed way:

$$\dot{s} = -\beta s x \qquad \dot{x} = \beta s x - \gamma x \qquad \dot{r} = \gamma x$$

subject to $s + x + r = 1$





- Setting $\gamma = 0$ in the SIR model reduces to an SI model with **logistic growth**

$$x(t) = \frac{x_0 e^{\beta t}}{1 - x_0 + x_0 e^{\beta t}}$$

of infected individuals which **always** results in an epidemic.

Epidemic Spreading in a Small World

[Moore-Newman '00]

- The SIR model can be solved as a quadrature

$$t = \frac{1}{\gamma}\int_0^r \frac{du}{1-u-s_0 e^{-\beta u/\gamma}}$$

- At late times

$$r \to 1 - e^{-\beta r/\gamma}$$

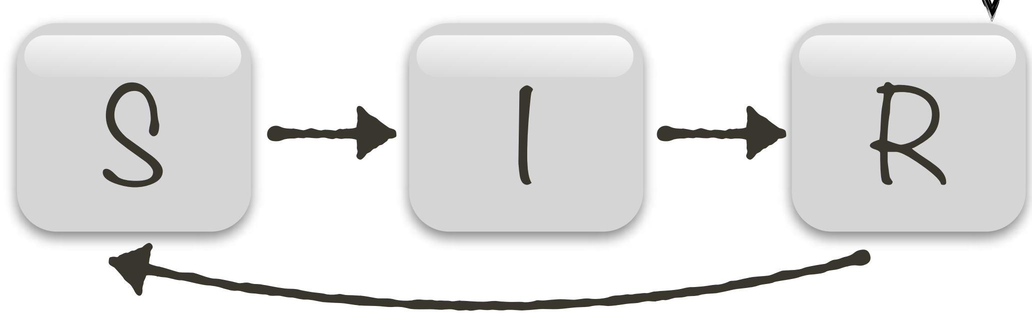
- The critical point $\beta = \gamma$ defines the **epidemic threshold** below which there is no epidemic.
- In the SIR model, the **basic reproduction number**

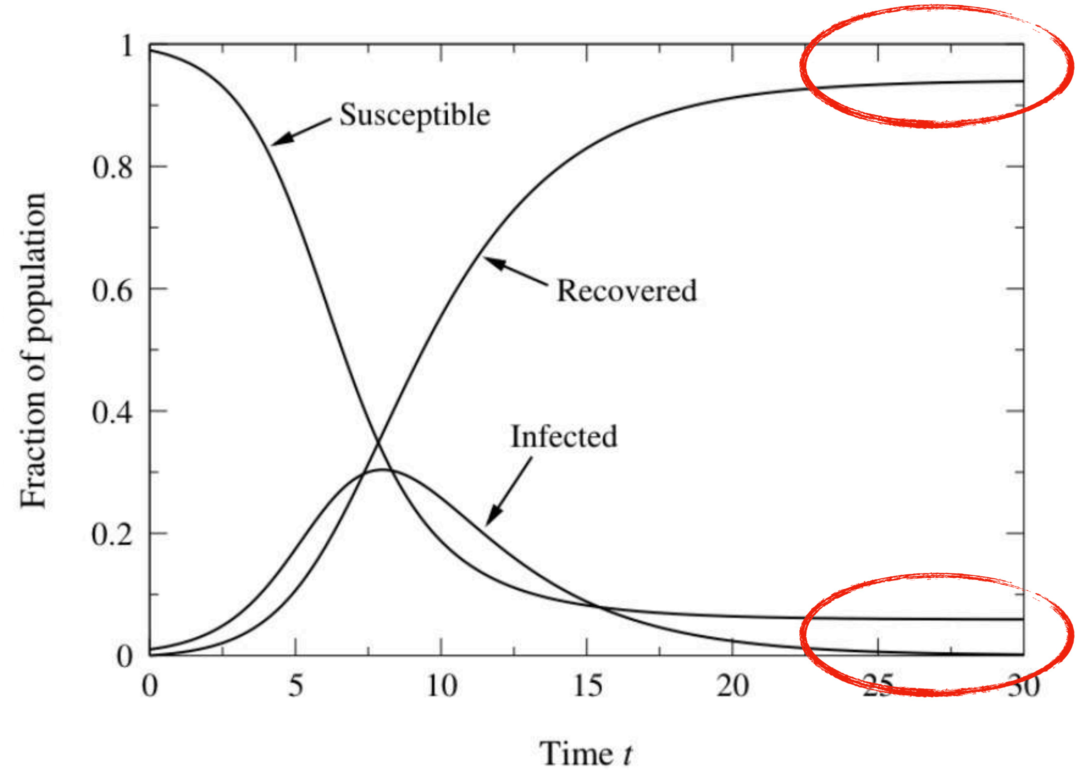
$$R_0 = \beta\gamma\int_0^\infty d\tau\,\tau e^{-\gamma\tau} = \frac{\beta}{\gamma}$$

- Traditional epidemic models **compartmentalise** the population into **S**usceptible, **I**nfectious and **R**emoved interacting in a fully mixed way:

$$\dot{s} = -\beta s x \qquad \dot{x} = \beta s x - \gamma x \qquad \dot{r} = \gamma x$$

subject to $s + x + r = 1$





- Setting $\gamma = 0$ in the SIR model reduces to an SI model with **logistic growth**

$$x(t) = \frac{x_0 e^{\beta t}}{1 - x_0 + x_0 e^{\beta t}}$$

of infected individuals which **always** results in an epidemic.

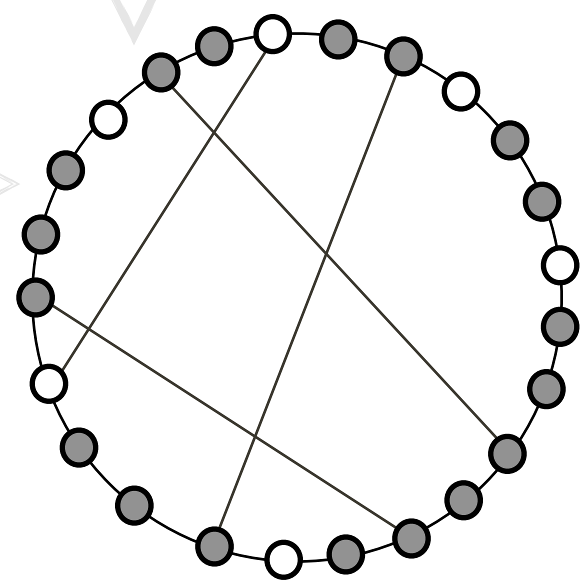
Jeff Murugan (UCT)

Epidemic Spreading in a Small World

- Important epidemiological parameters are **susceptibility** of the population and **transmissibility** of the disease.
- When an epidemic takes place can be mapped to a standard **percolation** problem on a small-world network!

[Moore-Newman '00]

- Numerical results show that:
  - Epidemics spread more rapidly in highly **susceptible populations**
  - The amount of time for an epidemic to spread is given by the average **radius of connected clusters** of susceptible individuals.
  - The infection **curve flattens** as ϕ is increased.
  - No epidemic outbreak takes place below the **percolation threshold.**

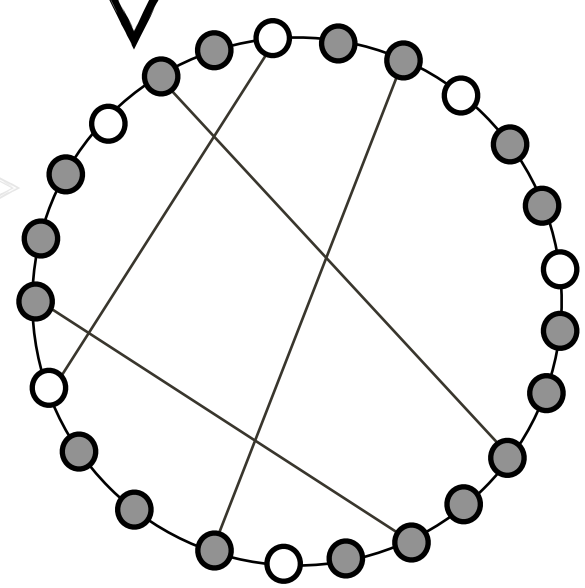


- A uniform probability of interaction corresponds to a **random network**.
- Real social networks exhibit **clustering**: two people are more likely to know each other if they have a common acquaintance.

Epidemic Spreading in a Small World

[Moore-Newman '00]

- Important epidemiological parameters are **susceptibility** of the population and **transmissibility** of the disease.
- When an epidemic takes place can be mapped to a standard **percolation** problem on a small-world network!

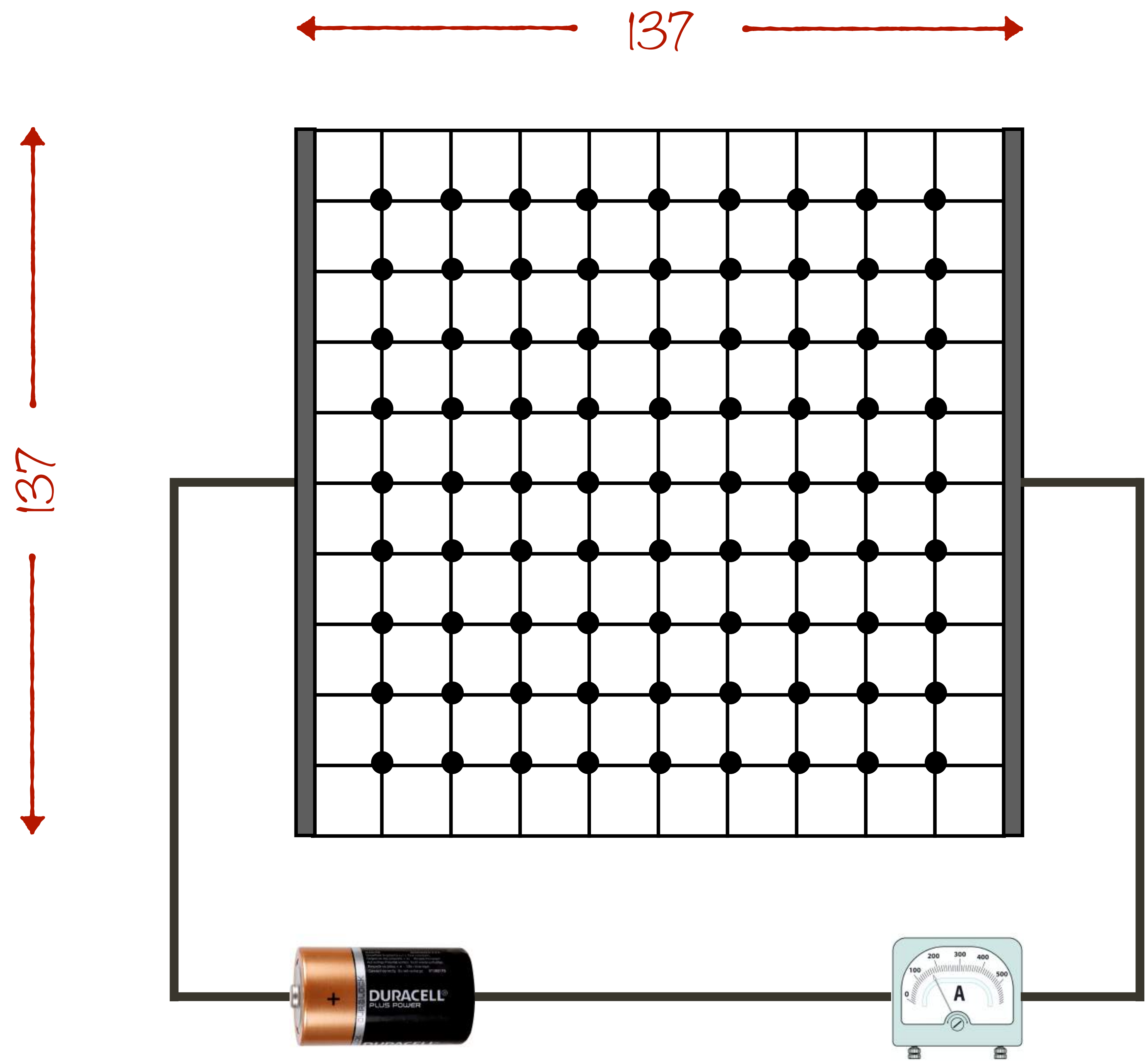


- A uniform probability of interaction corresponds to a **random network**.
- Real social networks exhibit **clustering**: two people are more likely to know each other if they have a common acquaintance.

- Numerical results show that:
  - Epidemics spread more rapidly in highly **susceptible populations**
  - The amount of time for an epidemic to spread is given by the average **radius of connected clusters** of susceptible individuals.
  - The infection **curve flattens** as ϕ is increased.
  - No epidemic outbreak takes place below the **percolation threshold**.

Percolation

[Watson-Leath '74]

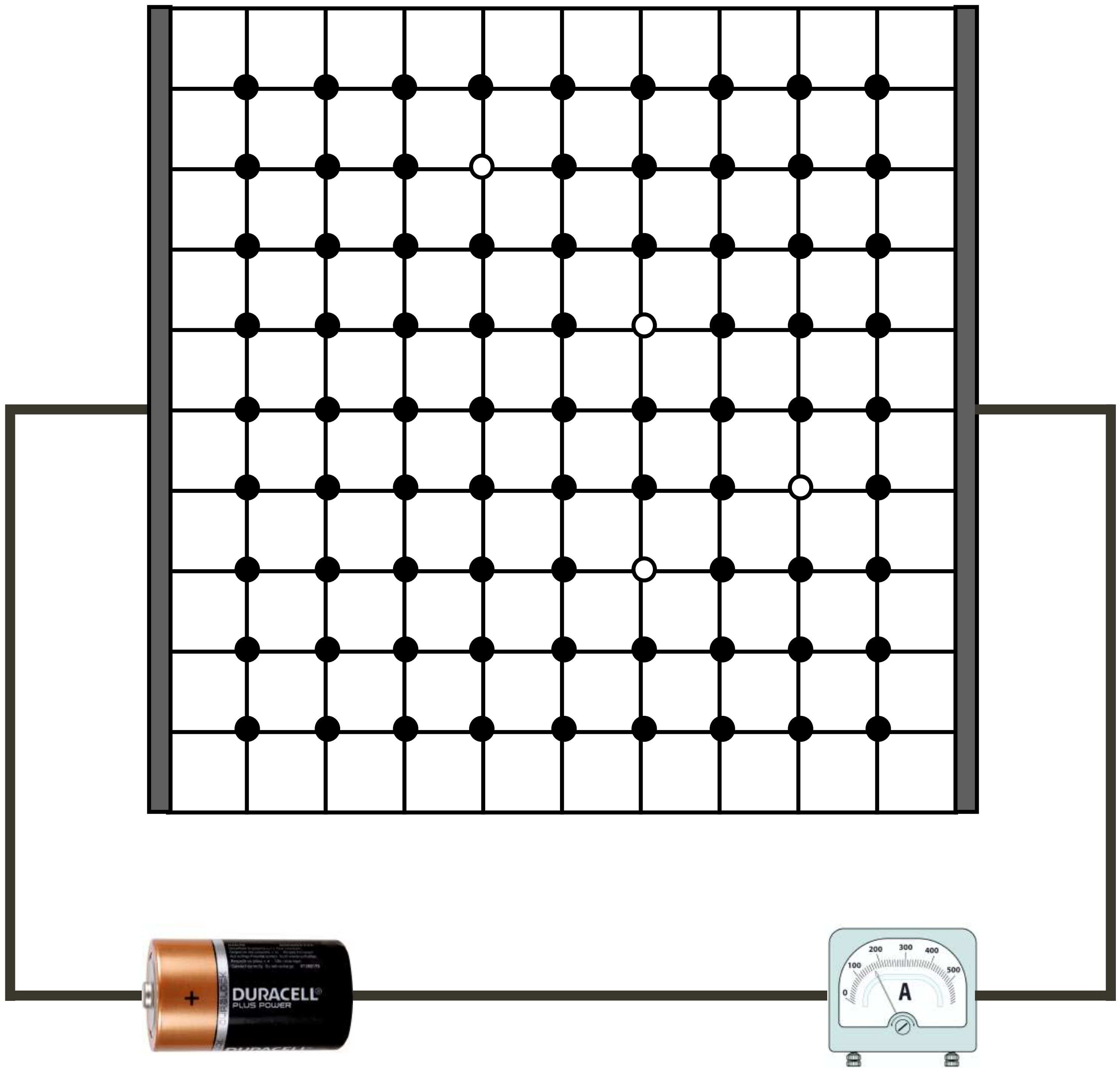


Jeff Murugan (UCT)

Percolation

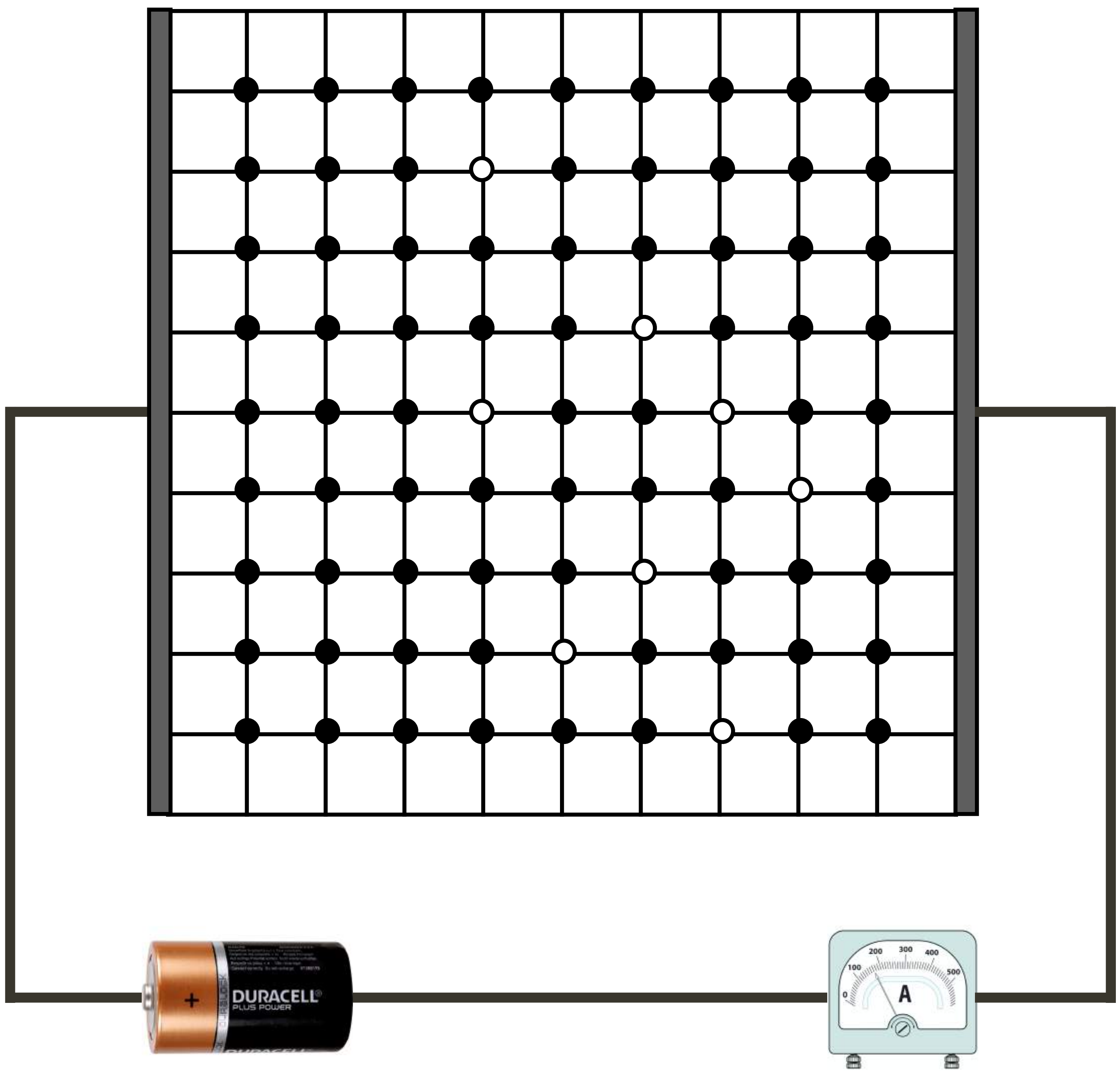
[Watson-Leath '74]

$\phi =$ Ratio of unblocked sites to total number of sites (137 x 137)



Percolation

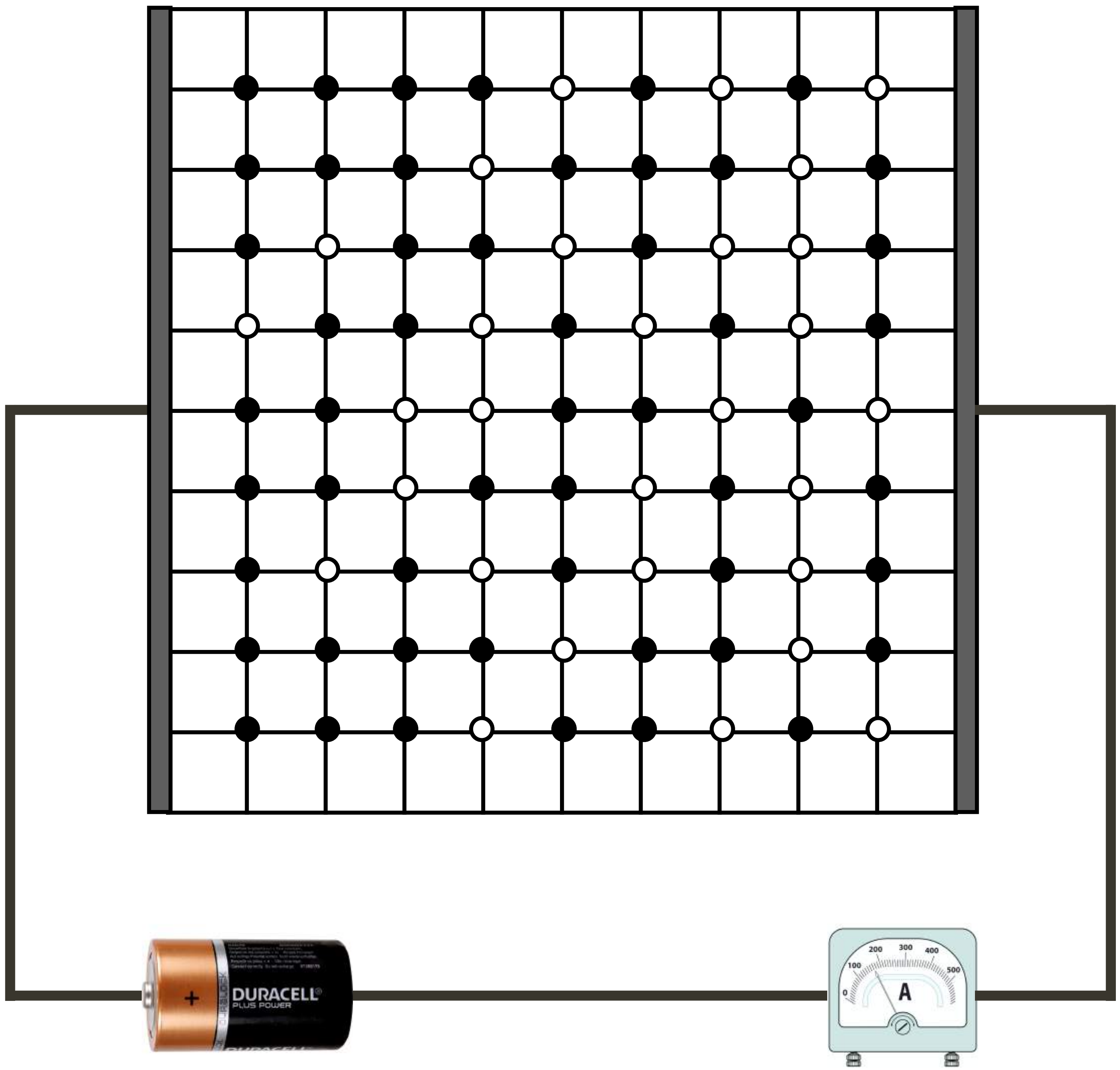
[Watson-Leath '74]



Percolation

[Watson-Leath '74]

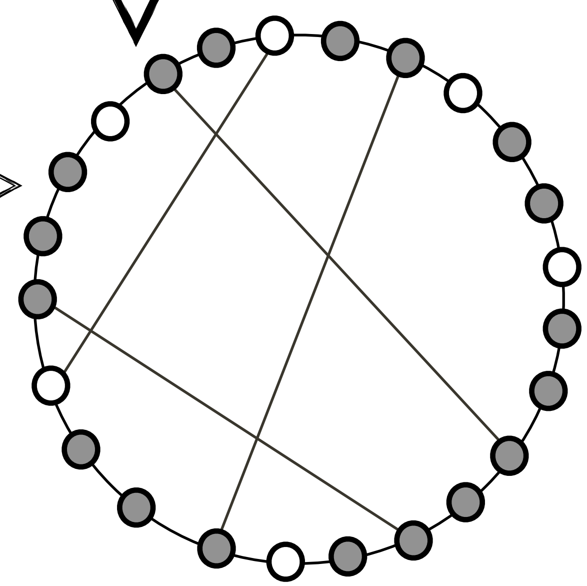
The smallest value of ϕ at which no current flows is the **percolation threshold** ϕ_c



Epidemic Spreading in a Small World

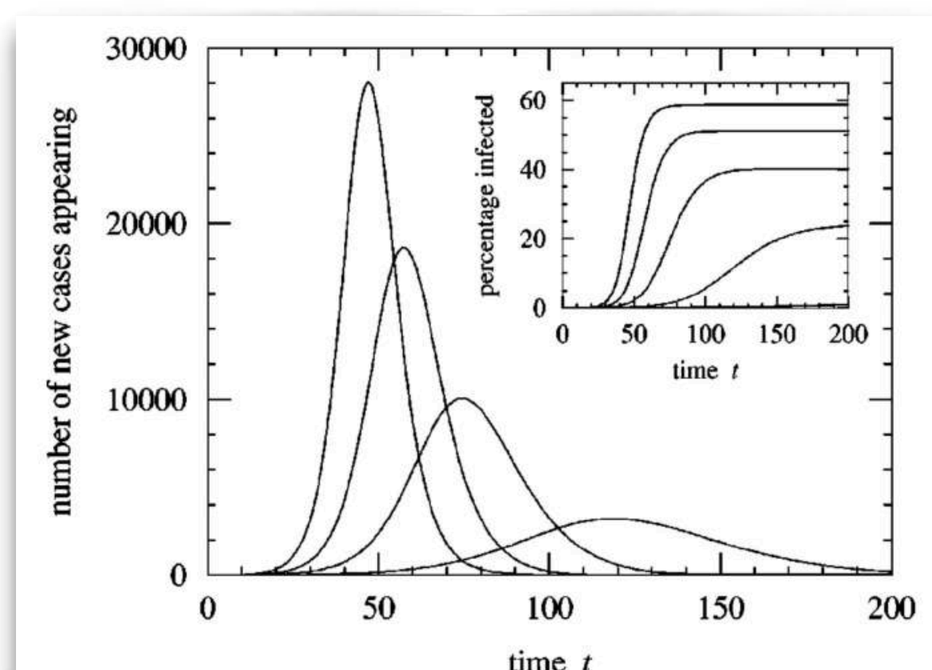
[Moore-Newman '00]

- Important epidemiological parameters are **susceptibility** of the population and **transmissibility** of the disease.
- When an epidemic takes place can be mapped to a standard **percolation** problem on a small-world network!



- A uniform probability of interaction corresponds to a **random network**.
- Real social networks exhibit **clustering**: two people are more likely to know each other if they have a common acquaintance.

- Numerical results show that:
  - Epidemics spread more rapidly in highly **susceptible populations**
  - The amount of time for an epidemic to spread is given by the average **radius of connected clusters** of susceptible individuals.
  - The infection **curve flattens** as ϕ is increased.
  - No epidemic outbreak takes place below the **percolation threshold.**



Jeff Murugan (UCT)

Quantum Small-Worlds I - Hamiltonian

[JM-Hartmann-Shock '19]

- For **nearest-neighbour** interactions this is the XXX Heisenberg spin chain, a well-known and much-loved integrable system.
- For more general (regular) couplings, we can still solve the eigenvalue problem for $H \subset GL(2^N \times 2^N, \mathbb{C})$

$$H = -\sum_{I=1}^{N}\sum_{j=I+1}^{N}\sum_{k=1}^{3} A_{ij} S_i^k S_j^k$$

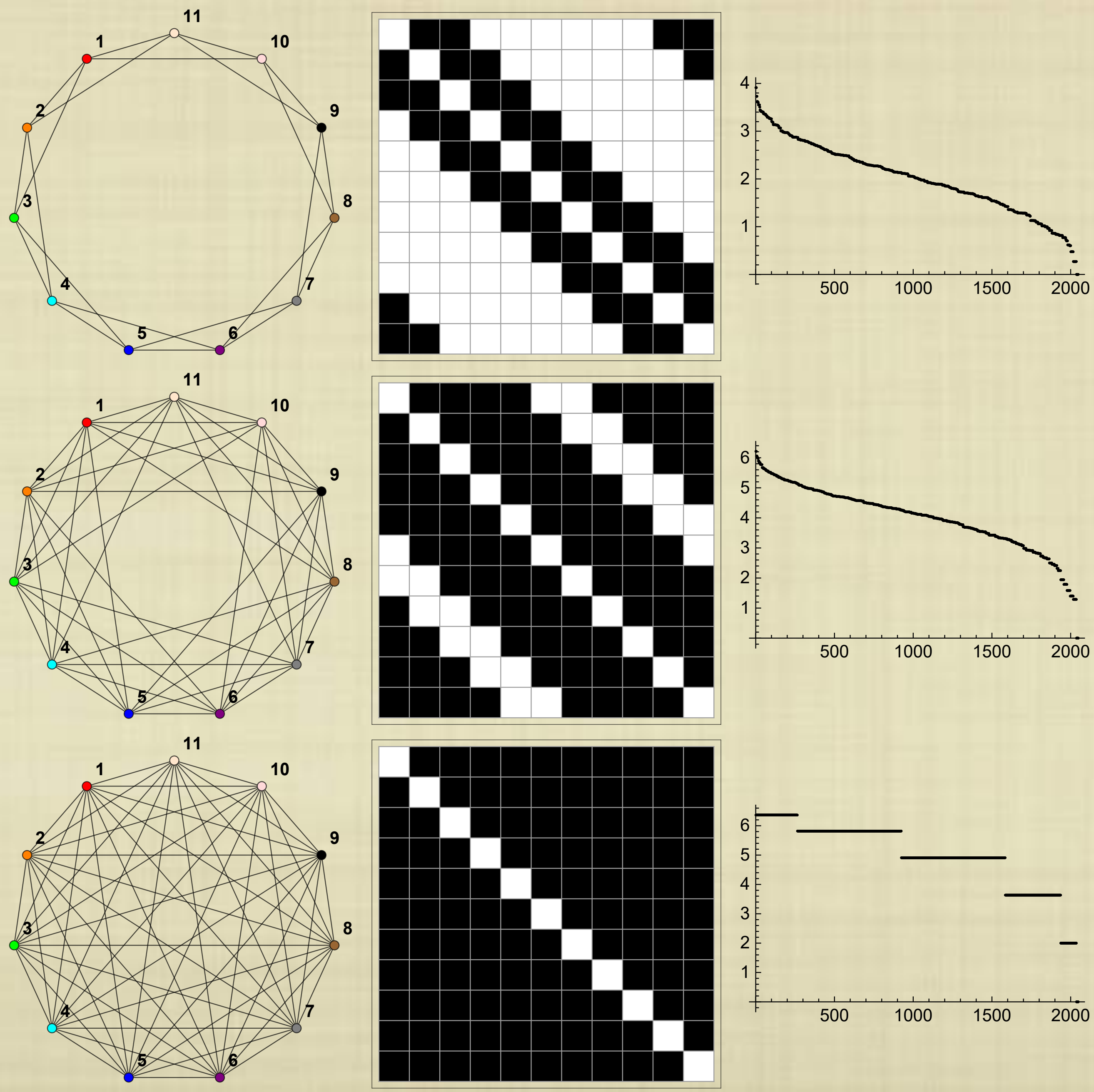
- Each vertex in the network accommodates a **spin-1/2** state.
- Edges represent **spin-exchange** interactions between states on the lattice.
- $S_i^k = \frac{1}{2}\sigma_i^k$ is the k'th Pauli spin matrix acting at site i.
- The **network topology** is encoded in the NxN **adjacency matrix** A_{ij} which will consist of either 1's or 0's if we normalise the couplings.

Quantum Small-Worlds I - Hamiltonian

[JM-Hartmann-Shock '19]

Adjacency matrix with black=1 and white=0

k-local regular lattices with N=11 sites. From top to bottom (k,C,L) = (4,0.5,1.8); (8,0.75,1.2) and (10,1.0,1.0)



Eigenvalue spectrum

Quantum Small-Worlds I - Hamiltonian

[JM-Hartmann-Shock '19]

- For **nearest-neighbour** interactions this is the XXX Heisenberg spin chain, a well-known and much-loved integrable system.
- For more general (regular) couplings, we can still solve the eigenvalue problem for $H \subset GL(2^N \times 2^N, \mathbb{C})$

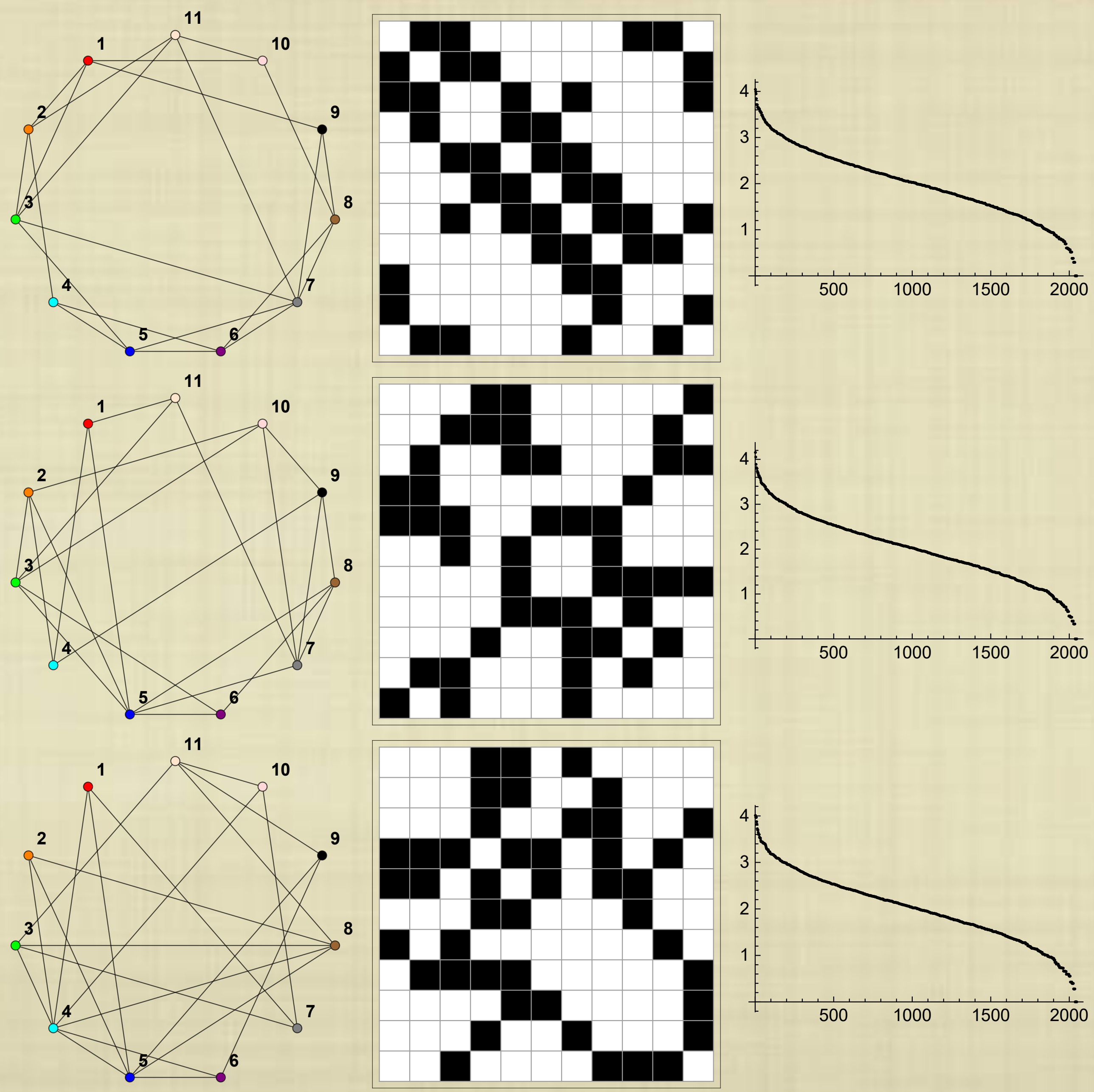
$$H = -\sum_{I=1}^{N} \sum_{j=I+1}^{N} \sum_{k=1}^{3} A_{ij} S_i^k S_j^k$$

- Each vertex in the network accommodates a **spin-1/2** state.
- Edges represent **spin-exchange** interactions between states on the lattice.
- $S_i^k = \frac{1}{2}\sigma_i^k$ is the k'th Pauli spin matrix acting at site i.
- The **network topology** is encoded in the NxN **adjacency matrix** A_{ij} which will consist of either 1's or 0's if we normalise the couplings.

Quantum Small-Worlds I - Hamiltonian

[JM-Hartmann-Shock '19]

Implementing the Watts-Strogatz protocol for fixed **k=4** and **N=11.** From top to bottom the spin-chains differ only in **re-wiring probability, p**



From top to bottom **(p,C,L)** = (0.1,0.45,0.76); (0.5,0.41,1.73); (0.75,0.23,1.67) Notice that even for large k the spectrum remains close to the regular chain

Diagnostic tools

[JM-Hartmann-Shock '19]

- To study scrambling in quantum small- world networks, we compute the infinite temperature **four-point OTOC**

$$C_{\beta=0}(t) = \langle\psi|S_i^z(0)S_j^z(t)S_i^z(0)S_j^z(t)|\psi\rangle_{\beta=0}$$

where $|\psi\rangle$ is some pure state and $S_j^z(t) = e^{iHt}S_j^z(0)e^{-iHt}$ is the time-evolved Heisenberg spin operator

How do quantum small world systems scramble information?

An alternative diagnostic is the **spectral form factor** which is the analytic continuation of the thermal partition function

$$g(t;\beta) = \frac{\langle|Z(\beta,t)|^2\rangle_J}{\langle Z(\beta)\rangle_J^2}$$

- The SFF exhibits late time **RMT behaviour** and is closer to the OTOC than standard RMT measures.

- **Scrambling** is the tendency of a many body quantum system to delocalize quantum information over all its degrees of freedom.
- It can be diagnosed by the **thermally averaged commutator** squared, C(t)
- Equivalently, through the OTOC $\langle A^\dagger(t)B^\dagger(0)A(t)B(0)\rangle$

Diagnostic tools

- To study scrambling in quantum small- world networks, we compute the infinite temperature **four-point OTOC**

$$C_{\beta=0}(t) = \langle\psi|S_i^z(0)S_j^z(t)S_i^z(0)S_j^z(t)|\psi\rangle_{\beta=0}$$

where $|\psi\rangle$ is some pure state and $S_j^z(t) = e^{iHt}S_j^z(0)e^{-iHt}$ is the time-evolved Heisenberg spin operator

How do quantum small world systems scramble information?

An alternative diagnostic is the **spectral form factor** which is the analytic continuation of the thermal partition function

$$g(t;\beta) = \frac{\langle|Z(\beta,t)|^2\rangle_J}{\langle Z(\beta)\rangle_J^2}$$

- The SFF exhibits late time **RMT behaviour** and is closer to the OTOC than standard RMT measures.

[JM-Hartmann-Shock '19]

- **Scrambling** is the tendency of a many body quantum system to delocalize quantum information over all its degrees of freedom.
- It can be diagnosed by the **thermally averaged commutator** squared, C(t)
- Equivalently, through the OTOC $\langle A^\dagger(t)B^\dagger(0)A(t)B(0)\rangle$

Diagnostic tools

[JM-Hartmann-Shock '19]

- To study scrambling in quantum small- world networks, we compute the infinite temperature **four-point OTOC**

$$C_{\beta=0}(t) = \langle\psi|S_i^z(0)S_j^z(t)S_i^z(0)S_j^z(t)|\psi\rangle_{\beta=0}$$

where $|\psi\rangle$ is some pure state and $S_j^z(t) = e^{iHt}S_j^z(0)e^{-iHt}$ is the time-evolved Heisenberg spin operator

How do quantum small world systems scramble information?

An alternative diagnostic is the **spectral form factor** which is the analytic continuation of the thermal partition function

$$g(t;\beta) = \frac{\langle|Z(\beta,t)|^2\rangle_J}{\langle Z(\beta)\rangle_J^2}$$

- The SFF exhibits late time **RMT behaviour** and is closer to the OTOC than standard RMT measures.

- **Scrambling** is the tendency of a many body quantum system to delocalize quantum information over all its degrees of freedom.
- It can be diagnosed by the **thermally averaged commutator** squared, C(t)
- Equivalently, through the OTOC $\langle A^\dagger(t)B^\dagger(0)A(t)B(0)\rangle$

Diagnostic tools

[JM-Hartmann-Shock '19]

- To study scrambling in quantum small- world networks, we compute the infinite temperature **four-point OTOC**

$$C_{\beta=0}(t) = \langle\psi|S_i^z(0)S_j^z(t)S_i^z(0)S_j^z(t)|\psi\rangle_{\beta=0}$$

where $|\psi\rangle$ is some pure state and $S_j^z(t) = e^{iHt}S_j^z(0)e^{-iHt}$ is the time-evolved Heisenberg spin operator

How do quantum small world systems scramble information?

An alternative diagnostic is the **spectral form factor** which is the analytic continuation of the thermal partition function

$$g(t;\beta) = \frac{\langle|Z(\beta,t)|^2\rangle_J}{\langle Z(\beta)\rangle_J^2}$$

- The SFF exhibits late time **RMT behaviour** and is closer to the OTOC than standard RMT measures.

- **Scrambling** is the tendency of a many body quantum system to delocalize quantum information over all its degrees of freedom.
- It can be diagnosed by the **thermally averaged commutator** squared, C(t)
- Equivalently, through the OTOC $\langle A^\dagger(t)B^\dagger(0)A(t)B(0)\rangle$

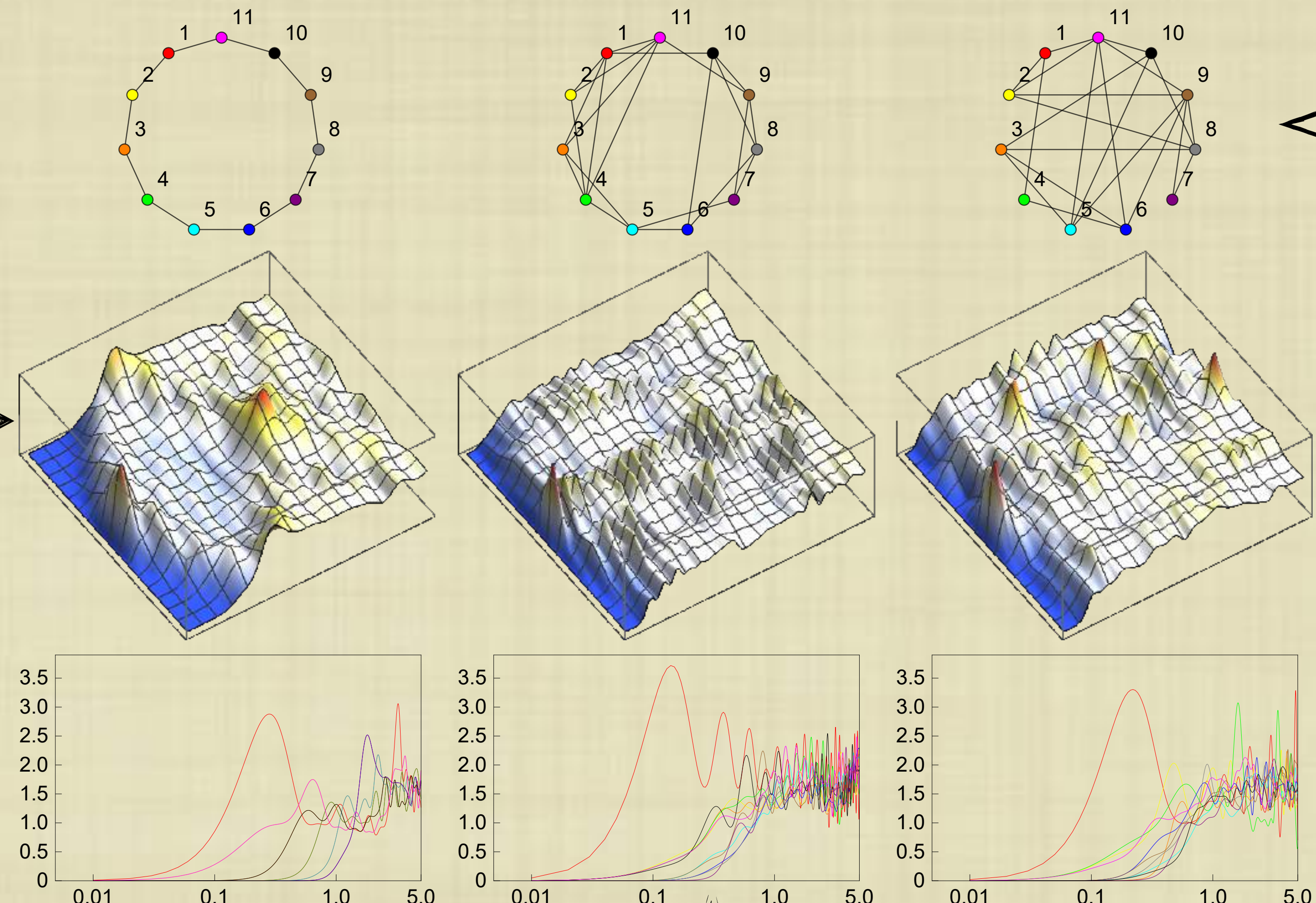
Quantum Small-Worlds II - OTOC & SFF

[JM-Hartmann-Rosa-Shock '19]

The OTOC

$C_{ij}(t) = 2(1 - Re(C_0(t)))$

numerically computed and plotted as a function of the vertex degree k and rewiring probability p for (p,k) = (2,0); (4,0.25); (4,0.75)



11-site lattices with random re-wirings of the lattice following the Watts-Strogatz protocol.

- Vertex-wise correlators $C_{1j}(t)$ for an initial disturbance at site 1.
- We note that $C_{1j}(t) \sim t^b$ for 1.76 < b < 6.23; 1.76 < b < 3.22 and 1.73 < b < 3.42 for p=0, 0.25 and 0.75 respectively.

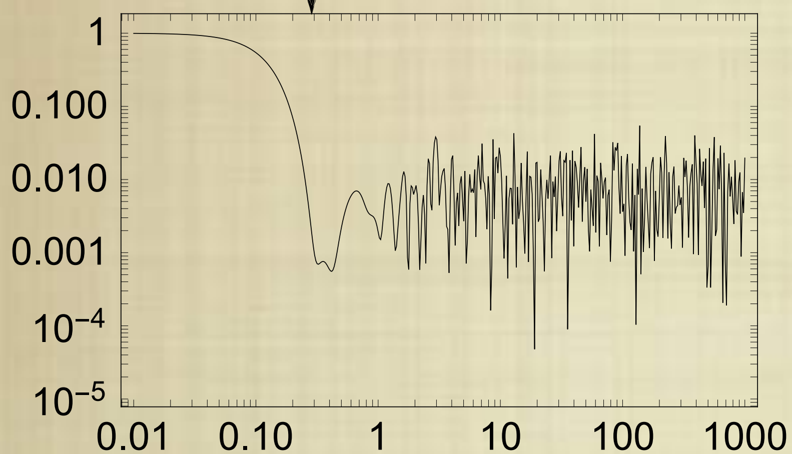
Jeff Murugan (UCT)

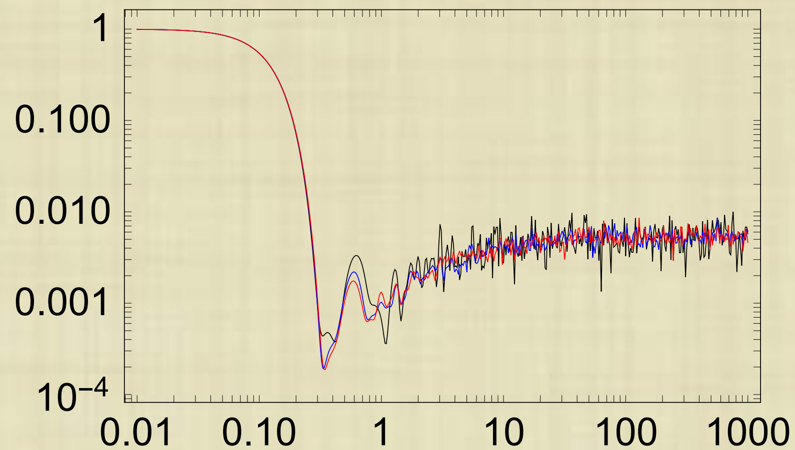
Quantum Small-Worlds II - OTOC & SFF

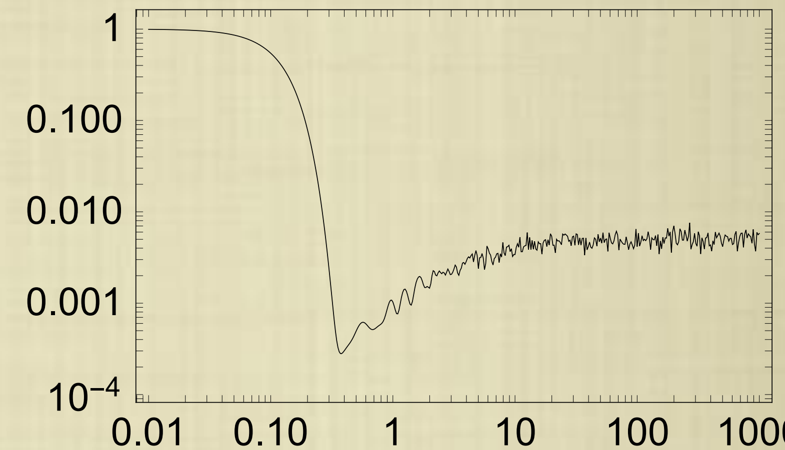
[JM-Hartmann-Rosa-Shock '19]

Infinite temperature SFF for **zero re-wirings** of the network (regular lattice. No DRP behaviour.

SFF computed for 20 re-wirings ($p \sim 1$). There is a clear Dip-Ramp-Plateau behaviour with the plateau setting in at around t=10, for N=7,8,9,...







SFF for 1 (black), 2 (blue) and 3 (red) re-wirings of the system. The **Dip-Ramp-Plateau** behaviour starts to manifest.

Jeff Murugan (UCT)

Quantum Small-Worlds III - Spectral Statistics

[JM-Hartmann-Rosa-Shock '19]

- Any short-range chaos diagnostic will crucially depend on **global symmetries** of the system.
- No level-repulsion between eigenvalues in different global symmetry sectors.
- Each symmetry sector must be analysed separately.

- Neighbouring **eigenvalues repel** each other in random matrix theory
- To study the spectral statistics we take a list of ordered, non-degenerate energy eigenvalues and compute $s_i = E_{I+1} - E_i$
- If the system is chaotic, $p_\beta(s) = a_\beta s^\beta \exp(-b_\beta s^2)$

How close to RMT is the Quantum Small World?

- Nearest neighbour distributions have two drawbacks:
  1. They require the spectrum to be **unfolded**
  2. Only tell us if the spectrum is **globally RMT** or not.

- **r-statistics** is a local spectral observable capable of telling whether small clusters of energy eigenvalues display RMT behaviour
- Given the energy spacings, we define $r_i = \mathrm{Min}(s_i, s_{I+1})/\mathrm{Max}(s_i, s_{I+1})$
- The ratios r_i take very specific values in RMT e.g. $r_{GOE} \approx 0.53590$ but are much smaller for integrable systems e.g. $r_P \approx 0.38629$

Quantum Small-Worlds III - Spectral Statistics

[JM-Hartmann-Rosa-Shock '19]

- Any short-range chaos diagnostic will crucially depend on **global symmetries** of the system.
- No level-repulsion between eigenvalues in different global symmetry sectors.
- Each symmetry sector must be analysed separately.

- Neighbouring **eigenvalues repel** each other in random matrix theory
- To study the spectral statistics we take a list of ordered, non-degenerate energy eigenvalues and compute $s_i = E_{I+1} - E_i$
- If the system is chaotic, $p_\beta(s) = a_\beta s^\beta \exp(-b_\beta s^2)$

How close to RMT is the Quantum Small World?

- Nearest neighbour distributions have two drawbacks:
1. They require the spectrum to be **unfolded**
2. Only tell us if the spectrum is **globally RMT** or not.

- **r-statistics** is a local spectral observable capable of telling whether small clusters of energy eigenvalues display RMT behaviour
- Given the energy spacings, we define $r_i = \mathrm{Min}(s_i, s_{I+1})/\mathrm{Max}(s_i, s_{I+1})$
- The ratios r_i take very specific values in RMT e.g. $r_{GOE} \approx 0.53590$ but are much smaller for integrable systems e.g. $r_P \approx 0.38629$

Quantum Small-Worlds III - Spectral Statistics

[JM-Hartmann-Rosa-Shock '19]

- Any short-range chaos diagnostic will crucially depend on **global symmetries** of the system.
- No level-repulsion between eigenvalues in different global symmetry sectors.
- Each symmetry sector must be analysed separately.

- Neighbouring **eigenvalues repel** each other in random matrix theory
- To study the spectral statistics we take a list of ordered, non-degenerate energy eigenvalues and compute $s_i = E_{I+1} - E_i$
- If the system is chaotic, $p_\beta(s) = a_\beta s^\beta \exp(-b_\beta s^2)$

How close to RMT is the Quantum Small World?

- Nearest neighbour distributions have two drawbacks:
1. They require the spectrum to be **unfolded**
2. Only tell us if the spectrum is **globally RMT** or not.

- **r-statistics** is a local spectral observable capable of telling whether small clusters of energy eigenvalues display RMT behaviour
- Given the energy spacings, we define $r_i = \mathrm{Min}(s_i, s_{I+1})/\mathrm{Max}(s_i, s_{I+1})$
- The ratios r_i take very specific values in RMT e.g. $r_{GOE} \approx 0.53590$ but are much smaller for integrable systems e.g. $r_P \approx 0.38629$

Quantum Small-Worlds III - Spectral Statistics

[JM-Hartmann-Rosa-Shock '19]

- Any short-range chaos diagnostic will crucially depend on **global symmetries** of the system.
- No level-repulsion between eigenvalues in different global symmetry sectors.
- Each symmetry sector must be analysed separately.

- Neighbouring **eigenvalues repel** each other in random matrix theory
- To study the spectral statistics we take a list of ordered, non-degenerate energy eigenvalues and compute $s_i = E_{I+1} - E_i$
- If the system is chaotic, $p_\beta(s) = a_\beta s^\beta \exp(-b_\beta s^2)$

How close to RMT is the Quantum Small World?

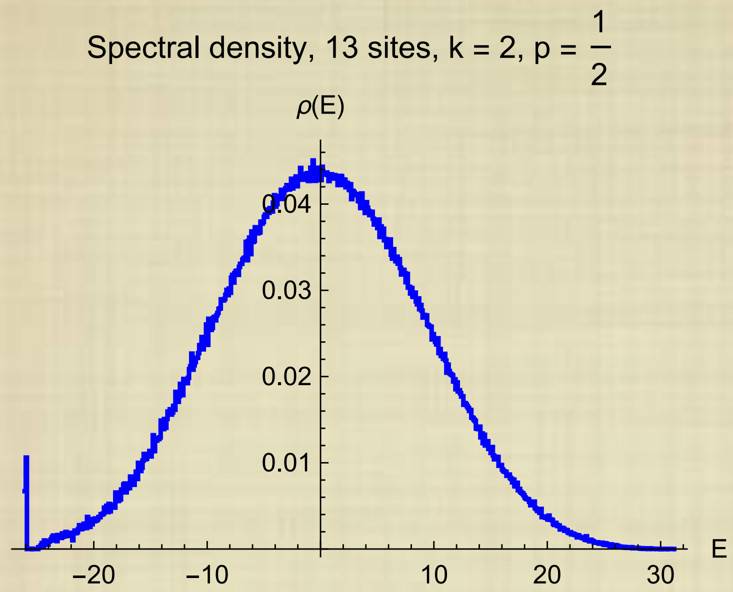
- Nearest neighbour distributions have two drawbacks:

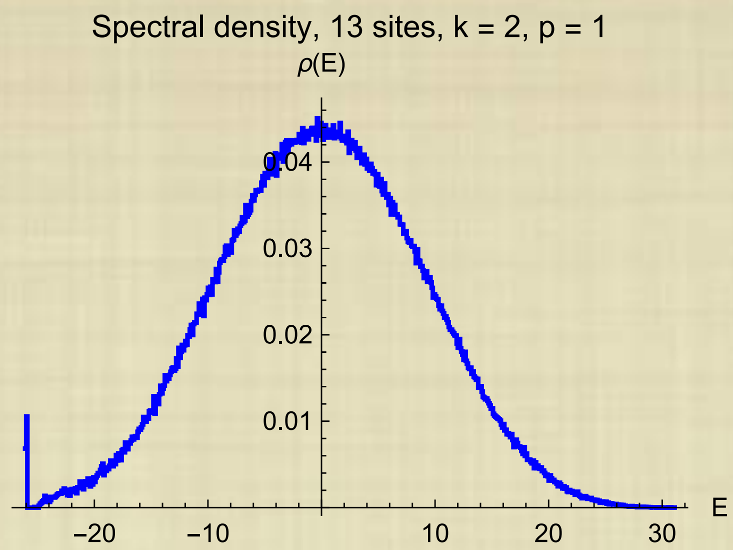
  1. They require the spectrum to be **unfolded**
  2. Only tell us if the spectrum is **globally RMT** or not.

- **r-statistics** is a local spectral observable capable of telling whether small clusters of energy eigenvalues display RMT behaviour
- Given the energy spacings, we define $r_i = \mathrm{Min}(s_i,\, s_{I+1})/\mathrm{Max}(s_i,\, s_{I+1})$
- The ratios r_i take very specific values in RMT e.g. $r_{GOE} \approx 0.53590$ but are much smaller for integrable systems e.g. $r_P \approx 0.38629$

Quantum Small-Worlds III - Spectral Statistics

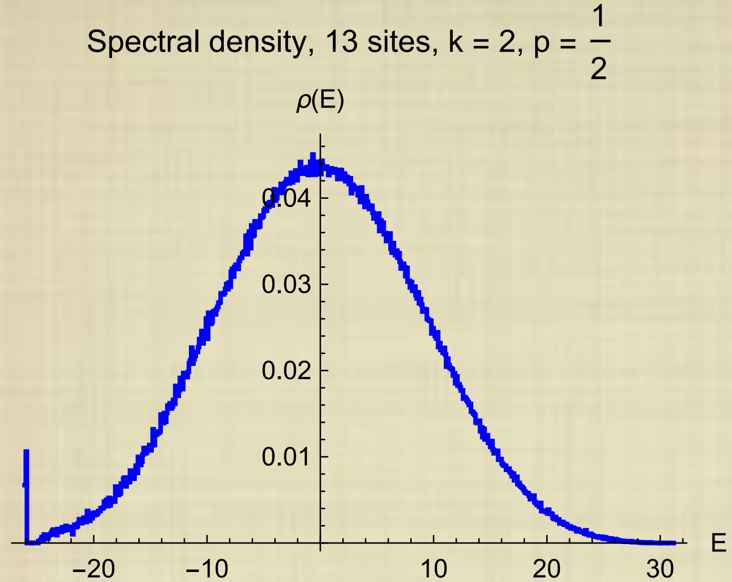
[JM-Hartmann-Rosa-Shock '19]

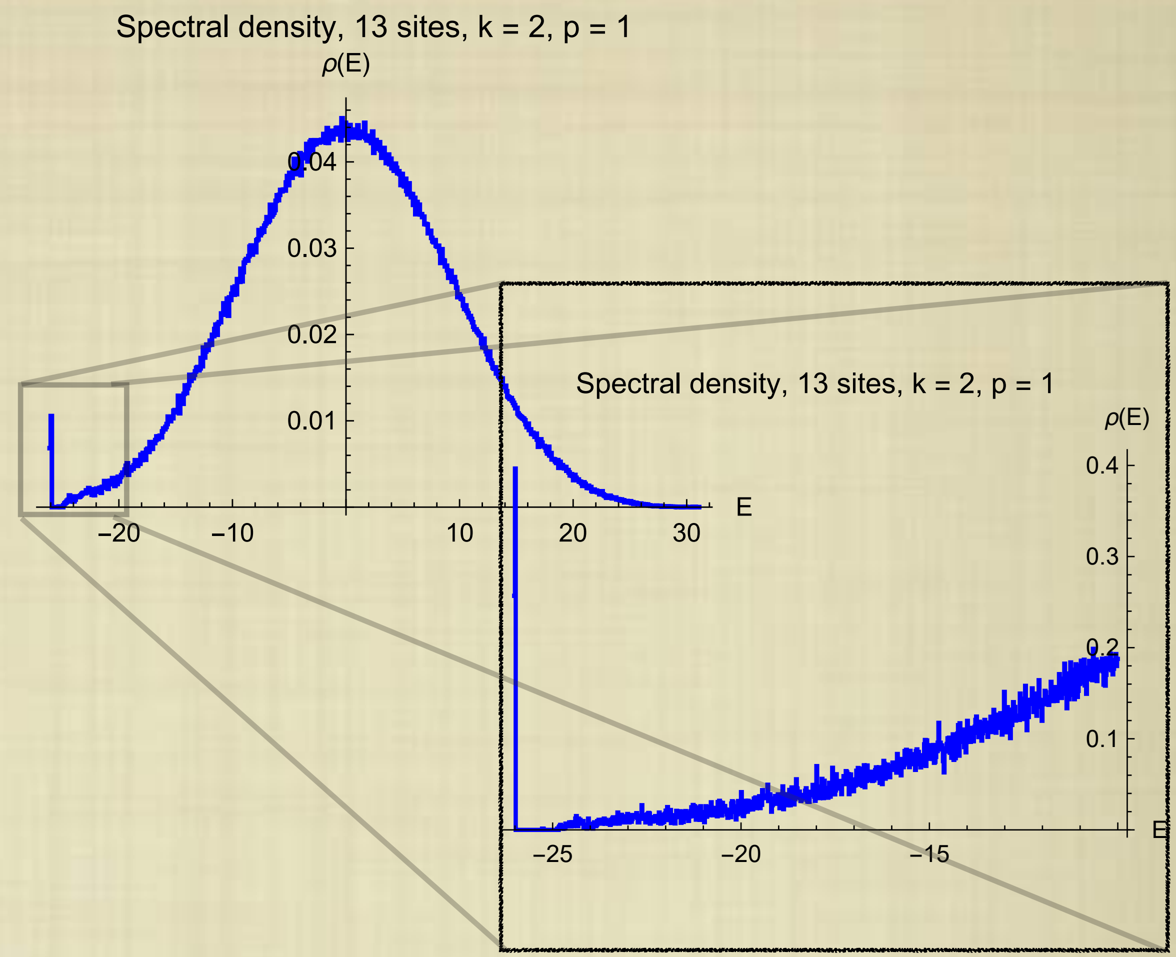




Quantum Small-Worlds III - Spectral Statistics

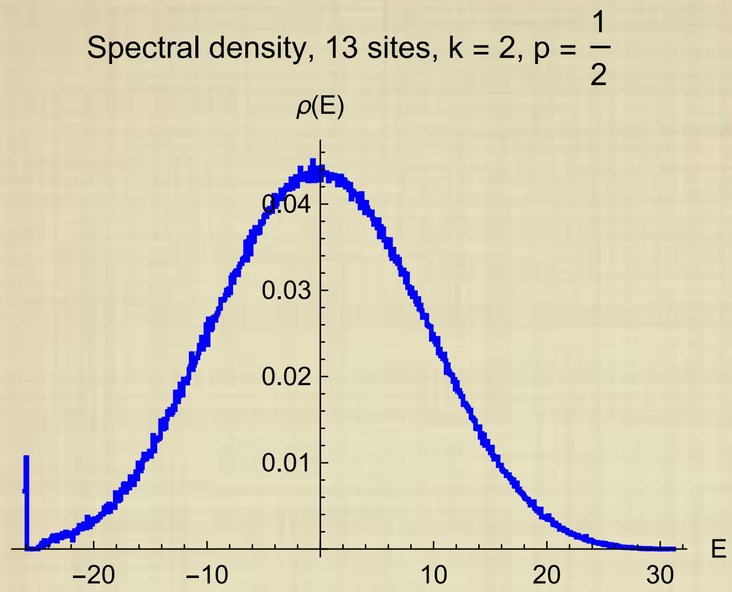
[JM-Hartmann-Rosa-Shock '19]

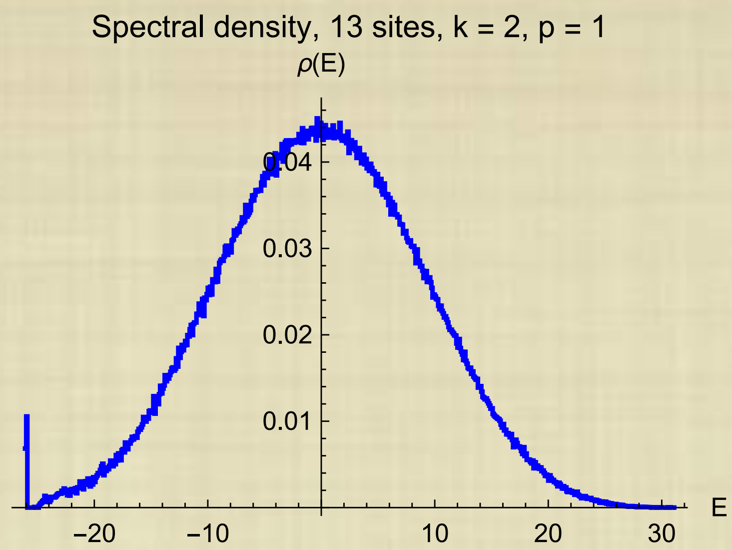


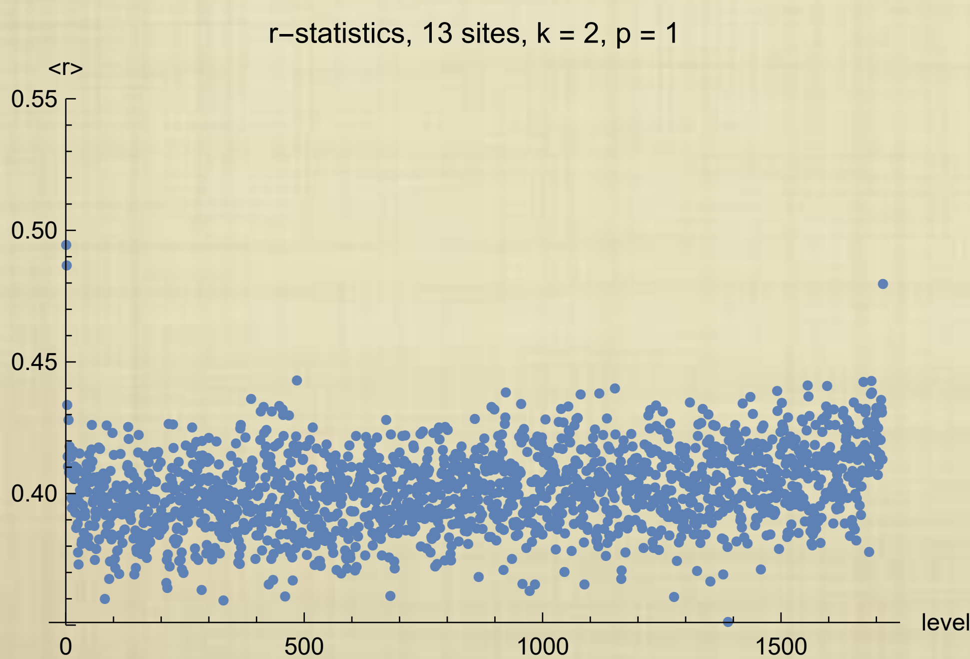


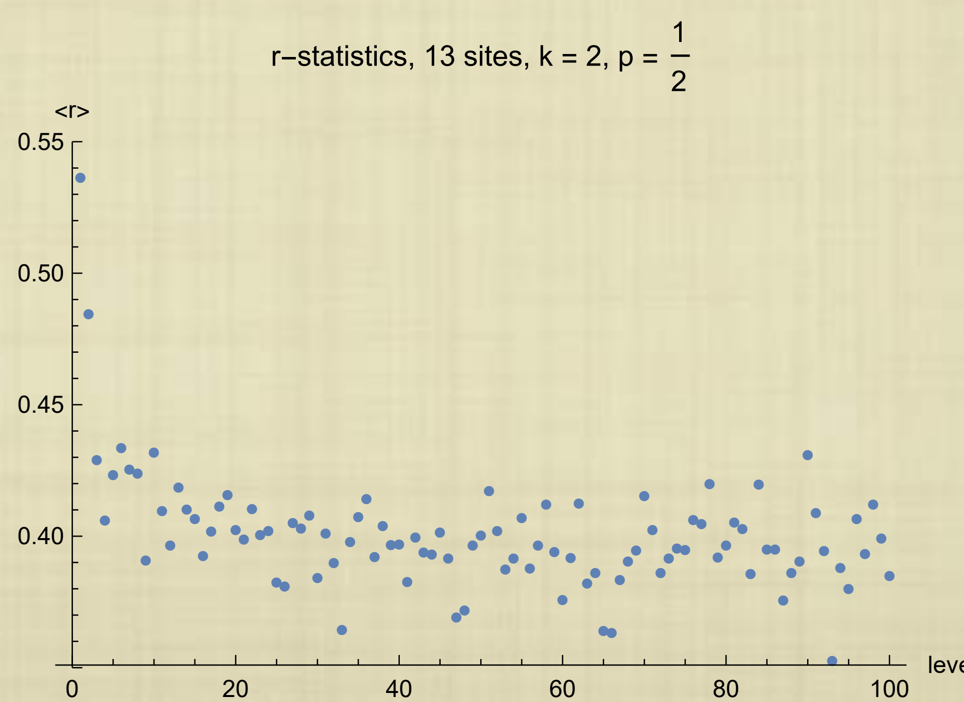
Quantum Small-Worlds III - Spectral Statistics

[JM-Hartmann-Rosa-Shock '19]









Jeff Murugan (UCT)

Conclusions and Future work

- **Quantum small-world systems** offer a novel class of many-body problems that parametrically interpolate between integrable (regular) and chaotic (random) systems

- The OTOC and SFF are best understood for large N (and in, for example SYK, large k). However our numerics are restricted to $k < N < 11$ so we have a few-body sparse quantum system. **We need to understand such systems for larger values of N**.

- Our computations are restricted to the very simple infinite temperature limit (especially for the SFF). **Finite temperature corrections** are important and subtle and needs to be understood.

- While this is clearly a toy model, it is similar to table-top cold-atom experiments in cavity QED studied in recent work by Swingle et.al. **Can such models be physically realised?**

Conclusions and Future work

- **Quantum small-world systems** offer a novel class of many-body problems that parametrically interpolate between integrable (regular) and chaotic (random) systems

- The OTOC and SFF are best understood for large N (and in, for example SYK, large k). However our numerics are restricted to $k < N < 11$ so we have a few-body sparse quantum system. **We need to understand such systems for larger values of N.**

- Our computations are restricted to the very simple infinite temperature limit (especially for the SFF). **Finite temperature corrections** are important and subtle and needs to be understood.

- While this is clearly a toy model, it is similar to table-top cold-atom experiments in cavity QED studied in recent work by Swingle et.al. **Can such models be physically realised?**

Conclusions and Future work

- **Quantum small-world systems** offer a novel class of many-body problems that parametrically interpolate between integrable (regular) and chaotic (random) systems

- The OTOC and SFF are best understood for large N (and in, for example SYK, large k). However our numerics are restricted to k < N < 11 so we have a few-body sparse quantum system. **We need to understand such systems for larger values of N**.

- Our computations are restricted to the very simple infinite temperature limit (especially for the SFF). **Finite temperature corrections** are important and subtle and needs to be understood.

- While this is clearly a toy model, it is similar to table-top cold-atom experiments in cavity QED studied in recent work by Swingle et.al. **Can such models be physically realised?**

Conclusions and Future work

- **Quantum small-world systems** offer a novel class of many-body problems that parametrically interpolate between integrable (regular) and chaotic (random) systems

- The OTOC and SFF are best understood for large N (and in, for example SYK, large k). However our numerics are restricted to $k < N < 11$ so we have a few-body sparse quantum system. **We need to understand such systems for larger values of N.**

- Our computations are restricted to the very simple infinite temperature limit (especially for the SFF). **Finite temperature corrections** are important and subtle and needs to be understood.

- While this is clearly a toy model, it is similar to table-top cold-atom experiments in cavity QED studied in recent work by Swingle et.al. **Can such models be physically realised?**

մերսի

Hristo! 감사합니다 спасибі! תודה

நன்றி Ndiyabulela! Ke a leboha! Paldies

σας ευχαριστώ! Gracies! Ngeyabonga! Baie Dankie!

Děkuji Ukhani! Thank You! Merci! Asante

Obrigado! Grazias!

Ihe edn! Inkomu! Siyabonga! Danke! Ďakujem Tak

धन्यवाद ¡Gracias धन्यवाद Grazie! ありがとう

Eskerrik asko!

Suksema! Juspajaraña شكرا Təşəkkür edirəm!

Dzięki Obrigadu! Дзякуй Благодаря Diolch

Dank Je Dankon Mahalo

Jeff Murugan (UCT)