

Feynman Integrals & Intersection Numbers

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Oxford 04.06.2020



Outline



On-shell and Unitarity-based methods



Integrand Decomposition and Maximum-cut Theorem



Feynman Integrals in Dim Reg



Integration-by-parts Identities



Basics of Intersection Theory



Vector Space Structure of Feynman Integrals



Integral Relations by Intersection Numbers



Special Functions



Feynman Integrals



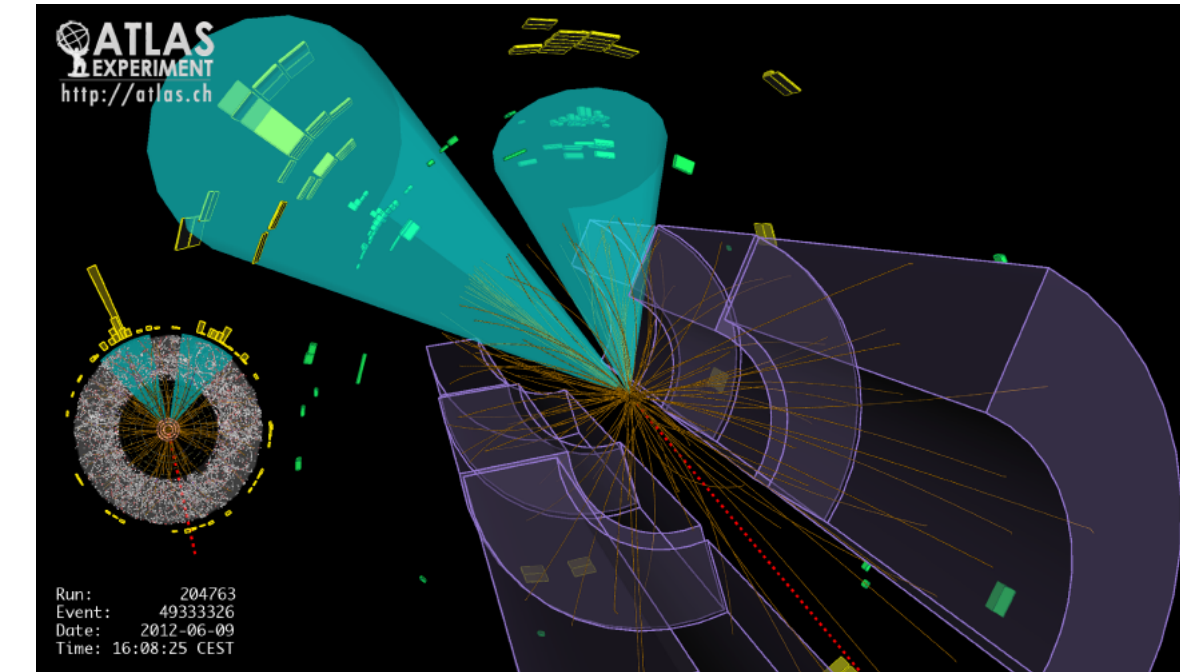
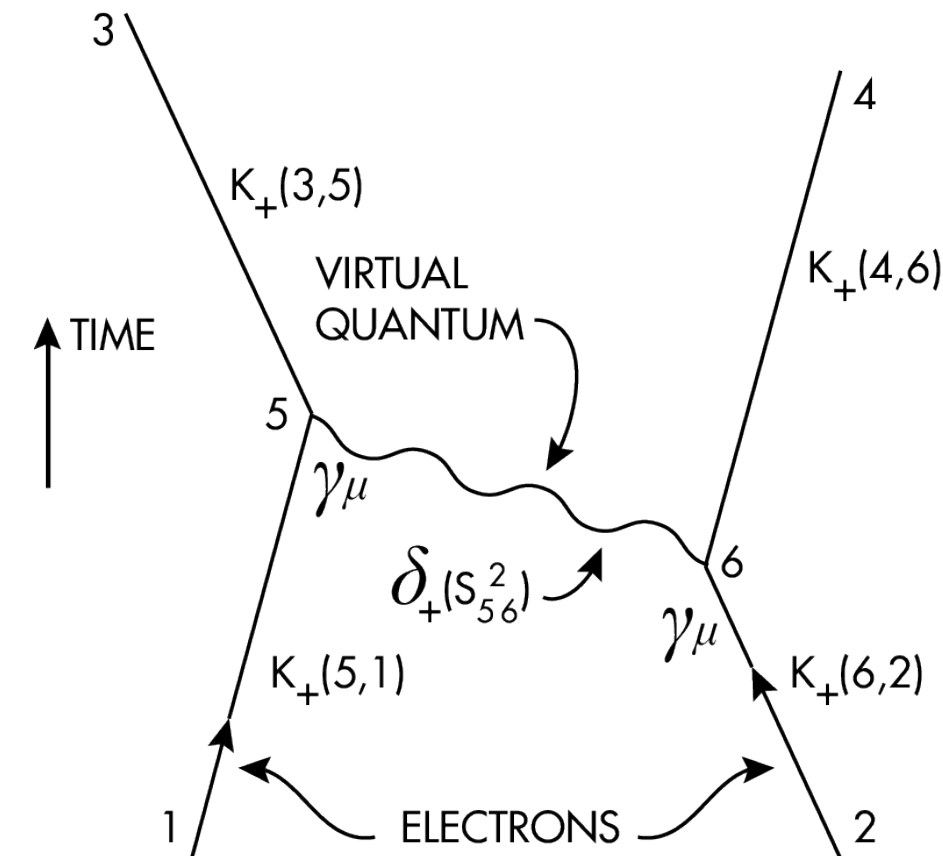
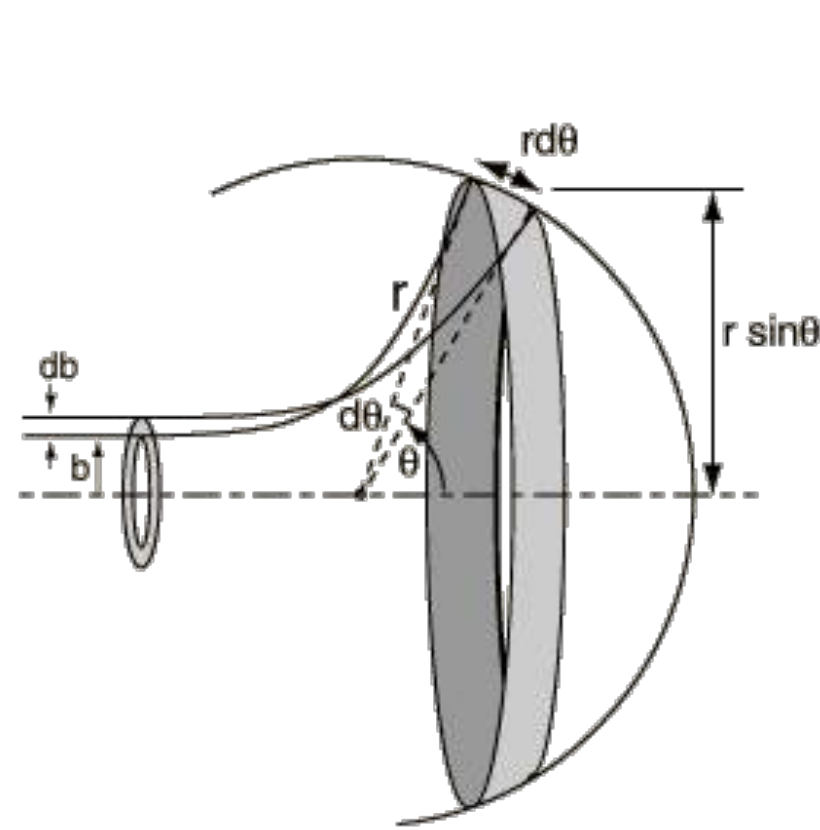
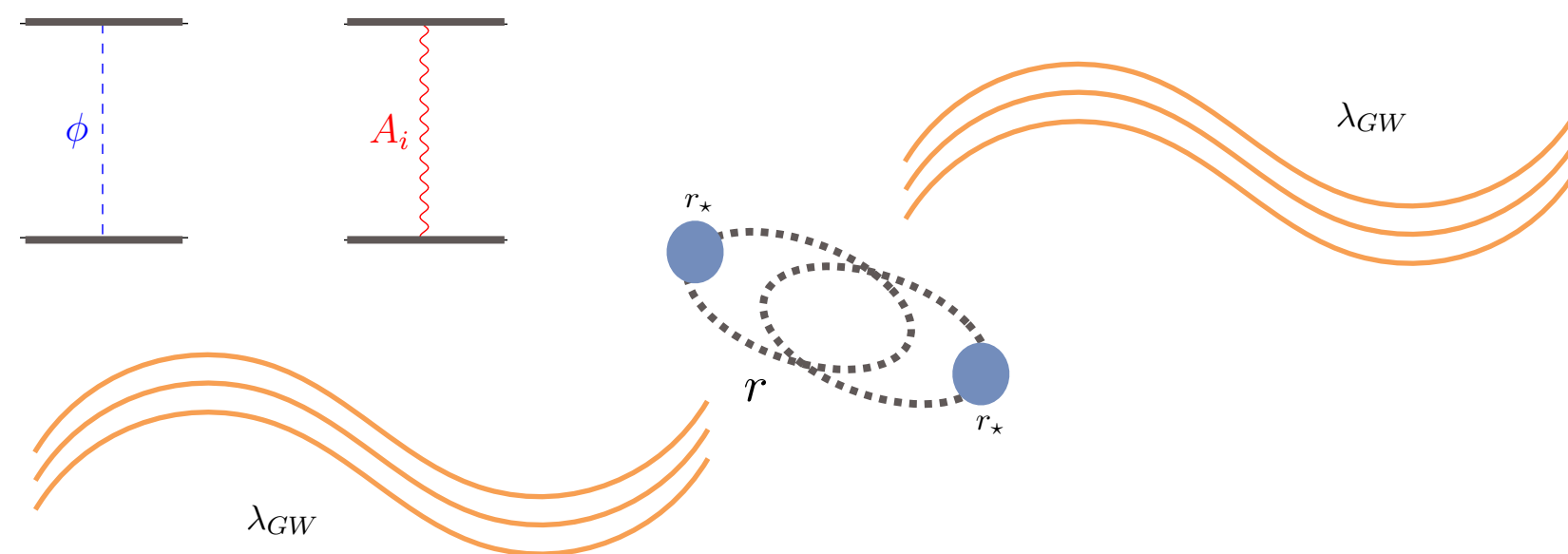
Conclusions

Based on:

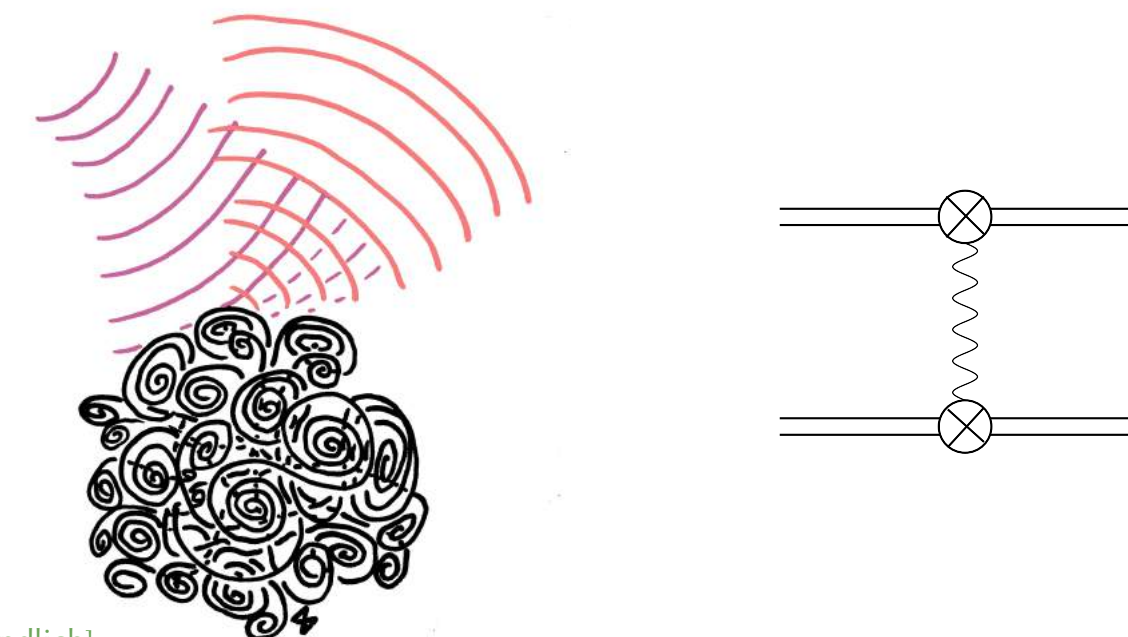
- **PM**, Mizera
Feynman Integral and Intersection Theory
JHEP 1902 (2019) 139 [arXiv: 1810.03818]
- Frellesvig, Gasparotto, Laporta, Mandal, **PM**, Mattiazzi, Mizera
Decomposition of Feynman Integrals in the Maximal Cut by Intersection Numbers
JHEP 1095 (2019) 153 [arXiv: 1901.11510]
- Frellesvig, Gasparotto, Mandal, **PM**, Mattiazzi, Mizera
Vector Space of Feynman Integrals and Multivariate Intersection Numbers
- Phys. Rev. Lett. 123 (2019) 20, 201602 [arXiv 1907.02000]

Particle Physics Approach (to several problems) and Feynman Integrals

Scattering in Perturbation Theory

 **Effective Field Theory approach to General Relativity**

Effective Field Theory approach to Fluid Dynamics

 Feynman Calculus @ Vortex-Sound interactions

[pic: S. Endlich]

Over the last 15 years we developed new competences to tackle **calculations considered inaccessible**, and to unveil **the *complex* structure of Scattering Amplitudes**

Definition. Physics is a part of mathematics devoted to the calculation of integrals of the form $\int g(x)e^{f(x)}dx$. Different branches of physics are distinguished by the range of the variable x and by the names used for $f(x)$, $g(x)$ and for the integral. For example, in classical statistical physics x runs over a symplectic manifold, $f(x)$ is called the Hamiltonian function and the integral has the meaning of a partition function or of a correlation function. In a d -dimensional quantum field theory x runs over the space of functions on a d -dimensional manifold (the space of fields) and $f(x)$ is interpreted as an action functional.

Of course this is a joke, physics is not a part of mathematics. However, it is true that the main mathematical problem of physics is the calculation of integrals of the form

$$I(g) = \int g(x)e^{-f(x)}dx$$

[...] If f can be represented as $f_0 + \lambda V$ where f_0 is a negative quadratic form, then the integral $\int g(x)e^{f(x)}dx$ can be calculated in the framework of perturbation theory with respect to the formal parameter λ . We will fix f and consider the integral as a functional $I(g)$ taking values in $\mathbb{R}[[\lambda]]$. It is easy to derive from the relation

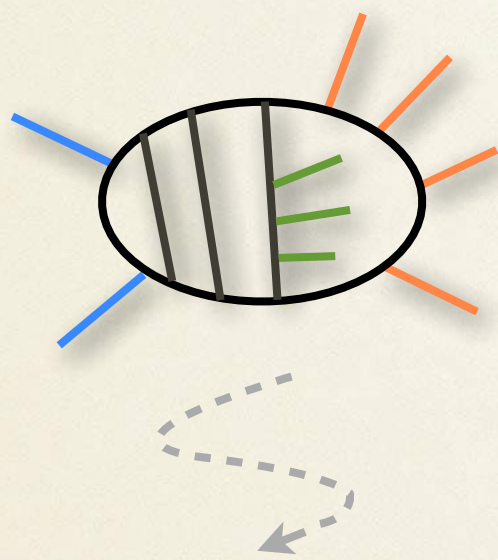
$$\int \partial_a(h(x)e^{f(x)})dx = 0$$

that the functional $I(g)$ vanishes in the case when g has the form

$$g = \partial_a h + (\partial_a f)h.$$

Feynman Integrals

● Momentum-space Representation



A Feynman diagram representing an N-denominator generic integral. It consists of an oval with several vertical lines inside. Two blue lines enter from the left, and two orange lines exit to the right. Three green lines enter from the right. A dashed arrow points from the diagram towards the text 'N-denominator generic Integral'.

$$= I_{a_1, \dots, a_N} = \int \prod_{i=1}^L d^d k_i \left(\prod_{n=1}^N \frac{1}{D_n^{a_n}} \right)$$

N-denominator
generic Integral

L loops, $E+1$ external momenta,
 $N = LE + \frac{1}{2}L(L+1)$ (generalised) denominators
total number of *reducible* and *irreducible*
scalar products
’t Hooft & Veltman

On-Shell and Unitarity-based Methods

Penrose, ..., Bern, Dixon, Dunbar, Kosower, ... , Witten, Cachazo, Svrzcek, Britto, Feng, Hodges, Spradlin, Volovich, ...

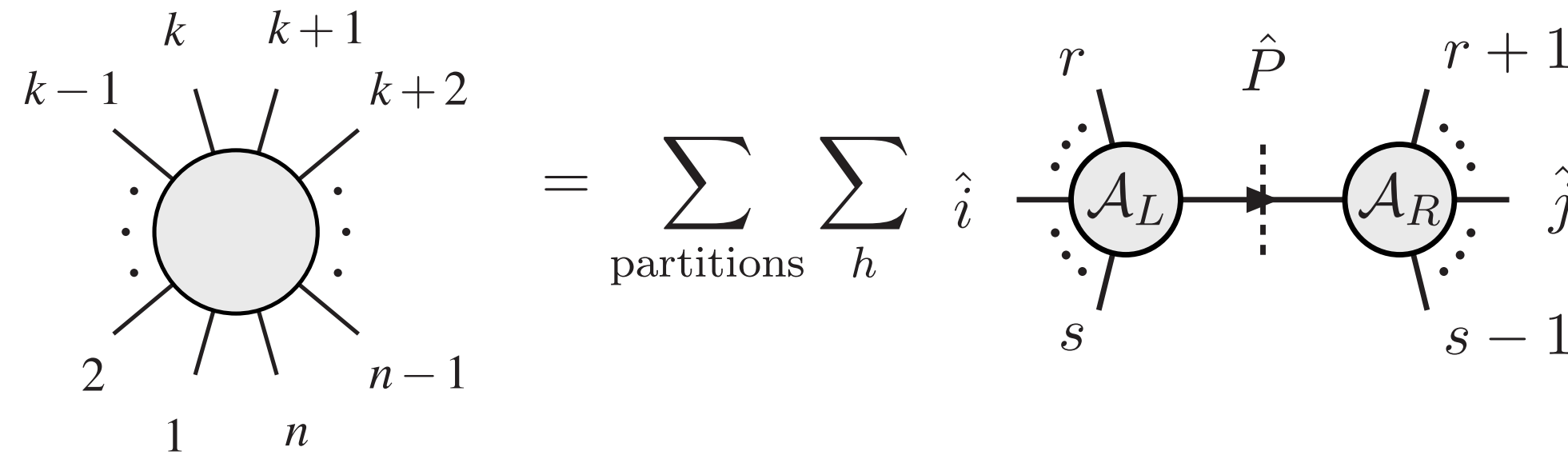
● BCFW Recurrence Relation and Cauchy's Residue theorem for holomorphic functions

Britto, Cachazo, Feng, Witten

$$\frac{1}{2\pi i} \oint \frac{\mathcal{A}_n(z)}{z} = \mathcal{A}_n(\infty) = -\mathcal{A}(0) - \sum_{\text{poles}} \text{Res} \mathcal{A}(z)$$

If $\mathcal{A}_n(\infty) = 0$, then one obtains the relation

$$\mathcal{A}_n(0) = - \sum_{\text{poles}} \text{Res} \mathcal{A}_n(z)$$



$$\mathcal{A}_n(p_1^{h_1}, \dots, p_n^{h_n}) = \sum_{\text{partition}} \sum_h \mathcal{A}_L(p_r, \dots, \hat{p}_i, \dots, p_s, -\hat{P}_{r:s}^h) \frac{1}{P^2} \mathcal{A}_R(\hat{P}_{r:s}^h, p_{s+1}, \dots, \hat{p}_j, \dots, p_{r-1})$$

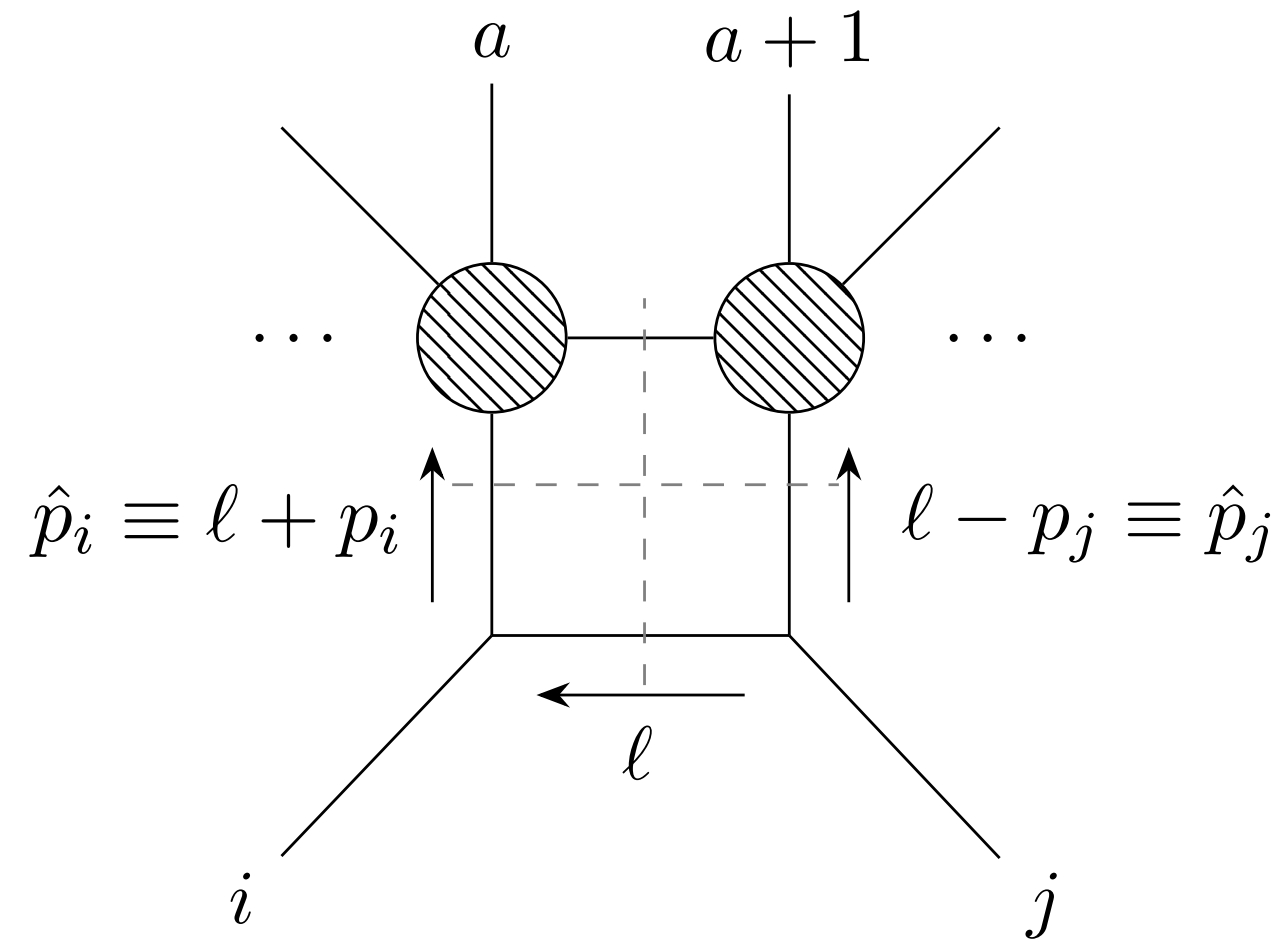
Complex deformation of on-shell momenta across the cut: $\hat{p}_i + \hat{p}_j = p_i + p_j$, $\hat{p}_i^2 = \hat{p}_j^2 = 0$.

$$1) \quad \begin{aligned} \hat{p}_i &= p_i + z\epsilon_{ij}, \\ \hat{p}_j &= p_j - z\epsilon_{ij}, \end{aligned} \quad \epsilon_{ij}^\mu = \frac{\langle i | \gamma^\mu | j \rangle}{2} \quad 2) \quad \begin{aligned} \hat{p}_i &= p_i + \bar{z}\epsilon_{ji}, \\ \hat{p}_j &= p_j - \bar{z}\epsilon_{ji}, \end{aligned} \quad \epsilon_{ji} = \frac{\langle j | \gamma^\mu | i \rangle}{2}$$

On-Shell and Unitarity-based Methods

● BCFW Recurrence Relation & Generalised Unitarity

Britto, Cachazo, Feng;



$$p_i^2 = p_j^2 = 0$$

$$\ell^\mu = x_1 p_i^\mu + x_2 p_j^\mu + x_3 \epsilon_{ij}^\mu + x_4 \epsilon_{ji}^\mu$$

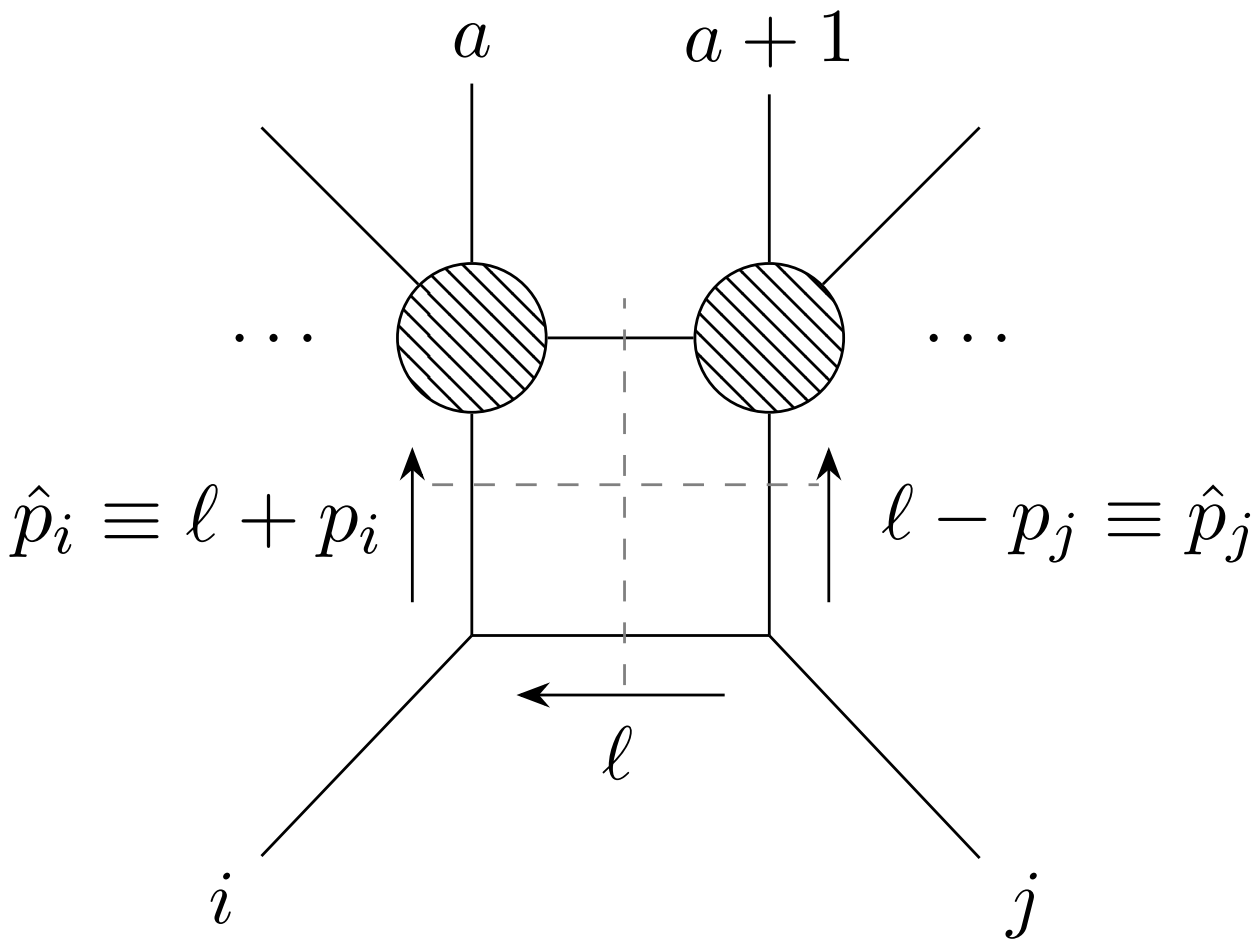
The **two** solutions of the 4ple-cut...

$$\ell_+^\mu = z \epsilon_{ij}^\mu, \quad \ell_-^\mu = \bar{z} \epsilon_{ji}^\mu$$

On-Shell and Unitarity-based Methods

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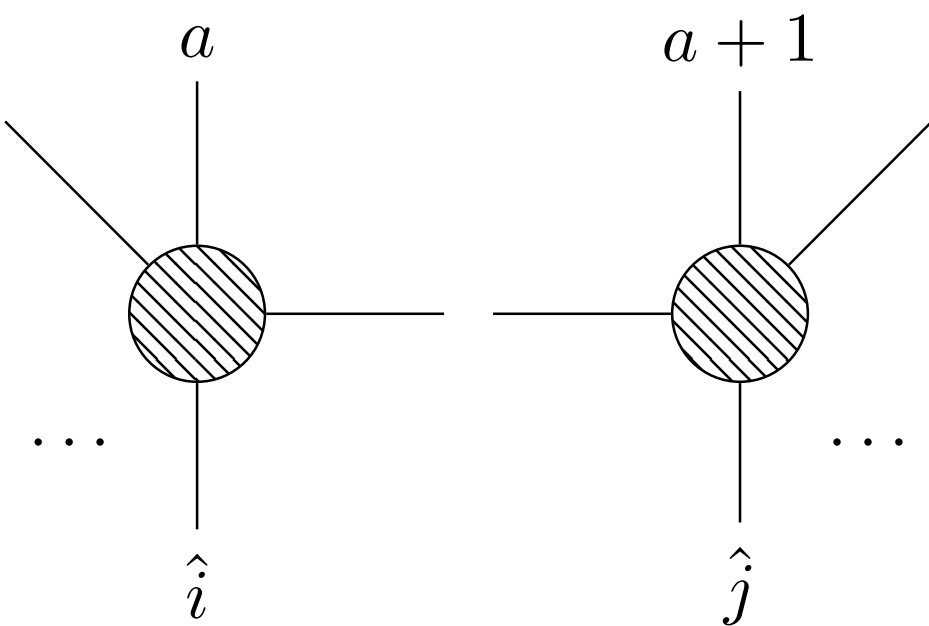
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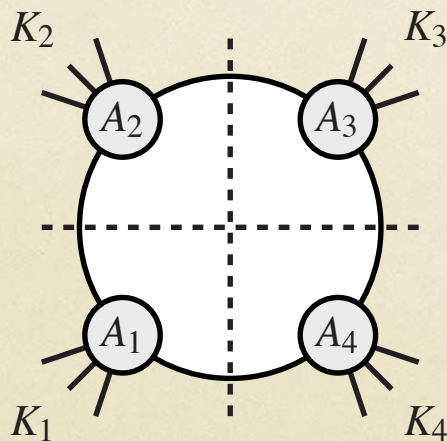


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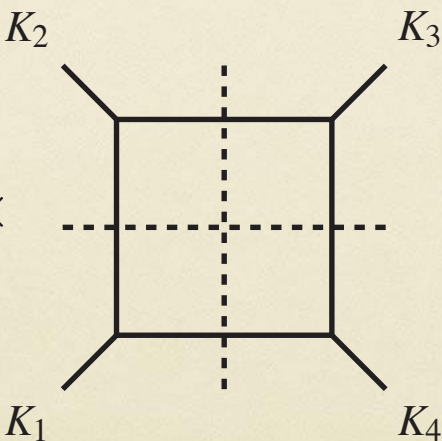
$$\ell_+^\mu = z \epsilon_{ij}^\mu \; , \qquad \ell_-^\mu = \bar{z} \epsilon_{ji}^\mu$$

...the **two** options for the BCFW deformation

● Coefficients of 1-loop box integrals from 4ple-cuts



$$= \; c_{[K_1|K_2|K_3|K_4]} \times$$



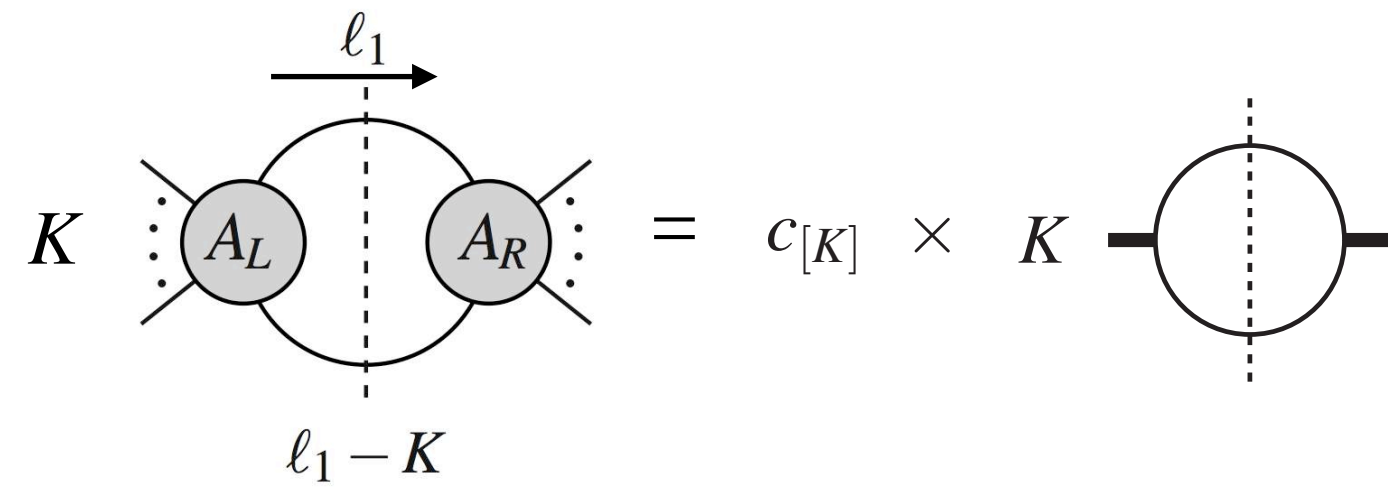
$$c_{[K_1|K_2|K_3|K_4]} \; = \; \frac{I_4(\ell_+) + I_4(\ell_-)}{2}$$

Double-Cut and Stokes' Theorem P.M. (2009)

- **Cauchy-Pompeieu's Formula** *alias Stokes' Theorem* for non-holomorphic functions

$$2\pi i \mathcal{F}(z_0) = \int_{\partial D} \frac{\mathcal{F}(z)}{z - z_0} dz - \iint_D \frac{\mathcal{F}_{\bar{z}}}{z - z_0} d\bar{z} \wedge dz.$$

- **Coefficients of 1-loop bubble integrals from 2ple-cuts**



Complex deformation of on-shell momenta across the cut:

$$K^\mu = p_i^\mu + p_j^\mu, \quad p_i^2 = 0 = p_j^2,$$

$$\ell_1^\mu \equiv t \ell^\mu$$

$$\ell^\mu = p_i^\mu + z\bar{z} p_j^\mu + z \epsilon_{ij}^\mu + \bar{z} \epsilon_{ji}^\mu$$

$$\int d^4 \ell_1 \delta(\ell_1^2) \delta((\ell_1 - K)^2) A_L(\ell_1) A_R(\ell_1) = \oint dz \int d\bar{z} \int dt t^2 \delta\left(t - \frac{1}{(1 + z\bar{z})}\right) A_L(t\ell) A_R(t\ell)$$

$$= \oint dz \int d\bar{z} f(z, \bar{z}) \quad f(z, \bar{z}) = \frac{A_L(z, \bar{z}) A_R(z, \bar{z})}{(1 + z\bar{z})^{2+\alpha_L+\alpha_R}}$$

$\alpha_{L,R}$ parametrizes the scaling behaviour of $A_{L,R}^{\text{tree}}$

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$\alpha_{L,R}$ parametrizes the scaling behaviour of $A_{L,R}^{\text{tree}}$

To begin with the integration, we find a primitive of f with respect to $\bar{z}$, say F , by keeping z as independent variable,

$$F(z, \bar{z}) = \int d\bar{z} \, f(z, \bar{z}) = \underbrace{F^{\text{rat}}(z, \bar{z})}_{\text{2-point function}} + \underbrace{F^{\text{log}}(z, \bar{z})}_{(n > 2)\text{-point functions}}$$

The z -integration will be performed by Cauchy's Residue Theorem

$$c_{[K]} = -\text{Res}_{z=0} F^{\text{rat}}(z, \bar{z}) - \text{Res}_{z\neq 0} F^{\text{rat}}(z, \bar{z}) \, .$$

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I computed my first intersection number, but I didn't know that!

Integrand Decomposition and Unitarity at integrand level

Ossola, Papadopoulos, Pittau (2005)
Ellis, Giele, Kunszt, Melnikov (2007)
... ..

- 1Loop Integrand Decomposition

$$\mathcal{A}_n^{\text{one-loop}} = c_{5,0} \text{pentagon} + c_{4,0} \text{square} + c_{4,4} \text{square}_{d+4} + c_{3,0} \text{triangle} + c_{3,7} \text{triangle}_{d+2} + c_{2,0} \text{bubble} + c_{2,9} \text{bubble}_{d+2} + c_{1,0} \text{circle}$$

- 2Loop Integrand Decomposition

Ossola & P.M. (2011) Mirabella, Ossola, Peraro, & P.M. (2012)

- Integrand Decomposition by Multivariate Polynomial Division

Zhang (2012); Badger Frellesvig Zhang (2012)
Mirabella, Ossola, Peraro, & P.M. (2012)

$$I = \int d^d \ell_1 \dots d^d \ell_L \mathcal{I}_{i_1 \dots i_n}$$
$$\mathcal{I}_{i_1 \dots i_n} = \frac{\mathcal{N}_{i_1 \dots i_n}}{D_{i_1} D_{i_2} \dots D_{i_n}} = \sum_{1=i_1 << i_{\max}}^n \frac{\Delta_{i_1 i_2 \dots i_{\max}}}{D_{i_1} D_{i_2} \dots D_{i_{\max}}} + \sum_{1=i_1 << i_{\max}-1}^n \frac{\Delta_{i_1 i_2 \dots i_{\max}-1}}{D_{i_1} D_{i_2} \dots D_{i_{\max}-1}} + \sum_{1=i_1 << i_{\max}-2}^n \frac{\Delta_{i_1 i_2 \dots i_{\max}-2}}{D_{i_1} D_{i_2} \dots D_{i_{\max}-2}} + \dots + \sum_{1=i_1 < i_2}^n \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{1=i_1}^n \frac{\Delta_{i_1}}{D_{i_1}} .$$

Δ 's are the **remainders** of the polynomial divisions (modulo Gröbner bases)

Independent Integrals emerge after integrating the Integrand Decomposition (non-vanishing term)

Independent Integrals <==> **Independent Monomials** appearing in the **Remainders**

warning: the number of Independent Monomials is non-minimal !!

The Maximum-Cut Theorem

Mirabella, Ossola, Peraro, & P.M. (2012)

At any loop ℓ , loops we define *maximum cut* as the set of vanishing denominators

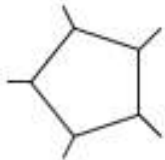
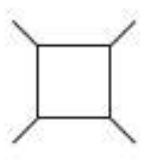
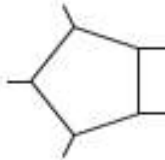
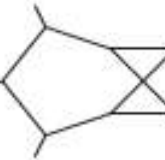
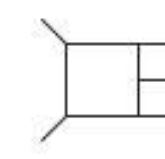
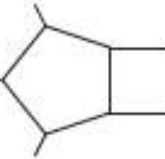
$$D_0 = D_1 = \dots = 0$$

which constrains completely the components of the loop momenta.

We assume that, in non-exceptional phase-space points, a maximum-cut has a finite number n_s of solutions, each with multiplicity one.

Then,

Theorem 4.1 (Maximum cut). *The residue at the maximum-cut is a polynomial parametrised by n_s coefficients, which admits a univariate representation of degree $(n_s - 1)$.*

diagram	Δ	n_s	diagram	Δ	n_s
	c_0	1		$c_0 + c_1 z$	2
	$\sum_{i=0}^3 c_i z^i$	4		$\sum_{i=0}^3 c_i z^i$	4
	$\sum_{i=0}^7 c_i z^i$	8		$\sum_{i=0}^7 c_i z^i$	8

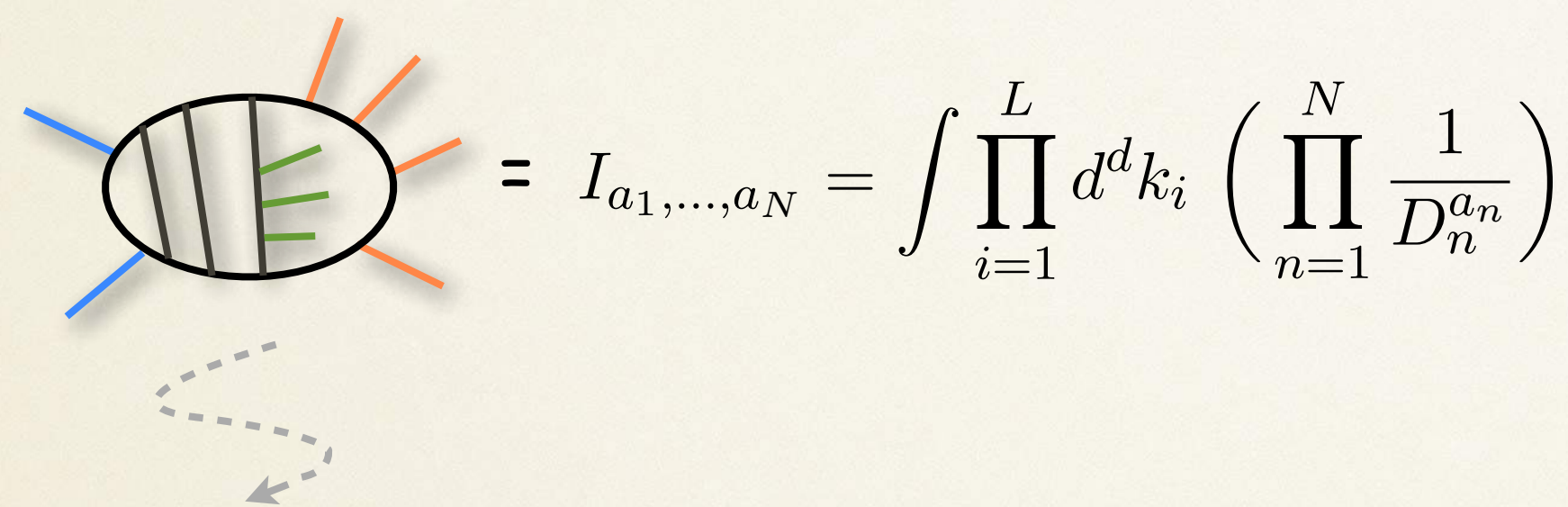
Based on the **Shape Lemma** in computational algebraic geometry:

The number n_s of solutions of a zero dimensional system, whose polynomials generate the ideal $I(V)$, with $V \subset \mathbb{C}^n$, corresponds to the dimension of the Quotient Ring:

$$n_s = \dim\left(\mathbb{C}[\mathbf{z}]/I(V)\right) = \dim\left(\mathbb{C}[\mathbf{z}]/\langle \mathcal{G} \rangle\right)$$

Feynman Integrals

● Momentum-space Representation



N-denominator
generic Integral

L loops, $E+1$ external momenta,
 $N = LE + \frac{1}{2}L(L+1)$ (generalised) denominators
total number of *reducible* and *irreducible*
scalar products
’t Hooft & Veltman

● Integration-by-parts Identities Tkachov; Chetyrkin & Tkachov

Laporta, Remiddi, Kuehn,
Baikov, Smirnov, Melnikov, Gehrmann, Weinzierl, Anastasiou,...

$$\int \prod_{i=1}^L d^d k_i \frac{\partial}{\partial k_j^\mu} \left(v_\mu \prod_{n=1}^N \frac{1}{D_n^{a_n}} \right) = 0$$

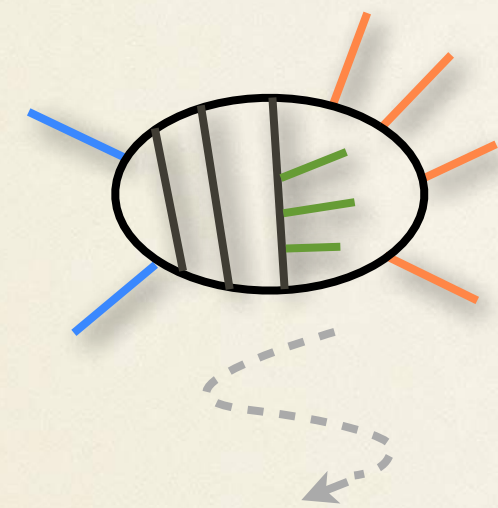
$v_\mu = v_\mu(p_i, k_j)$ arbitrary

● IBP identities

$$\sum_i b_i I_{a_1, \dots, a_i \pm 1, \dots, a_N} = 0$$

Parametric Representation(s)

- Upon a change of integration variables



$$= I_{a_1, \dots, a_N} = \int_{\mathcal{C}} u(\mathbf{z}) \varphi_N(\mathbf{z})$$

N-denominator
generic Integral

$$\varphi_N(\mathbf{z}) = \hat{\varphi}(\mathbf{z}) d^N \mathbf{z} \quad \text{differential } N\text{-form}$$

$$d^N \mathbf{z} = dz_1 \wedge \dots \wedge dz_N$$

$$\hat{\varphi}_N(\mathbf{z}) = f(\mathbf{z}) \prod_i z_i^{-a_i}$$

$$u(\mathbf{z}) = \mathcal{P}(\mathbf{z})^\gamma$$

$$\mathcal{P}(\mathbf{z}) = \text{graph-Polynomial}$$

$$\gamma(d) = \text{generic exponent}$$

- **Integration-by-parts:** two situations may occur

$$\int_{\mathcal{C}} d \left(u(\mathbf{z}) \varphi_N(\mathbf{z}) \right) \quad \begin{cases} \neq 0, \\ = 0, \end{cases} \quad u(\partial \mathcal{C}) = 0.$$

- Schwinger representation, Lee-Pomeranski repr'n
- Baikov representation, or other repr'ns

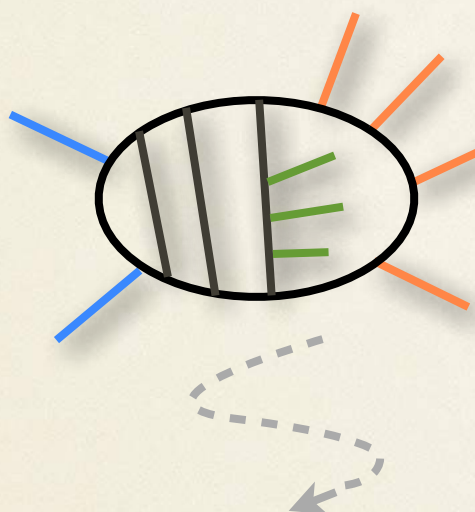
- **IBP identities**

$$\sum_i b_i I_{a_1, \dots, a_i \pm 1, \dots, a_N} = 0$$

Feynman Integrals :: Baikov Representation

● Denominators as integration variables Baikov (1996)

$\{D_1, \dots, D_N\} \rightarrow \{z_1, \dots, z_N\} \equiv \mathbf{z}$



N-denominator
generic Integral

$$= I_{a_1, \dots, a_N} \equiv K(d, s_{ij}) \int_{\mathcal{C}} d\mathbf{z} \, B(\mathbf{z})^\gamma \prod_{i=1}^N \frac{1}{z_i^{a_i}}$$



Volume²

$$B(\mathbf{z}) = \det(q_i \cdot q_j)$$

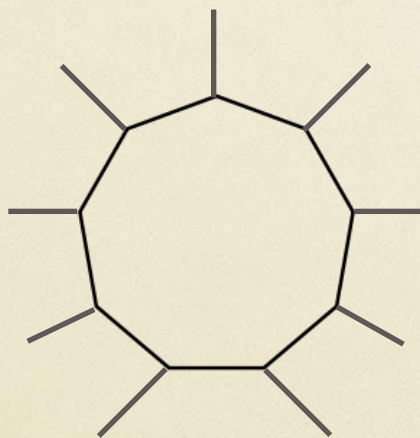
$$\gamma \equiv (d - E - L - 1)/2$$

$$q = \{p_i, k_j\}$$

$$B(\partial\mathcal{C} = 0)$$

Fundamental property

● 1-loop Nonagon

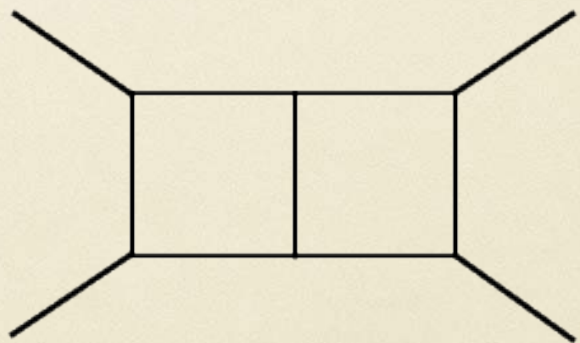


$$N = LE + \frac{1}{2}L(L + 1)$$

$$\int_{\mathcal{C}} dz_1 \wedge \dots \wedge dz_9 \frac{B(\mathbf{z})^\gamma}{z_1^{n_1} \dots z_9^{n_9}}$$

$B(\mathbf{z}), \mathcal{C}, \gamma$ depend on the graph.

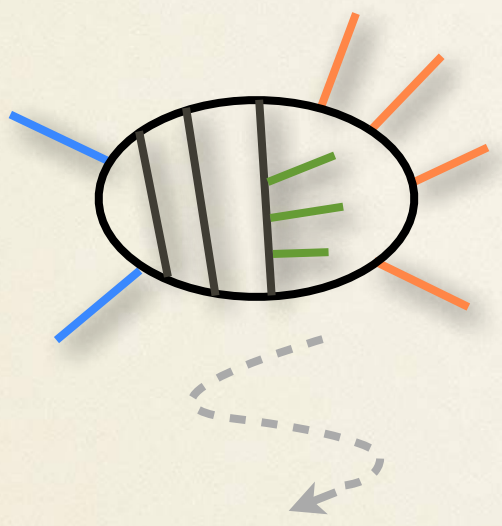
● 2-loop Box



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N-denominator
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Volume²

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$$\gamma \equiv (d - E - L - 1)/2$$
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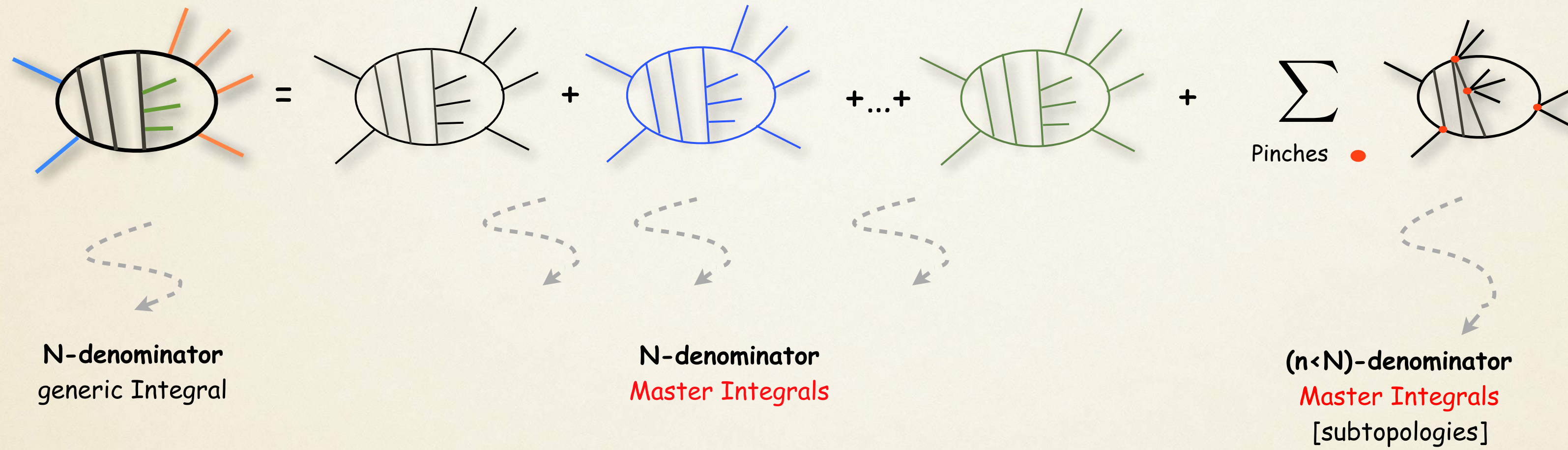
Fundamental property

● Integration-by-parts Identites Zhang, Larsen; Lee; Frellesvig, Papadopoulos

$$\int_{\mathcal{C}} d \left(h(\mathbf{z}) \, B(\mathbf{z})^\gamma \prod_{i=1}^N \frac{1}{z_i^{a_n}} \right) = 0 \qquad h(\mathbf{z}) \text{ arbitrary rational function}$$
$$B(\partial\mathcal{C}) = 0$$

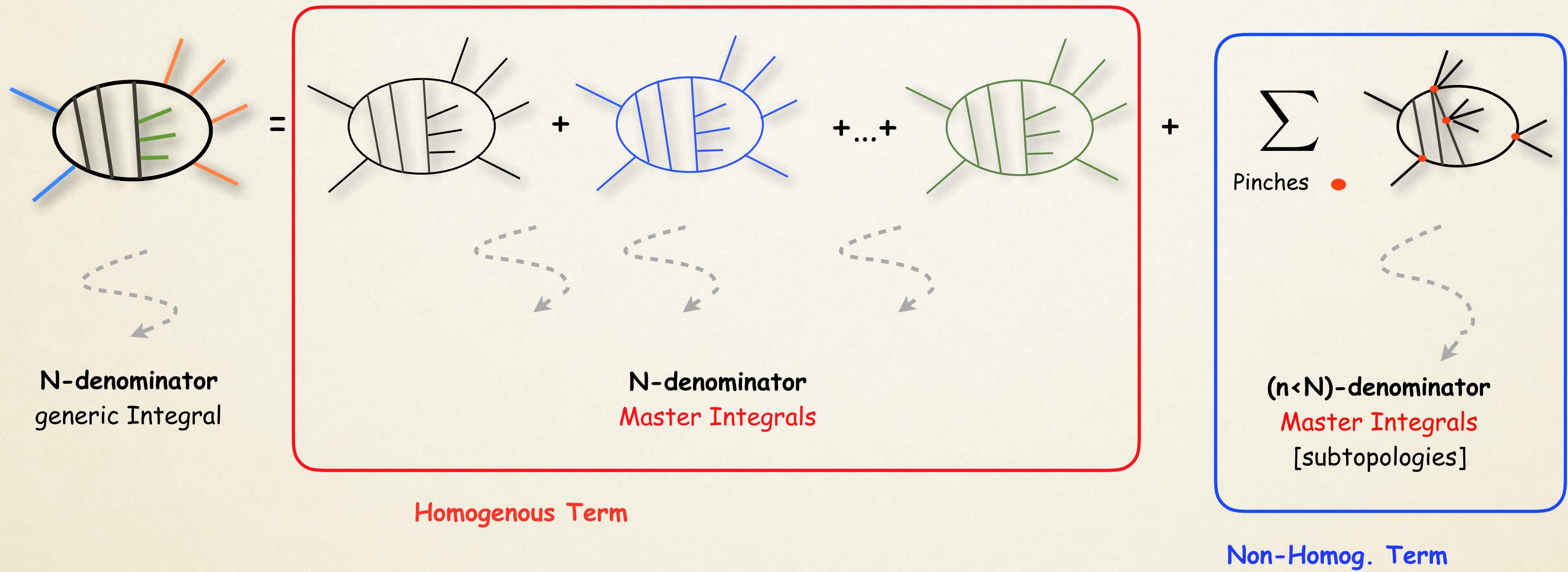
Integration-by-parts identities

- Relations among Integrals in dim. reg.



Integration-by-parts identities

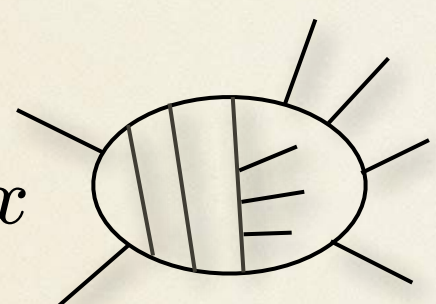
- Relations among Integrals in dim. reg.



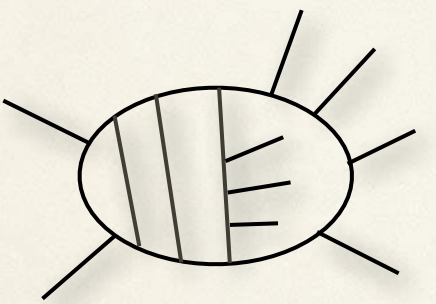
Integration-by-parts identities :: *byproducts*

● **1st order Differential Equations for MIs** Barucchi, Ponzano; Kotikov; Remiddi, & Gerhmann; Bonciani, Remiddi & P.M. ...Weinzierl, Adams, Bogner ... Henn; Lee; Argeri, diVita, Mirabella, Schubert, Tancredi, Schlenck & P.M.; ...

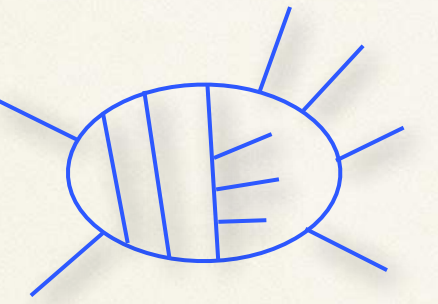
∂_x



=



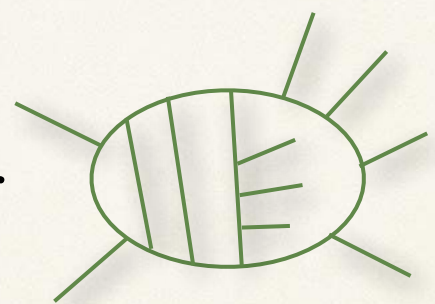
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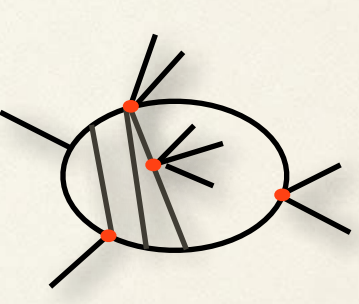
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$\sum$

Pinches

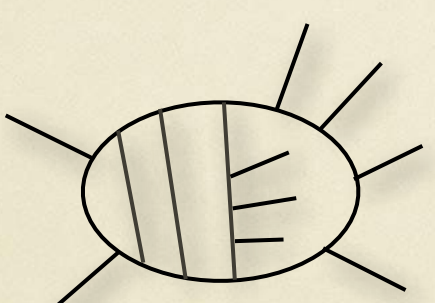


Homogenous Term

Non-Homog. Term

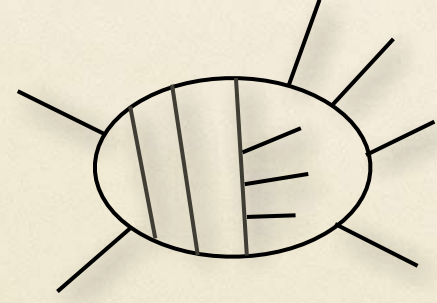
● **Dimension-Shift relations and Gram determinant relations**

Bern Dixon Kosower; Tarasov; Lee; Bernstein-Tkachov;
Passarino et al.; Gluza, Kajda, Kosower; Remiddi, Tancredi

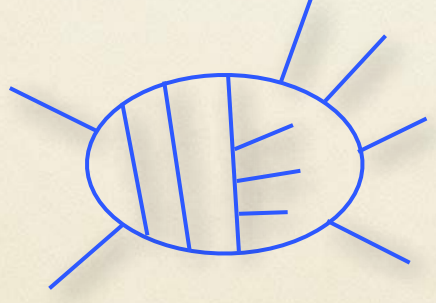


$\Big|_{d+2}$

=



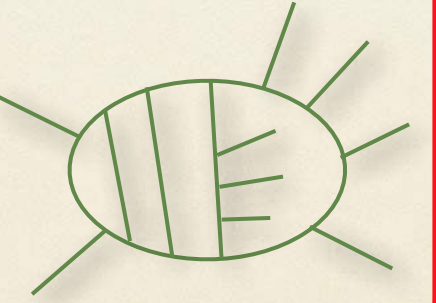
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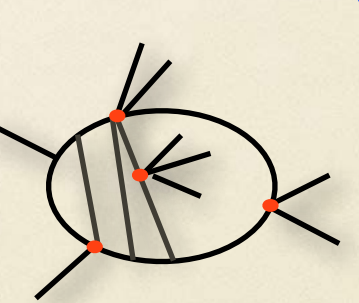
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$\sum$

Pinches

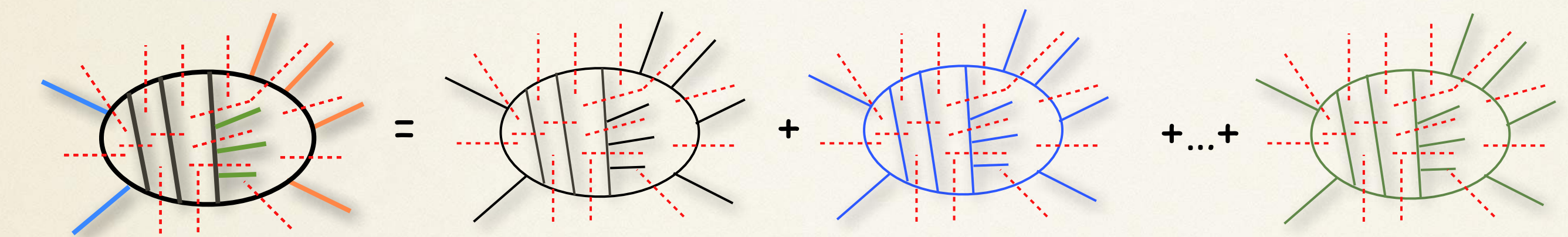


Homogenous Term

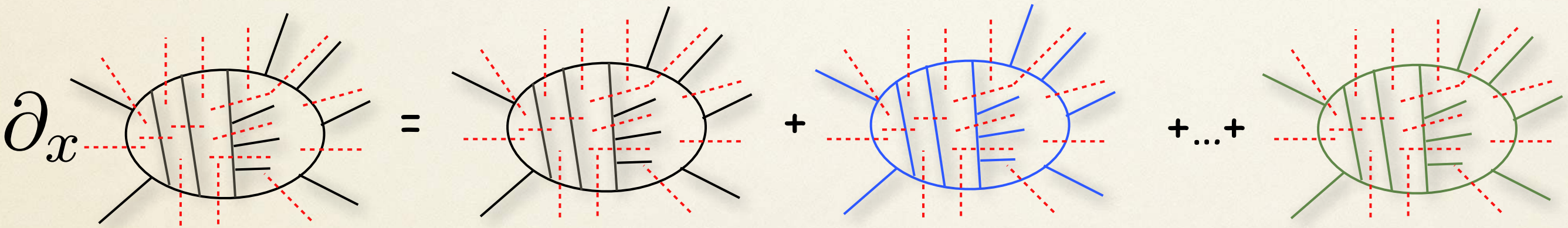
Non-Homog. Term

Integral Relations on *Maximal Cuts*

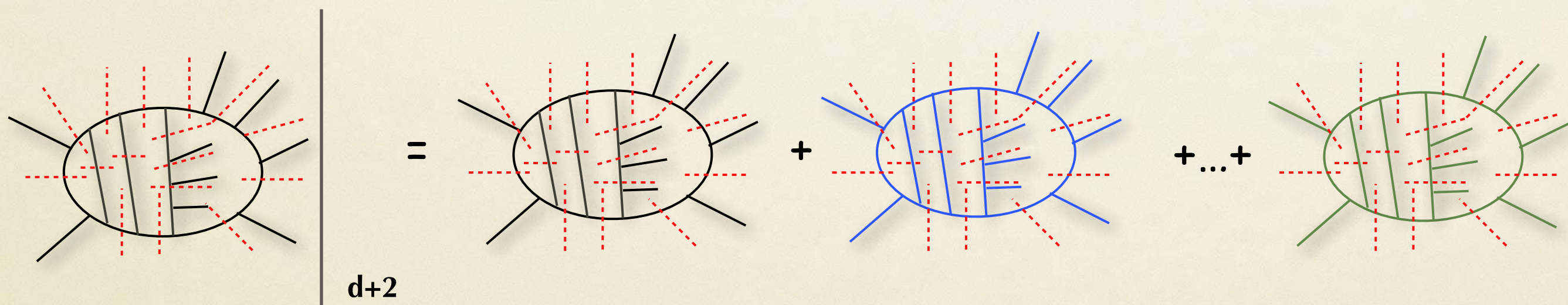
● cutting m (all) internal lines $D_1 = \dots, D_m = 0 \iff z_1 = \dots, z_m = 0, \quad m \leq N$



Larsen & Kosower;
+ CaronHuot;



Remiddi & Tancredi;
Primo & Tancredi;



Lee & Smirnov;

Homogenous Term only

Vanishing
Non-Homog. Term

- Maximal cuts are solutions of the Homogenous Differential Equations
- Maximal cuts are solutions of the Homogenous Dimensional Recurrence Relations

What are we looking for?

Vector Space Structure

- **Vector decomposition**

$$I = \sum_{i=1}^{\nu} c_i J_i$$

ν = dimension of the vector space

- **Projections**

$$c_i = \begin{cases} I \cdot J_i , & J_i \cdot J_j = \delta_{ij} \\ \sum_{j=1}^{\nu} I \cdot J_j (C^{-1})_{ji} , & J_i \cdot J_j = C_{ij} \neq \delta_{ij} \end{cases}$$

- **Completeness**

$$\sum_{i,j} J_j (C^{-1})_{ji} J_i = \mathbb{I}_{\nu \times \nu}$$

Vector Space Structure

- **Vector decomposition**

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- **Completeness**

$$\sum_{i,j} J_j (C^{-1})_{ji} J_i = \mathbb{I}_{\nu \times \nu}$$

The two questions:

- 1) what is the vector space dimension ν ?
- 2) what is the *scalar product* “.” between integrals ?

Basics of Intersection Theory

Aomoto, Brown, Cho, Goto, Kita, Matsubara-Heo, Mazumoto, Mimachi, Mizera, Yoshida,...

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \dots, z_m)$

$$I = \underbrace{\int_{\mathcal{C}} u(\mathbf{z})}_{\text{twisted cycle}} \underbrace{\varphi_m(\mathbf{z})}_{\text{twisted cocycle}}$$

$\varphi_m(\mathbf{z})$ is a differential m -form
 $u(\mathbf{z})$ is a multivalued function (regulating *all* poles of φ_m)
 $u(\partial\mathcal{C}) = 0$

Related topics:

- KLT, CHY, Color/Kin. relations
- Coaction of Feynman Integrals
- D-module Theory
- Motivic Periods and Double Copy
- Monodromy Relations
- Correlation Functions on Lattice

Mizera (2017)

Abreu, Britto, Duhr, Gardi, Matthew (2018-19)

Bitoun, Bogner, Klausen, Panzer (2018)

Brown, Dupont (2019)

Casali, Mizera, Tourkine (2019-20)

Weinzierl (2020)

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$\varphi_m(\mathbf{z})$ is a differential m -form
 $u(\mathbf{z})$ is a multivalued function (regulating *all* poles of φ_m)
 $u(\partial\mathcal{C}) = 0$

There could exist many forms φ_m that upon integration give the same result I

● **Covariant Derivatives** $\omega \equiv d \log u$ $\nabla_{\omega} \equiv d + \omega \wedge \equiv u^{-1} \cdot d \cdot u$ $\nabla_{-\omega} \equiv d - \omega \wedge \equiv u \cdot d \cdot u^{-1}$

● **Integrals** $I = \int_{\mathcal{C}} u \varphi_m = \int_{\mathcal{C}} u \left(\varphi_m + \nabla_{\omega} \varphi_{m-1} \right)$

● **Dual Integrals** $\tilde{I} = \int_{\mathcal{C}} u^{-1} \phi_m = \int_{\mathcal{C}} u^{-1} \left(\phi_m + \nabla_{-\omega} \phi_{m-1} \right)$

Pairings of Cycles and Co-cycles

- **Co-Homology Groups**

$$H_{\omega}^m \equiv \{m\text{-forms } \varphi_m \mid \nabla_{\omega} \varphi_m = 0\} / \{\nabla_{\omega} \varphi_{m-1}\}$$

$$H_{-\omega}^m \equiv \{m\text{-forms } \phi_m \mid \nabla_{-\omega} \phi_m = 0\} / \{\nabla_{-\omega} \phi_{m-1}\}$$

- **Basic building blocks**

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H_{\omega}^m$$

$$|\varphi_R\rangle \equiv \varphi_R(\mathbf{z}) \in H_{-\omega}^m$$

$$|\mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z})$$

$$[\mathcal{C}_L| \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1}$$

Pairings of Cycles and Co-cycles

- **Co-Homology Groups**

$$H_{\omega}^m \equiv \{m\text{-forms } \varphi_m \mid \nabla_{\omega} \varphi_m = 0\} / \{\nabla_{\omega} \varphi_{m-1}\}$$

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- **Basic building blocks**

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H_{\omega}^m \quad | \varphi_R \rangle \equiv \varphi_R(\mathbf{z}) \in H_{-\omega}^m \quad | \mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \quad [\mathcal{C}_L | \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1}$$

- **Integrals :: pairings of cycles and co-cycles**

$$\langle \varphi_L | \mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \varphi_L(\mathbf{z}) = I$$

- **Dual Integrals :: pairings of cycles and co-cycles**

$$[\mathcal{C}_L | \varphi_R \rangle \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \varphi_R(\mathbf{z}) = \tilde{I}$$

- **Intersection numbers for cycles :: pairings of cycles**

$$[\mathcal{C}_L | \mathcal{C}_R] \equiv \text{intersection number}$$

- **Intersection numbers for co-cycles :: pairings of co-cycles**

$$\langle \varphi_L | \varphi_R \rangle \equiv \int_{\mathcal{C}} \iota(\varphi_L) \wedge \varphi_R$$

Linear Relations

Feynman Integrals and Intersection Theory

Mizera & P.M. (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

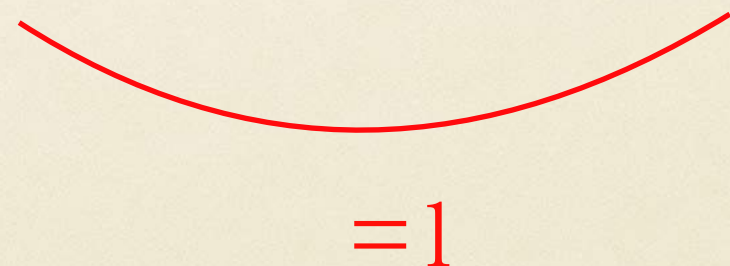
Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

$$I = \langle \varphi | \mathcal{C} \rangle$$

Consider a set of ν MIs,

$$J_i = \int_{\mathcal{C}} u(\mathbf{z}) e_i(\mathbf{z}) = \langle e_i | \mathcal{C} \rangle, \quad i = 1, \dots, \nu,$$

$$I = \sum_{i=1}^{\nu} c_i J_i$$

$$\langle \varphi | = \sum_{i=1}^{\nu} c_i \langle e_i | = \sum_{i,j=1}^{\nu} \langle \varphi | e_j \rangle (\mathbf{C}^{-1})_{ji} \langle e_i |$$


$$c_i = \sum_{j=1}^{\nu} \langle \varphi | e_j \rangle (\mathbf{C}^{-1})_{ji}$$

Quadratic Relations

Riemann Twisted Periods Relations (RTPR)

- Completeness for forms

$$\sum_{i,j=1}^{\nu} |e_j\rangle (\mathbf{C}^{-1})_{ji} \langle e_i| = \mathbb{I}_c$$

$$\mathbf{C}_{ij} \equiv \langle e_i | e_j \rangle$$

- Completeness for contours

$$\sum_{i,j=1}^{\nu} [\mathcal{C}_j] (\mathbf{H}^{-1})_{ji} [\mathcal{C}_i] = \mathbb{I}_h$$

$$\mathbf{H}_{ij} \equiv [\mathcal{C}_i | \mathcal{C}_j]$$

Riemann Twisted Periods Relations (RTPR)

- Completeness for forms

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$$\mathbf{H}_{ij} \equiv [\mathcal{C}_i | \mathcal{C}_j]$$

- Riemann Twisted Period Relations

Cho, Matsumoto (1995)

$$\langle \varphi_L | \varphi_R \rangle = \sum_{i,j} \langle \varphi_L | \mathcal{C}_{R,j}] [\mathcal{C}_{L,j} | \mathcal{C}_{R,i}]^{-1} [\mathcal{C}_{L,i} | \varphi_R \rangle$$

$$[\mathcal{C}_L | \mathcal{C}_R] = \sum_{i,j} [\mathcal{C}_L | \varphi_{R,j} \rangle \langle \varphi_{L,j} | \varphi_{R,i} \rangle^{-1} \langle \varphi_L | \mathcal{C}_R]$$

Vector spaces of differential forms

● **Space Dimensions = Number of Master Integrals**

ν = number of independent *master* integrals

Chetyrkin, Tkachov (1981); Remiddi, Laporta (1996); Laporta (2000)

= is finite Smirnov, Petuckhov (2010)

= number of critical points of graph polynomials

Lee, Pomeranski (2013)

= is related to Euler characteritics χ_E

Aluffi, Marcolli (2008)

Bitoun, Bogner, Klausen, Panzer (2018)

= number of independent integration contours

Bosma, Sogaard, Zhang (2017)

Primo, Tancredi (2017)

= number of independent forms

Mizera & P.M. (2018)

= $\dim H^m_{\pm\omega}$

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

Master Decomposition Formula

Mizera & P.M. (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

● Basis :: bra

$$\langle e_i | \quad i = 1, 2, \dots, \nu$$

● dual-Basis :: ket

$$|h_j\rangle \quad j = 1, 2, \dots, \nu$$

● Metric Matrix

$$\mathbf{C}_{ij} = \langle e_i | h_j \rangle$$

intersection number

● Decomposition of differential forms

projecting $\langle \varphi |$ onto a basis of $\langle e_i |$

$$\langle \varphi | = \sum_{i,j=1}^{\nu} \langle \varphi | h_j \rangle \underbrace{(\mathbf{C}^{-1})_{ji}}_{=1} \langle e_i |$$

$$c_i = \sum_{j=1}^{\nu} \langle \varphi | h_j \rangle (\mathbf{C}^{-1})_{ji}$$

Univariate Intersection Number Matsumoto (1996,98)

● A natural pairing for twisted cocycles

$$\int_X \varphi_L \wedge \varphi_R \quad \rightarrow \quad \langle \varphi_L | \varphi_R \rangle \equiv \int_X \varphi_L^c \wedge \varphi_R$$

$$\varphi_L^c \equiv \varphi_L - \sum_{z_i \in \mathcal{P}_\omega} \nabla_\omega(h_i \psi_i) \quad \text{with compact support}$$

$$= \varphi_L - (\psi_i(dh_i) + h_i \nabla_\omega \psi_i)$$

the regulating function

$$h_i = h_i(z, \bar{z}) \equiv \begin{cases} 0, & \text{on } X \setminus U_i \\ 0 \leq h_i \leq 1, & \text{on } D_i = U_i \setminus V_i \\ 1, & \text{on } V_i \end{cases}$$

Iff

$$\nabla_\omega \psi_i = \varphi_L,$$

for $z \rightarrow z_i$,

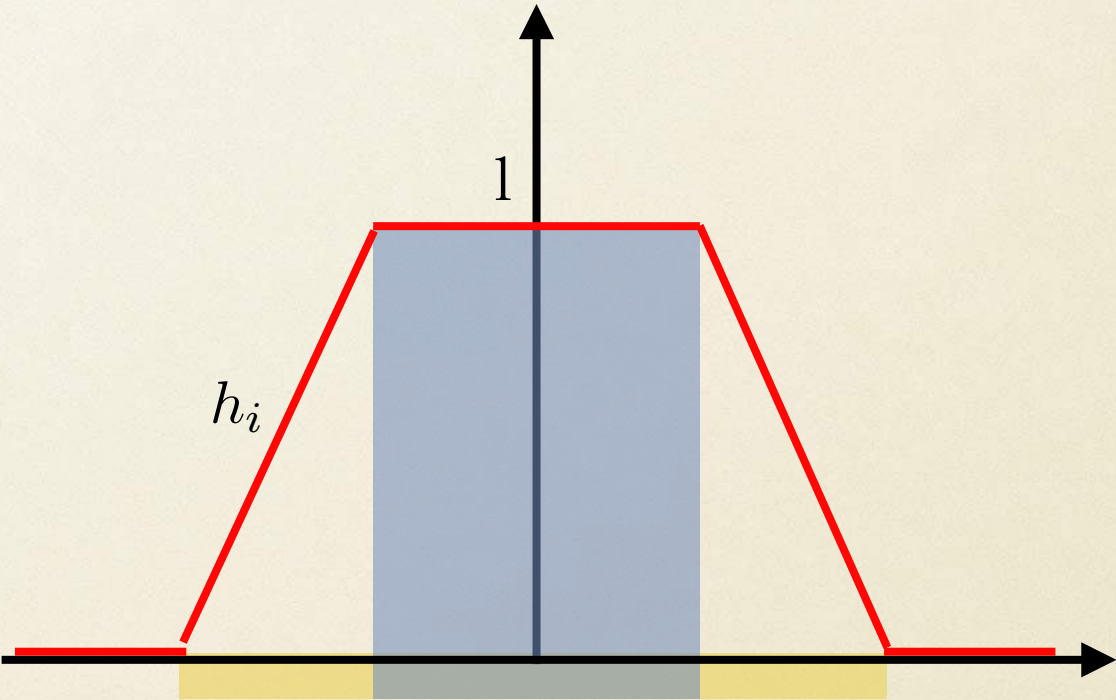
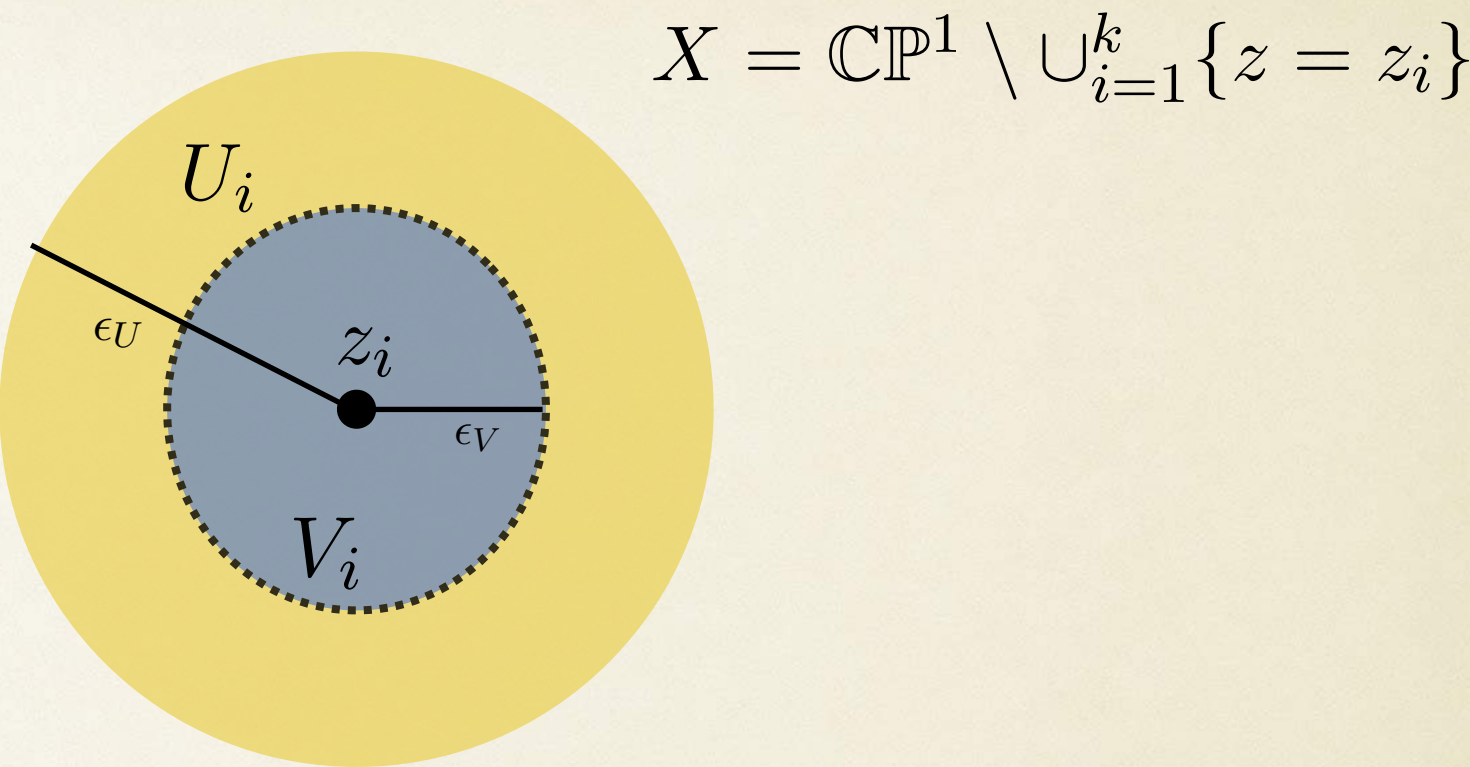
namely on $U_i \setminus \{z_i\}$

then

$$\varphi_L^c \equiv \begin{cases} 0, & \text{on } V_i \\ \varphi_L - (\psi_i(dh_i) + h_i \varphi_L), & \text{on } D_i = U_i \setminus V_i \\ \varphi_L, & \text{on } X \setminus U_i \end{cases}$$



$$\begin{aligned} \langle \varphi_L | \varphi_R \rangle &= - \int_{D_i} \psi_i(dh_i) \wedge \varphi_R = - \int_{D_i} d(h_i \psi_i \varphi_R) = - \int_{\partial D_i} h_i \psi_i \varphi_R \\ &= \int_{\partial V_i} \psi_i \varphi_R = \oint \psi_i \varphi_R \end{aligned}$$



Intersection Numbers :: *1-forms*

● **1-form** $\langle \varphi | \equiv \hat{\varphi}(z) \, dz$ $\hat{\varphi}(z)$ rational function

● **Zeroes and Poles of ω**

$$\omega \equiv d \log u$$

$\nu = \{ \text{the number of solutions of } \omega = 0 \}$

$\mathcal{P} \equiv \{ z \mid z \text{ is a pole of } \omega \}$

$\mathcal{P}$ can also include the pole at infinity if $\text{Res}_{z=\infty}(\omega) \neq 0$.

● **Intersection Numbers (for cocycles)**

Matsumoto (1996, 98)

1-forms φ_L and φ_R

$$\langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathcal{P}} \text{Res}_{z=p} \left(\psi_p \varphi_R \right)$$

ψ_p is a function (0-form), solution to the differential equation $\nabla_\omega \psi = \varphi_L$, around p

Intersection Numbers :: *1-forms*

- Solving a 1st ODE

$$\nabla_{\omega}\psi = \varphi_L$$

$$\frac{d}{dz}\psi + \omega\psi = \varphi_L$$

- Way-1 :: Laurent expansions

$$\tau \equiv z - p$$

known: $\varphi_{L,p}$ and ω_p

ansatz:
$$\psi_p = \sum_{j=\min}^{\max} \psi_p^{(j)} \tau^j + \mathcal{O}(\tau^{\max+1})$$

- Fixing the coefficients ==>
==> solving a simple, triangular system

- Log forms

Iff φ_L and ω contain only *simple poles*

~~$$\frac{d}{dz}\psi + \omega\psi = \varphi_L$$~~

$$\psi_p = \left. \frac{\varphi_L}{\omega} \right|_{z \rightarrow z_p}$$

- Way-2 :: Variation of parameters

$$\psi = \frac{1}{u} \int u \varphi_L$$

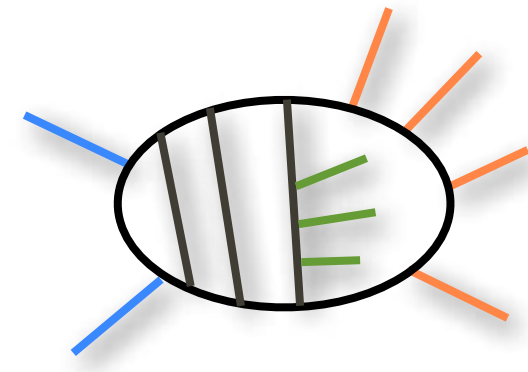
$$\langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathcal{P}} \text{Res}_{z=p} \left\{ \left(\int u \varphi_L \right) \left(u^{-1} \varphi_R \right) \right\}$$

- left term :: series expansion + integration
- right term :: series expansion
- Residue extraction :: no system-solving required!

Feynman Integrals & Intersection Theory

Mizera & P.M. (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)



$$= I_{a_1, a_2, \dots, a_N} \equiv \int \prod_{i=1}^L \frac{d^d k_i}{\pi^{d/2}} \prod_{j=1}^N \frac{1}{D_j^{a_j}}$$

$$\equiv K \int_{\mathcal{C}} u \varphi \equiv K \langle \varphi | \mathcal{C} \rangle$$

- **Baikov representation**

$$u = B^\gamma, \quad \gamma \equiv (d - E - L - 1)/2$$

$$\omega \equiv d \log(u) = \gamma d \log(B)$$

$$\varphi \equiv \hat{\varphi} d^N \mathbf{z}, \quad \hat{\varphi} \equiv \frac{1}{z_1^{a_1} z_2^{a_2} \cdots z_N^{a_N}},$$

$$d^N \mathbf{z} \equiv dz_1 \wedge dz_2 \wedge \cdots \wedge dz_N$$

- **General formalism, which can be applied to other integral representations**

Mizera , Pokraka (2019); CaronHuot, Pokraka (2019)

Integrals reduction and Master Integrals

Mizera & P.M. (2018)

$\nu = \{\text{the number of solutions of } \omega = 0\}$

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

- Basis of Master Forms

$$\langle e_i | \quad | h_j \rangle \quad i = 1, 2, \dots, \nu$$

- Master Integrals

$$J_i \equiv K E_i \text{ , with } E_i \equiv \langle e_i | \mathcal{C}$$

- Integral Decomposition

$$I = K \langle \varphi | \mathcal{C} \rangle = \sum_{i=1}^{\nu} c_i J_i$$

$$\langle \varphi | = \sum_{i,j=1}^{\nu} \langle \varphi | h_j \rangle (\mathbf{C}^{-1})_{ji} \langle e_i |$$

$$c_i \equiv \sum_{j=1}^{\nu} \langle \varphi | h_j \rangle (\mathbf{C}^{-1})_{ji}$$

Basis choices

for $i = 1, 2, \dots, \nu$

- **dLog Basis**

$$\langle e_i | = \langle \varphi_i | \equiv \frac{dz}{z - z_i} \quad z_i \text{ are poles of } \omega$$

- **Monomial Basis**

$$\langle e_i | = \langle \phi_i | \equiv z^{i-1} dz$$

- **Orthonormal Basis**

$\mathcal{P} = \{z_1, z_2, \dots, z_{\nu+1}, z_{\nu+2}\}$ pick two special ones, say $z_{\nu+1}$ and $z_{\nu+2}$

$$\langle e_i | \equiv d \log \frac{z - z_i}{z - z_{\nu+1}}, \quad |h_i\rangle \equiv \text{Res}_{z=z_i}(\omega) d \log \frac{z - z_i}{z - z_{\nu+2}}$$

$$\mathbf{C}_{ij} = \delta_{ij} \quad \langle \varphi | = \sum_{i=1}^{\nu} \langle \varphi | h_i \rangle \langle e_i |$$

- **Gram-Schmidt method**

- **...or any arbitrary rational basis...**

Two special applications:

Dimensional Recurrence Relation

- **MLs in (d+2n) dimensions**

$$J_i^{(d+2n)} \equiv K(d+2n) E_i^{(d+2n)} \quad E_i^{(d+2n)} \equiv \langle B^n e_i | \mathcal{C} \rangle = \int_{\mathcal{C}} u (B^n e_i) , \quad u = B^\gamma , \quad \gamma \equiv (d-E-L-1)/2$$

- **Master Decomposition Formula**

$$\langle B^\nu e_i | = \sum_{n=0}^{\nu-1} c_n \langle B^n e_i | \quad n = 0, 1, \dots, \nu - 1$$

- **Recurrence Relations for Master Forms**

$$\sum_{n=0}^{\nu} c_n \langle B^n e_i | = 0 , \quad c_\nu \equiv -1$$

- **Recurrence Relations for Master Integrals**

$$\sum_{n=0}^{\nu} \alpha_n J_i^{(d+2n)} = 0 \quad \alpha_n \equiv c_n / K(d+2n)$$

System of Differential Equations

- External Derivative

$$\partial_x I = \partial_x \langle \varphi | \mathcal{C} \rangle = \partial_x \int_{\mathcal{C}} u \varphi = \int_{\mathcal{C}} u \left(\frac{\partial_x u}{u} \wedge + \partial_x \right) \varphi = \langle (\partial_x + \sigma) \varphi | \mathcal{C} \rangle$$

- External (connection) dLog-form

$$\nabla_{x,\sigma} \equiv \partial_x + \sigma \quad \sigma = \partial_x \log u$$

- Derivative of Master Forms

$$\partial_x \langle e_i | = \langle (\partial_x + \sigma \wedge) e_i | = \langle (\partial_x + \sigma \wedge) e_i | h_k \rangle (\mathbf{C}^{-1})_{kj} \langle e_j | = \boldsymbol{\Omega}_{ij} \langle e_j |$$

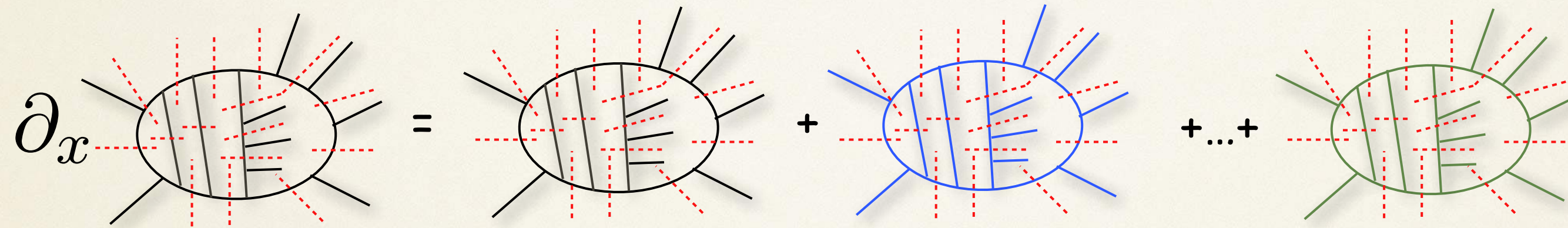
- System of DEQ for Master Forms

$$\partial_x \langle e_i | = \boldsymbol{\Omega}_{ij} \langle e_j | , \quad \boldsymbol{\Omega} = \boldsymbol{\Omega}(d, x)$$

An analogous System of DEQ can be derived for dual forms: $u \rightarrow u^{-1} \implies \nabla_{x,\sigma} \rightarrow \nabla_{x,-\sigma}$

System of Differential Equations

- (Homogenous) Solutions



- Period Matrix

$\nu \times \nu$ matrix $\mathbf{P}$

For each i , the ν independent solutions

$$\mathbf{P}_{ij} = \langle e_i | \mathcal{C}_j \rangle = \int_{\mathcal{C}_j} u e_i , \quad i, j = 1, 2, \dots, \nu ,$$

- Example :: Derivative basis

ν -dimensional basis formed by $\langle e_i |$ and its derivatives up the $(\nu - 1)^{\text{th}}$ -order

$\mathbf{P}$ = Wronski matrix

Contiguity relations for Special Functions

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & **P.M.** (2019)

Euler Beta Integrals

$$I_n \equiv \int_{\mathcal{C}} u \, z^n dz \, , \quad u \equiv B^\gamma \, , \quad B \equiv z(1 - z) \, , \quad \mathcal{C} \equiv [0, 1]$$

- **Direct Integration**

$$I_n = \frac{\Gamma(1 + \gamma)\Gamma(1 + \gamma + n)}{\Gamma(2 + 2\gamma + n)}$$

- **Integral relation**

a relation between I_n and I_0

$$I_n = \frac{\Gamma(1 + \gamma + n)\Gamma(2 + 2\gamma)}{\Gamma(1 + \gamma)\Gamma(2 + 2\gamma + n)} I_0$$

- **Special case**

$$n = 1$$

$$I_1 = \frac{1}{2} I_0$$

Euler Beta Integrals

$$I_n \equiv \int_{\mathcal{C}} u \, z^n dz \, , \quad u \equiv B^\gamma \, , \quad B \equiv z(1 - z) \, , \quad \mathcal{C} \equiv [0, 1]$$

- IBP identities

$$\int_{\mathcal{C}} d(B^{\gamma+1} z^{n-1}) = 0$$

$$(\gamma + n)I_{n-1} - (1 + 2\gamma + n)I_n = 0$$

$$I_n = \frac{(\gamma + n)}{(1 + 2\gamma + n)} I_{n-1}$$

- Special case

$$n = 1 \qquad I_1 = \frac{1}{2} I_0$$

Euler Beta Integrals

- Intersection Theory

$$I_n \equiv \int_{\mathcal{C}} u \phi_{n+1} \equiv \langle \phi_{n+1} | \mathcal{C} \rangle, \quad \phi_{n+1} \equiv z^n dz$$

$$u = B^\gamma \quad B = z(1-z), \quad \omega = d \log u = \gamma \left(\frac{1}{z} + \frac{1}{z-1} \right) dz \quad \nu = 1, \quad \mathcal{P} = \{0, 1, \infty\}$$

- Monomial Basis

$$1 \text{ master integral} \quad I_0 = \langle \phi_1 | \mathcal{C} \rangle$$

- Integral relation

$$I_1 = c_1 I_0 \quad \Longleftrightarrow \quad \langle \phi_2 | = c_1 \langle \phi_1 |$$

$$c_1 = \langle \phi_2 | \phi_1 \rangle \langle \phi_1 | \phi_1 \rangle^{-1}$$

- Master Decomposition Formula

$\mathbf{C}_{ij}$ has just one element $\mathbf{C}_{11} = \langle \phi_1 | \phi_1 \rangle$

$$\langle \phi_1 | \phi_1 \rangle = \text{Res}_{z=\infty} (\psi_\infty \phi_1) = \frac{\gamma}{2(2\gamma-1)(2\gamma+1)}$$

$$\langle \phi_2 | \phi_1 \rangle = \text{Res}_{z=\infty} (\psi_\infty \phi_1) = \frac{\gamma}{4(2\gamma-1)(2\gamma+1)}$$

$$c_1 = \frac{1}{2}$$

● dLog Basis

$$\varphi_1 = d \log \frac{z}{z-1} = \left(\frac{1}{z} - \frac{1}{z-1} \right) dz ,$$

and let us decompose both $\langle \phi_1 |$ and $\langle \phi_2 |$ in the basis of $\langle \varphi_1 |$,

$$\langle \phi_1 | = \langle \phi_1 | \varphi_1 \rangle \langle \varphi_1 | \varphi_1 \rangle^{-1} \langle \varphi_1 | ,$$

$$\langle \phi_2 | = \langle \phi_2 | \varphi_1 \rangle \langle \varphi_1 | \varphi_1 \rangle^{-1} \langle \varphi_1 | .$$

We need the intersection numbers,

$$\langle \varphi_1 | \varphi_1 \rangle = \frac{2}{\gamma}, \quad \langle \phi_1 | \varphi_1 \rangle = \frac{1}{2\gamma + 1}, \quad \langle \phi_2 | \varphi_1 \rangle = \frac{1}{2(2\gamma + 1)}$$

Therefore

$$\langle \phi_1 | = \frac{\gamma}{2(2\gamma + 1)} \langle \varphi_1 | , \quad \langle \phi_2 | = \frac{\gamma}{4(2\gamma + 1)} \langle \varphi_1 |$$

from which

$$\langle \phi_2 | = 1/2 \langle \phi_1 |$$

● Observation:

$$\mathbf{C}^{-1} = \langle \varphi_1 | \varphi_1 \rangle^{-1} = \gamma/2 \quad \gamma \text{ factorizing out}$$

Gauss ${}_2F_1$ Hypergeometric Functions

$$\begin{aligned} \beta(b, c-b) {}_2F_1(a, b, c; x) &= \int_0^1 z^{b-1} (1-z)^{c-b-1} (1-xz)^{-a} dz \\ &= \int_{\mathcal{C}} u \varphi = \omega \langle \varphi | \mathcal{C} \rangle \qquad u = z^{b-1} (1-xz)^{-a} (1-z)^{-b+c-1}, \quad \varphi = dz \end{aligned}$$

$$\omega = d \log u = \frac{xz^2(c-a-2) + z(ax-c+x+2) - bxz + b-1}{(z-1)z(xz-1)} dz, \quad \boxed{\nu = 2}, \quad \mathcal{P} = \{0, 1, \tfrac{1}{x}, \infty\}$$

Gauss ${}_2F_1$ Hypergeometric Functions

● Monomial Basis

$$\{\langle \phi_i | \rangle\}_{i=1,2} \quad \phi_{n+1} \equiv z^n dz$$

● Metric

$$\mathbf{C} = \begin{pmatrix} \langle \phi_1 | \phi_1 \rangle & \langle \phi_1 | \phi_2 \rangle \\ \langle \phi_2 | \phi_1 \rangle & \langle \phi_2 | \phi_2 \rangle \end{pmatrix}$$

$$\begin{aligned} \langle \phi_1 | \phi_1 \rangle &= \left(x^2 (-(a-b+1))(b-c+1) - 2ax(-b+c-1) + a(c-2) \right) / \left(x^2 (a-c+1)(a-c+2)(a-c+3) \right), \\ \langle \phi_1 | \phi_2 \rangle &= \left(x^3 (-(a-b+1))(a-b+2)(b-c+1) - ax^2(-b+c-1)(2a-3b+c+2) + ax(a+2c-5)(-b+c-1) - a(c-3)(c-2) \right) / \left(x^3 (a-c+1) \right. \\ &\quad \left. (a-c+2)(a-c+3)(a-c+4) \right), \\ \langle \phi_2 | \phi_1 \rangle &= \left(x^3 (-(a-b))(a-b+1)(b-c+1) - ax^2(-b+c-1)(2a-3b+c) + ax(a+2c-3)(-b+c-1) - a(c-2)(c-1) \right) / \left(x^3 (a-c)(a-c+1) \right. \\ &\quad \left. (a-c+2)(a-c+3)(a-c+4) \right), \\ \langle \phi_2 | \phi_2 \rangle &= \left(-ax^2(a^2b - a^2c + a^2 - 3ab^2 + 7abc - 8ab - 4ac^2 + 9ac - 5a - 3b^2c + 6b^2 + 4bc^2 - 10bc + 6b - c^3 + 2c^2 - c) + x^4 (-(a^3 - 3a^2b + 3a^2 + 3ab^2 \right. \\ &\quad \left. + 2ax^3(a-b+1)(ab-ac+a-2b^2+3bc-2b-c^2+c) + 2a(c-2)x(a+c-2)(b-c+1) + a(c^3-6c^2+11c-6)) \right) / \left(x^4 (a-c)(a-c+1)(a-c+2)(a-c+3)(a-c+4) \right)^{c+1} \end{aligned}$$

● Master Decomposition Formula

$$\langle \phi_n | = \sum_{i,j=1}^2 \langle \phi_n | \phi_j \rangle \left(\mathbf{C}^{-1} \right)_{ji} \langle \phi_i |$$

● Gauss' contiguity relation

$$\begin{aligned} \langle \phi_3 | \mathcal{C}] &\equiv \beta(b+2, c-b) {}_2F_1(a, b+2, c+2; x) \\ &= \left(\frac{b}{x(a-c-1)} \right) \beta(b, c-b) {}_2F_1(a, b, c; x) + \left(\frac{(b-a+1)x+c}{x(c-a+1)} \right) \beta(b+1, c-b) {}_2F_1(a, b+1, c+1; x) \end{aligned}$$

Gauss ${}_2F_1$ Hypergeometric Functions

● **dLog Basis**

$$\varphi_1 = \left(\frac{1}{z} - \frac{1}{z-1}\right) dz \qquad \varphi_2 = \left(\frac{1}{z-1} - \frac{x}{xz-1}\right) dz.$$

$$I_1 = \langle \varphi_1 | \mathcal{C} \rangle = {}_2F_1(a, b-1, c-2; x), \qquad I_2 = \langle \varphi_2 | \mathcal{C} \rangle = \frac{(b-1)(x-1)}{c-2} {}_2F_1(a+1, b, c-1; x)$$

$$\mathbf{C}_{ij} = \langle \varphi_i | \varphi_j \rangle \qquad \mathbf{C} = \frac{1}{c-b-1} \begin{pmatrix} \frac{c-2}{b-1} & -1 \\ -1 & \frac{a+b-c+1}{a} \end{pmatrix}$$

● **Canonical System of Differential Equations**

$$a = -\gamma, \, b = \gamma+1, \, c = 2(\gamma+1)$$

$$\partial_x I_i = \mathbf{A}_{ij} I_j \qquad \mathbf{A} = \gamma \begin{pmatrix} 0 & \frac{-1}{x-1} \\ \frac{-1}{x} & \frac{2}{x-1} - \frac{2}{x} \end{pmatrix} \qquad \bullet \text{ Canonical}$$

Gauss ${}_2F_1$ Hypergeometric Functions

- **Mixed Basis**

$$\langle e_i | = \langle \phi_i |, \quad |h_j\rangle = |\varphi_j\rangle$$

$$\mathbf{C} = \langle \phi_i | \varphi_j \rangle = \begin{pmatrix} \frac{1}{c-a-1} & \frac{x-1}{(1+a-c)x} \\ \frac{a-ax+bx}{(a-c)(1+a-c)x} & \frac{(x-1)(1-c+ax-bx)}{(a-c)(1+a-c)x^2} \end{pmatrix}$$

- **Master Decomposition Formula**

$$\langle \varphi | = \sum_{i,j=1}^{\nu} \langle \varphi | h_j \rangle (\mathbf{C}^{-1})_{ji} \langle e_i |$$

$$\langle \varphi | = \langle \phi_3 |$$

$$\langle \phi_3 | \varphi_1 \rangle = \frac{a(x-1)(c+(2b-a+1)x) - b(1+b)x^2}{(a-c-1)(a-c)(1+a-c)x^2}$$

$$\langle \phi_3 | \varphi_2 \rangle = \frac{(x-1)(bx(a+c+(b-2a+1)x-1) + (c-ax-1)(c+x-ax))}{(a-c-1)(a-c)(1+a-c)x^3}$$

- **Gauss' contiguity relation as before!**

Appell F_1 Functions

$$\beta(a, c - a) \, F_1(a, b_1, b_2, c; x, y) = \int_{\mathcal{C}} z^{a-1} (1 - z)^{-a+c-1} (1 - xz)^{-b_1} (1 - yz)^{-b_2} dz$$

$$= \int_{\mathcal{C}} u \, \varphi = \, \omega \langle \varphi | \mathcal{C} \rangle \qquad \mathcal{C} = [0, 1]$$

$$u = z^{a-1} (1 - z)^{-a+c-1} (1 - xz)^{-b_1} (1 - yz)^{-b_2},$$

$$\omega = \left(\frac{-a + c - 1}{z - 1} + \frac{a - 1}{z} - \frac{b_1 x}{xz - 1} - \frac{b_2 y}{yz - 1} \right) dz, \qquad \boxed{\nu = 3}, \qquad \mathcal{P} = \left\{ 0, 1, \frac{1}{x}, \frac{1}{y}, \infty \right\}$$

● **dLog Basis**

$$\varphi_1 = \left(\frac{1}{z} - \frac{1}{z - 1} \right) dz, \qquad \varphi_2 = \left(\frac{1}{z - 1} - \frac{x}{xz - 1} \right) dz, \qquad \varphi_3 = \left(\frac{x}{xz - 1} - \frac{y}{yz - 1} \right) dz$$

$$\mathbf{C} = \frac{1}{c - a - 1} \begin{pmatrix} \frac{c-2}{a-1} & -1 & 0 \\ -1 & \frac{a-c+b_1+1}{b_1} & \frac{-a+c-1}{b_1} \\ 0 & \frac{-a+c-1}{b_1} & \frac{(a-c+1)(b_1+b_2)}{b_1 b_2} \end{pmatrix}$$

Lauricella F_D Functions

$$\beta(a, c - a) F_D(a, b_1, b_2, \dots, b_m, c; x_1, \dots, x_m) = \int_{\mathcal{C}} u \varphi = \omega \langle \varphi | \mathcal{C} \rangle$$

$$u = z^{a-1} (1 - z)^{-a+c-1} \prod_{i=1}^m (1 - x_i z)^{-b_i},$$

$$\mathcal{C} = [0, 1], \quad \varphi = dz, \quad \omega = d \log(u),$$

$$\nu = m+1, \quad \mathcal{P} = \left\{ 0, \frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_m}, 1, \infty \right\}$$

$$\nu = \dim H_{\pm\omega}^1 = [\text{number of P-poles} - 2] = [\text{number of P-poles} - (1+1)]$$

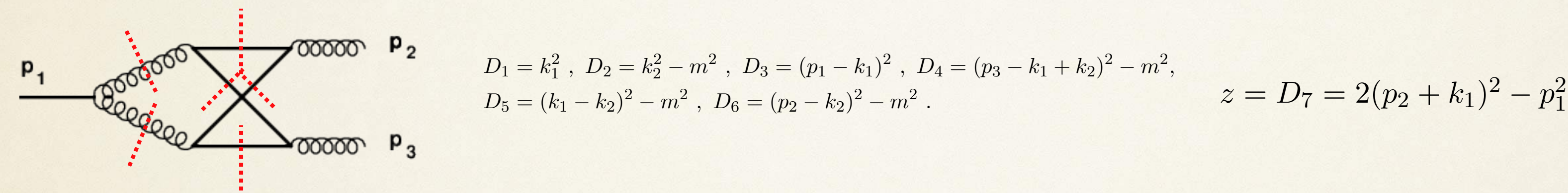
Is this relation accidental?

Feynman Integrals Decomposition :: on the maximal cut :: 1-forms

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

- 📌 On the maximal cut :: simpler integrals
- 📌 1-forms :: univariate integral representations
- 📌 Operation required :: Intersection Numbers for 1-forms

Two-Loop Non-Planar Triangle



$$u = B^\gamma, \quad B = (z^2 - \tau_1^2)(z^2 - \tau_2^2), \quad \tau_1 = s\sqrt{1 + (4m)^2/s}, \quad \tau_2 = s,$$

$$\gamma = \frac{d-5}{2} \quad \omega = \frac{2\gamma z (2z^2 - \tau_1^2 - \tau_2^2)}{(z^2 - \tau_1^2)(z^2 - \tau_2^2)} dz, \quad \nu = 3, \quad \mathcal{P} = \{-\tau_1, -\tau_2, \tau_2, \tau_1, \infty\}$$

dlog-basis.

$$\varphi_1 = \left(\frac{1}{\tau_1 + z} - \frac{1}{\tau_2 + z}\right) dz, \quad \varphi_2 = \left(\frac{1}{\tau_2 + z} - \frac{1}{z - \tau_2}\right) dz, \quad \varphi_3 = \left(\frac{1}{z - \tau_2} - \frac{1}{z - \tau_1}\right) dz,$$

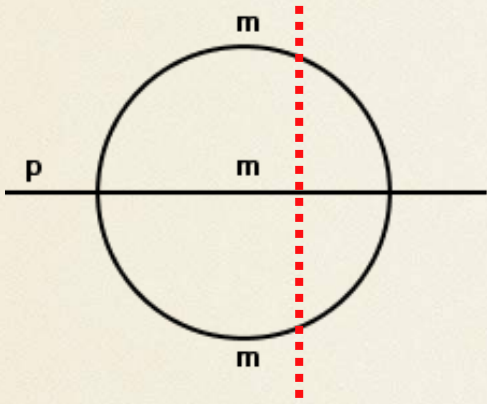
$$\mathbf{C} = \begin{pmatrix} \langle \varphi_1 | \varphi_1 \rangle & \langle \varphi_1 | \varphi_2 \rangle & \langle \varphi_1 | \varphi_3 \rangle \\ \langle \varphi_2 | \varphi_1 \rangle & \langle \varphi_2 | \varphi_2 \rangle & \langle \varphi_2 | \varphi_3 \rangle \\ \langle \varphi_3 | \varphi_1 \rangle & \langle \varphi_3 | \varphi_2 \rangle & \langle \varphi_3 | \varphi_3 \rangle \end{pmatrix} = \frac{1}{\gamma} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \mathbf{C}^{-1} = \gamma \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}$$

System of Differential Equations

$$x \equiv \frac{\tau_1}{\tau_2} \quad \sigma(x) = \partial_x \log \left(B(z, x)^\gamma \right) = -\frac{2\gamma \tau_2^2 x}{z^2 - \tau_2^2 x^2}.$$

$$\partial_x \langle \varphi_i | = \langle (\partial_x + \sigma(x)) \varphi_i | = \boldsymbol{\Omega}_{ij} \langle \varphi_j | \quad \boldsymbol{\Omega} = \gamma \begin{pmatrix} \frac{4x^2+x-1}{(x-1)x(x+1)} & \frac{1}{x} & \frac{1}{x(x+1)} \\ -\frac{2}{(x-1)(x+1)} & \frac{2}{x+1} & -\frac{2}{(x-1)(x+1)} \\ \frac{1}{x(x+1)} & \frac{1}{x} & \frac{4x^2+x-1}{(x-1)x(x+1)} \end{pmatrix} \quad \bullet \text{ Canonical}$$

Massive Sunrise



$$\begin{aligned} p^2 &= s \text{ and } m^2 = 1 \\ D_1 &= k_1^2 - 1 \; , \quad D_2 = (k_1 - k_2)^2 - 1 \; , \quad D_3 = (k_2 - p)^2 - 1 \\ z &= D_4 = k_2^2. \end{aligned}$$

$$\begin{aligned} u &= z^{-\frac{1}{2}} \left(z-4\right)^{\frac{d-3}{2}} \left(z-s-2\sqrt{s}-1\right)^{\frac{d-3}{2}} \left(z-s+2\sqrt{s}-1\right)^{\frac{d-3}{2}} \; , \\ \omega &= \frac{1}{2} \left(\frac{d-3}{z-s-2\sqrt{s}-1} + \frac{d-3}{z-s+2\sqrt{s}-1} + \frac{d-3}{z-4} - \frac{1}{z} \right) dz \; , \end{aligned}$$

$\nu = 3,$

$$\mathcal{P} = \{0, \, 4, \, s-2\sqrt{s}+1, \, s+2\sqrt{s}+1, \, \infty\}.$$

dlog Basis.

$$J_1 = \langle \varphi_1 | \mathcal{C}], \quad J_2 = \langle \varphi_2 | \mathcal{C}] \text{ and } J_3 = \langle \varphi_3 | \mathcal{C}],$$

$$\varphi_1 = \left(\frac{1}{z-s+2\sqrt{s}-1}\right) dz, \qquad \varphi_2 = \left(\frac{1}{z-s-2\sqrt{s}-1}\right) dz, \qquad \varphi_3 = \left(\frac{1}{z-4}\right) dz.$$

$$\mathbf{C} = \begin{pmatrix} \langle \varphi_1 | \varphi_1 \rangle & \langle \varphi_1 | \varphi_2 \rangle & \langle \varphi_1 | \varphi_3 \rangle \\ \langle \varphi_2 | \varphi_1 \rangle & \langle \varphi_2 | \varphi_2 \rangle & \langle \varphi_2 | \varphi_3 \rangle \\ \langle \varphi_3 | \varphi_1 \rangle & \langle \varphi_3 | \varphi_2 \rangle & \langle \varphi_3 | \varphi_3 \rangle \end{pmatrix} = -\frac{2}{3d-10} \begin{pmatrix} -\frac{2d-7}{d-3} & 1 & 1 \\ 1 & -\frac{2d-7}{d-3} & 1 \\ 1 & 1 & -\frac{2d-7}{d-3} \end{pmatrix}$$

System of Differential Equations

$$x = \sqrt{s}.$$

$$\sigma(x) = \partial_x \log \Big(B(z,x)^\gamma \Big) = \frac{2(d-3)x \left(x^2-z-1\right)}{x^4-2x^2(z+1)+(z-1)^2}$$

$$\partial_x \langle \varphi_i | = \langle (\partial_x + \sigma(x)) \varphi_i | = \boldsymbol{\Omega}_{ij} \langle \varphi_j |$$

$$\boldsymbol{\Omega} = \begin{pmatrix} \frac{d-4}{x-1} + \frac{d-3}{2(x-3)} + \frac{d-3}{2x} + \frac{d-3}{2(x+1)} & \frac{3-d}{2x} + \frac{d-3}{x-1} & \frac{3-d}{2(x-3)} + \frac{3-d}{2(x+1)} + \frac{d-3}{x-1} \\ \frac{3-d}{2x} + \frac{d-3}{x+1} & \frac{d-4}{x+1} + \frac{d-3}{2(x-1)} + \frac{d-3}{2x} + \frac{d-3}{2(x+3)} & \frac{3-d}{2(x-1)} + \frac{3-d}{2(x+3)} + \frac{d-3}{x+1} \\ \frac{3-d}{2(x-3)} + \frac{3-d}{2(x+1)} & \frac{3-d}{2(x-1)} + \frac{3-d}{2(x+3)} & \frac{d-3}{2(x-3)} + \frac{d-3}{2(x-1)} + \frac{d-3}{2(x+1)} + \frac{d-3}{2(x+3)} \end{pmatrix}.$$

Other Applications :: *proof of concepts*

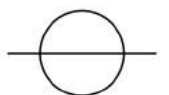
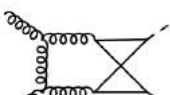
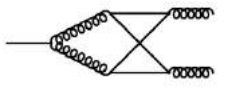
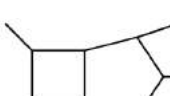
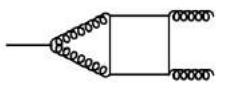
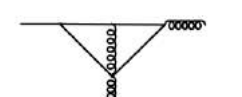
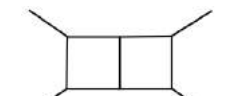
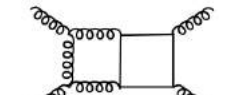
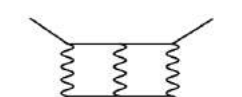
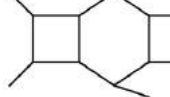
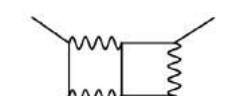
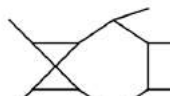
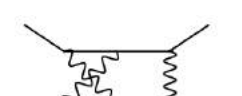
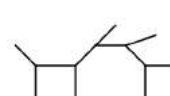
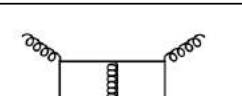
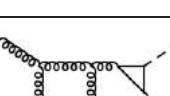
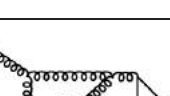
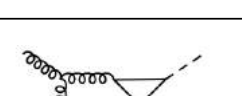
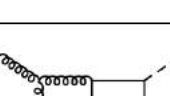
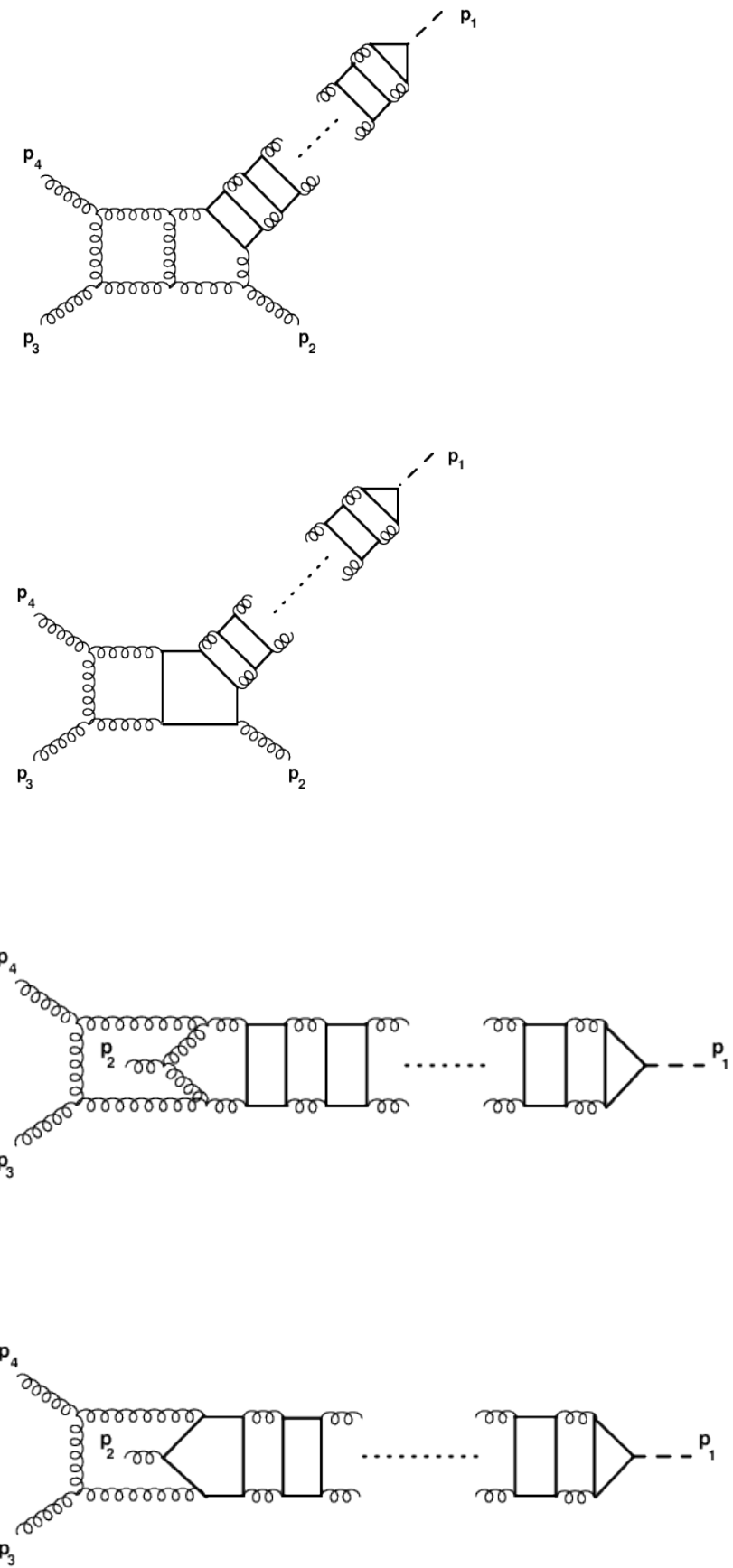
Integral family	Sec.	ν_{LBL}	ν_{std}	Integral family	Sec.	ν_{LBL}	ν_{std}
	7	1	1		14.3	4	6
	8	3	3		15.1	3	3
	9	1	1		15.2	3	3
	10	2	1		16	3	3
	11	2	2		16	3	3
	12	3	4		16	3	3
	13.1	2	2		16	3	3
	13.2	3	4		16	3	3
	13.3	3	4		16.1	3	3
	14.1	4	4		17.1	2	2
	14.1	4	4		17.2	3	3
	14.2	4	6		17.3	3	4

Table 1: Comparisons of the number of masters obtained by the LP criterion, from Loop-by-Loop (ν_{LBL}) and standard Baikov parametrization (ν_{std}).



Complete Feynman Integrals Decomposition

:: **n**-forms ::

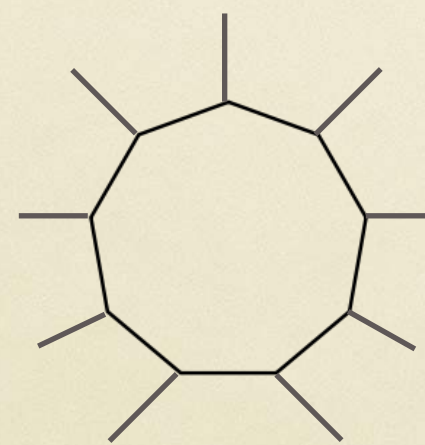
Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M.
arXiv:1907.02000

📌 **n**-forms :: **n**-variable integral representations

📌 Operation required :: Intersection Numbers for **n**-forms

📌 **n** steps down in the decomposition

● 1-loop Nonagon

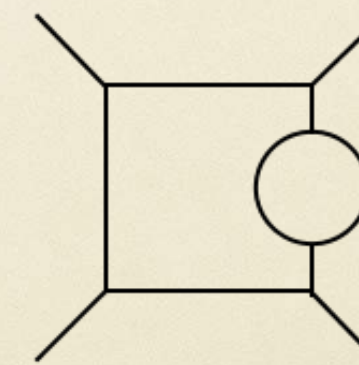


📌 Int. Num. for 8-forms

$$N = LE + \frac{1}{2}L(L + 1)$$

$$\int_C dz_1 \wedge \cdots \wedge dz_9 \frac{B(\mathbf{z})^\gamma}{z_1^{n_1} \cdots z_9^{n_9}}$$

● 2-loop Box



📌 Int. Num. for 7-forms

Intersection Numbers for *Logarithmic* n-Forms

Matsumoto (1998), Mizera (2017)

If $\langle \varphi_L |$ and $\langle \varphi_R |$ are dLog n -forms (hence contain only simple poles)

$$\begin{aligned} \langle \varphi_L | \varphi_R \rangle &= \int dz_1 \cdots dz_n \delta(\omega_1) \cdots \delta(\omega_n) \hat{\varphi}_L \hat{\varphi}_R = \\ &= \sum_{(z_1^*, \dots, z_n^*)} \det^{-1} \begin{bmatrix} \frac{\partial \omega_1}{\partial z_1} & \cdots & \frac{\partial \omega_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \omega_n}{\partial z_1} & \cdots & \frac{\partial \omega_n}{\partial z_n} \end{bmatrix} \hat{\varphi}_L \hat{\varphi}_R \Big|_{(z_1, \dots, z_n) = (z_1^*, \dots, z_n^*)} \end{aligned}$$

[Global Residue Theorem]

$(z_1^*, \dots, z_n^*)$ *critical points*, namely the solutions of the system $\omega_i = 0, \quad i = 1, \dots, n.$

In the 1-variate case:

$$\begin{aligned} \langle \varphi_L | \varphi_R \rangle &= \text{Res}_{z \in \mathcal{P}_{\omega_1}} \left(\frac{\hat{\varphi}_L \hat{\varphi}_R}{\omega} \right) \\ &= \int dz_1 \delta(\omega_1) \hat{\varphi}_L \hat{\varphi}_R = \sum_{(z_1^*)} \frac{\hat{\varphi}_L \hat{\varphi}_R}{\partial \omega_1 / \partial z_1} \end{aligned}$$

[Residue Theorem]

Nested Integrations

- Multivariate integral decomposition

$$I = \int dz_n \dots \int dz_3 \int dz_2 \int dz_1 f(z_n, \dots, z_3, z_2, z_1)$$

$$I = \sum_{i=1}^{\nu} c_i J_i$$

- Independent (Master) Integrals

$$J_i \equiv \int dz_n \dots \int dz_3 \int dz_2 \int dz_1 f_i(z_n, \dots, z_1)$$

● Cascade of Master Integrals

$$I = \int dz_n \dots \int dz_3 \int dz_2 \underbrace{\int dz_1 f(z_n, \dots, z_3, z_2, z_1)}_{\exists \nu^{(1)} \text{ master integrals in } z_1}$$

$$I = \int dz_n \dots \int dz_3 \int dz_2 \sum_{i_1=1}^{\nu^{(1)}} c_{i_1}(z_n, \dots, z_3, z_2) J_{i_1}(z_n, \dots, z_3, z_2)$$

● Cascade of Master Integrals

$$I = \int dz_n \dots \int dz_3 \int dz_2 \underbrace{\int dz_1 f(z_n, \dots, z_3, z_2, z_1)}_{\exists \nu^{(1)} \text{ master integrals in } z_1}$$

$$I = \int dz_n \dots \int dz_3 \underbrace{\int dz_2 \sum_{i_1=1}^{\nu^{(1)}} c_{i_1}(z_n, \dots, z_3, z_2) J_{i_1}(z_n, \dots, z_3, z_2)}_{\exists \nu^{(2)} \text{ master integrals in } z_2}$$

$$I = \int dz_n \dots \int dz_3 \sum_{i_2=1}^{\nu^{(2)}} c_{i_2}(z_n, \dots, z_3) J_{i_2}(z_n, \dots, z_3)$$

● Cascade of Master Integrals

$$I = \int dz_n \dots \int dz_3 \int dz_2 \underbrace{\int dz_1 f(z_n, \dots, z_3, z_2, z_1)}_{\exists \nu^{(1)} \text{ master integrals in } z_1}$$

$$I = \int dz_n \dots \int dz_3 \underbrace{\int dz_2 \sum_{i_1=1}^{\nu^{(1)}} c_{i_1}(z_n, \dots, z_3, z_2) J_{i_1}(z_n, \dots, z_3, z_2)}_{\exists \nu^{(2)} \text{ master integrals in } z_2}$$

$$I = \int dz_n \dots \underbrace{\int dz_3 \sum_{i_2=1}^{\nu^{(2)}} c_{i_2}(z_n, \dots, z_3) J_{i_2}(z_n, \dots, z_3)}_{\exists \nu^{(3)} \text{ master integrals in } z_3}$$

⋮

$$I = \underbrace{\int dz_n \sum_{i_n=1}^{\nu^{(n-1)}} c_{i_n}(z_n) J_{i_n}(z_n)}_{\exists \nu \text{ master integrals in } z_n}$$

$$I = \sum_{i=1}^{\nu} c_i J_i$$

Intersection Numbers for *Rational* n-Forms

$$I = \int_{\mathcal{C}} u(\mathbf{z}) \varphi(\mathbf{z}) \qquad \varphi(\mathbf{z}) = \hat{\varphi}(\mathbf{z}) d^n \mathbf{z} \, , \qquad d^n \mathbf{z} \equiv dz_1 \wedge \dots \wedge dz_n$$

● **Number of Master Integrals**

$$\nu \equiv \dim H^n_{\pm \omega}$$

1) **Counting Critical Points** *Lee, Pomeransky (2013)*

$$\omega \equiv d\log u(\mathbf{z}) = \sum_{i=1}^n \hat{\omega}_i dz_i$$

$$\nu \equiv \text{number of solutions of the system of equations}$$
$$\hat{\omega}_i \equiv \partial_{z_i} \log u(\mathbf{z}) = 0 \, , \qquad i = 1, \dots, n$$

2) **Shape Lemma ~ Ideal Saturation**

The number ν corresponds to the dimension of the Quotient Ring:

$$\nu = \dim \Big(\mathbb{C}[\mathbf{z}] / \langle \hat{\omega}_1, \dots, \hat{\omega}_n \rangle \Big) = \dim \Big(\mathbb{C}[\mathbf{z}] / \langle \mathcal{G} \rangle \Big)$$

3) **Euler Characteristics**

Aluffi, Marcolli (2008)
Bitoun, Bogner, Klausen, Panzer (2017)

$$\nu \; = \; (-1)^n \, (n+1 - \chi(\mathcal{P}_\omega))$$

in terms of the Euler characteristic $\chi(\mathcal{P}_\omega)$ of the projective variety $\mathcal{P}_\omega$ defined as the set of poles of ω .

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

Multivariate Intersection Numbers

Mizera (2019)
Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

● (n-1)-form Vector Space: known!

$$\nu_{\mathbf{n}-1} \quad \langle e_i^{(\mathbf{n}-1)} | \quad | h_i^{(\mathbf{n}-1)} \rangle \quad (\mathbf{C}_{(\mathbf{n}-1)})_{ij} \equiv \nu_{\mathbf{n}-1} \langle e_i^{(\mathbf{n}-1)} | h_j^{(\mathbf{n}-1)} \rangle$$

● n-form decomposition: $\mathbf{n} = (\mathbf{n}-1) + (n)$

$$\langle \varphi_L^{(\mathbf{n})} | = \sum_{i=1}^{\nu_{\mathbf{n}-1}} \langle e_i^{(\mathbf{n}-1)} | \wedge \langle \varphi_{L,i}^{(n)} | \ ,$$
$$\langle \varphi_{L,i}^{(n)} | = \langle \varphi_L^{(\mathbf{n})} | h_i^{(\mathbf{n}-1)} \rangle \ (\mathbf{C}_{(\mathbf{n}-1)}^{-1})_{ji} \ ,$$
$$\langle \varphi_{L,i}^{(n)} | (C_{(\mathbf{n}-1)})_{ij} = \langle \varphi_L^{(\mathbf{n})} | h_j^{(\mathbf{n}-1)} \rangle$$

$$|\varphi_R^{(\mathbf{n})} \rangle = \sum_{i=1}^{\nu_{\mathbf{n}-1}} | h_i^{(\mathbf{n}-1)} \rangle \wedge | \varphi_{R,i}^{(n)} \rangle \ ,$$
$$|\varphi_{R,i}^{(n)} \rangle = (\mathbf{C}_{(\mathbf{n}-1)}^{-1})_{ij} \ \langle e_j^{(\mathbf{n}-1)} | \varphi_R^{(\mathbf{n})} \rangle \ ,$$
$$(C_{(\mathbf{n}-1)})_{ij} |\varphi_{R,j}^{(n)} \rangle = \langle e_i^{(\mathbf{n}-1)} | \varphi_R^{(\mathbf{n})} \rangle$$

Multivariate Intersection Numbers

Mizera (2019)
Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

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$$\langle \varphi_{L,i}^{(n)} | = \langle \varphi_L^{(\mathbf{n})} | h_i^{(\mathbf{n}-1)} \rangle (\mathbf{C}_{(\mathbf{n}-1)}^{-1})_{ji} ,$$
$$\langle \varphi_{L,i}^{(n)} | (C_{(\mathbf{n}-1)})_{ij} = \langle \varphi_L^{(\mathbf{n})} | h_j^{(\mathbf{n}-1)} \rangle$$

$$| \varphi_R^{(\mathbf{n})} \rangle = \sum_{i=1}^{\nu_{\mathbf{n}-1}} | h_i^{(\mathbf{n}-1)} \rangle \wedge | \varphi_{R,i}^{(n)} \rangle ,$$
$$| \varphi_{R,i}^{(n)} \rangle = (\mathbf{C}_{(\mathbf{n}-1)}^{-1})_{ij} \langle e_j^{(\mathbf{n}-1)} | \varphi_R^{(\mathbf{n})} \rangle ,$$
$$(C_{(\mathbf{n}-1)})_{ij} | \varphi_{R,j}^{(n)} \rangle = \langle e_i^{(\mathbf{n}-1)} | \varphi_R^{(\mathbf{n})} \rangle$$

🔧 Intersection Numbers for **n**-forms :: Recursive Formula

$$\begin{aligned} \langle \varphi_L^{(\mathbf{n})} | \varphi_R^{(\mathbf{n})} \rangle &= \sum_{i,j} \langle \varphi_L^{(\mathbf{n})} | h_j^{(\mathbf{n}-1)} \rangle (C_{(\mathbf{n}-1)})_{ji}^{-1} \langle e_i^{(\mathbf{n}-1)} | \varphi_R^{(\mathbf{n})} \rangle \\ &= \sum_{i,j} \langle \varphi_{L,i}^{(n)} | (C_{(\mathbf{n}-1)})_{ij} \varphi_{R,j}^{(n)} \rangle \end{aligned}$$

$$\partial_{z_n} \psi_i^{(n)} + \psi_j^{(n)} \hat{\Omega}_{ji}^{(n)} = \hat{\varphi}_{L,i}^{(n)} ,$$
$$\hat{\Omega}^{(n)} \text{ is a } \nu_{\mathbf{n}-1} \times \nu_{\mathbf{n}-1} \text{ matrix, whose entries are given by}$$
$$\hat{\Omega}_{ji}^{(n)} = \langle (\partial_{z_n} + \hat{\omega}_n) e_j^{(\mathbf{n}-1)} | h_k^{(\mathbf{n}-1)} \rangle (\mathbf{C}_{(\mathbf{n}-1)}^{-1})_{ki}$$

Multivariate Intersection Numbers

Mizera (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

● Property of Intersection Number

invariance under differential forms redefinition within the same equivalence classes,

$$\langle \varphi_L | \varphi_R \rangle = \langle \varphi'_L | \varphi'_R \rangle , \quad \varphi'_L = \varphi_L + \nabla_{\omega} \xi_L , \quad \varphi'_R = \varphi_R + \nabla_{-\omega} \xi_R$$

● Interesting Observation Weinzierl (2020)

choose ξ_L and ξ_R , to build φ'_L and φ'_R that contain only simple poles, and if $\hat{\Omega}^{(n)}$ is reduced to Fuchsian form



the computation of multivariate intersection number can benefit of the evaluation of intersection numbers for dlog forms at each step of the iteration.

Contiguity relations for Special Functions

Hypergeometric ${}_3F_2$

$$\beta(a_1, b_1 - a_1) \beta(a_2, b_2 - a_2) {}_3F_2 \left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; x \right) = \int_{\mathcal{C}} u \, d^2 \mathbf{z} = \langle 1^{(12)} | \mathcal{C} \rangle$$

$$u = (1 - z_1 z_2 x)^{-a_3} \prod_{i=1}^2 z_i^{a_i - 1} (1 - z_i)^{b_i - a_i - 1}, \quad d^2 \mathbf{z} = dz_1 \wedge dz_2, \quad \mathcal{C} \text{ is the square with } z_i \in [0, 1]$$

(z_1, z_2) -space

$$\hat{\omega}_1 = \hat{\omega}_2 = 0 \qquad \nu_{(12)} = 3 \qquad \hat{e}_1^{(12)} = \frac{1}{z_1}, \quad \hat{e}_2^{(12)} = \frac{1}{z_2}, \quad \hat{e}_3^{(12)} = \frac{1}{1 - z_2},$$
$$\hat{h}_i^{(12)} = \hat{e}_i^{(12)} \quad (i = 1, 2, 3)$$

z_1 -subspace

$$\hat{\omega}_1 = 0 \text{ (w.r.t. } z_1), \quad \nu_{(1)} = 2 \qquad \hat{e}_1^{(1)} = \hat{h}_1^{(1)} = \frac{1}{z_1}, \quad \hat{e}_2^{(1)} = \hat{h}_2^{(1)} = \frac{1}{1 - z_1}$$

● Integral Decomposition ~ Contiguity Equation

$$\langle 1^{(12)} | = \sum_{i=1}^3 c_i \langle e_i^{(12)} |$$

$${}_3F_2 \left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; x \right) = \alpha_1 {}_3F_2 \left(\begin{matrix} a_1 - 1, a_2, a_3 \\ b_1 - 1, b_2 \end{matrix}; x \right) + \alpha_2 {}_3F_2 \left(\begin{matrix} a_1, a_2 - 1, a_3 \\ b_1, b_2 - 1 \end{matrix}; x \right) + \alpha_3 {}_3F_2 \left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 - 1 \end{matrix}; x \right)$$

Mathematica: ok

● **System of Differential Equations**

Within the same class of functions, let us consider:

$$u(\mathbf{z}) = ((1 - z_1)z_1(1 - z_2)z_2(1 - xz_1z_2))^\gamma;$$

$$\omega \equiv d \log u(\mathbf{z}) = \sum_{i=1}^2 \hat{\omega}_i \, dz_i;$$

a. *Number of MIs ::* I choose the ordering as $\{z_1, z_2\}$.

$$\nu_2 = 2, \quad \{\omega_2 = 0\}$$

$$\nu_{12} = 3, \quad \{\omega_1 = 0, \omega_2 = 0\}$$

b. *Choice of bases ::*

$$e^{(2)} = \left\{ \frac{1}{z_2}, \frac{1}{z_2 - 1} \right\}$$

$$e^{(12)} = \left\{ \frac{1}{z_2(z_1 - \frac{1}{x})}, \frac{1}{(z_1 - 1)(z_2 - 1)}, \frac{1}{z_1(z_2 - xz_1)} \right\}$$

$$\partial_x \langle \hat{e}_i^{(12)} | = \langle \partial_x \hat{e}_i^{(12)} + \sigma \hat{e}_i^{(12)} | = \Omega_{ij} \langle \hat{e}_j^{(12)} |$$

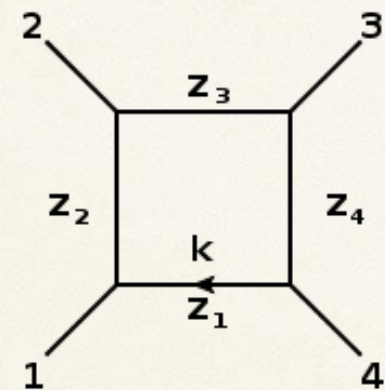
$$\sigma = \frac{d \operatorname{Log} u}{dx} = \frac{\gamma z_1 z_2}{x z_1 z_2 - 1}$$

$$\mathbf{\Omega} = \gamma \begin{pmatrix} -\frac{x-2}{4(x-1)x} & \frac{3x+10}{20(x-1)x} & \frac{13}{20} \frac{x-10}{(x-1)x} \\ \frac{3}{4(x-1)x} & \frac{20x+19}{20(x-1)x} & \frac{9}{20(x-1)} \frac{1}{x} \\ 0 & 0 & \frac{1}{x} \end{pmatrix}$$

● Canonical

Feynman Integrals Reduction

Massless Box



$$u(\mathbf{z}) = \left((st - sz_4 - tz_3)^2 - 2tz_1(s(t + 2z_3 - z_2 - z_4) + tz_3) + s^2z_2^2 + t^2z_1^2 - 2sz_2(t(s - z_3) + z_4(s + 2t)) \right)^{\frac{d-5}{2}}$$

● **Integral Decomposition**

$$\text{Box} = c_1 \text{Box} + c_2 \text{CutBox} + c_3 \text{Bubble}$$

The diagram shows the integral decomposition of a massless box. On the left is a shaded square box with four external legs. This is equal to the sum of three terms: c_1 times an unshaded square box, $+ c_2$ times a circle with two external legs (representing a cut), and $+ c_3$ times a circle with two external legs (representing a bubble).

Example.

$$\text{Diagram} = \int_{\mathcal{C}} \frac{u \, d^4 \mathbf{z}}{z_1^2 \, z_2^2 \, z_3 \, z_4}$$

$$\bullet \text{ Cut}_{\{1,3\}} : \quad \text{Diagram} = \int_{\mathcal{C}} u_{1,3} \, \varphi_{1,3} , \qquad \varphi_{1,3} = \hat{\varphi}_{1,3} \, dz_4 \wedge dz_2 \, , \qquad \hat{\varphi}_{1,3} = \frac{\hat{\omega}_1}{z_2^2 z_4}$$

$$u_{1,3} = z_2^{\rho_2} z_4^{\rho_4} u(0, z_2, 0, z_4)$$

$$\nu_{(42)} = 2, \qquad \hat{e}_1^{(42)} = \frac{1}{z_2 z_4}, \quad \hat{e}_2^{(42)} = 1 ,$$

$$\nu_{(4)} = 2 \qquad \hat{e}_1^{(4)} = \frac{1}{z_4}, \quad \hat{e}_2^{(4)} = 1 .$$

● Integral Decomposition

$$\text{Diagram} = c_1 \text{Diagram} + c_2 \text{Diagram}$$

$$c_1 = \sum_{j=1}^2 \langle \varphi_{1,3} | e_j^{(42)} \rangle (\mathbf{C}_{(42)}^{-1})_{j1} = \frac{(d-6)(d-5)}{st} ,$$

$$c_2 = \sum_{j=1}^2 \langle \varphi_{1,3} | e_j^{(42)} \rangle (\mathbf{C}_{(42)}^{-1})_{j2} = -\frac{4(d-5)(d-3)}{s^3 t} .$$

$$\bullet \text{ Cut}_{\{2,4\}} : \quad \text{Diagram} = c_1 \text{Diagram} + c_3 \text{Diagram}$$

$$c_3 = c_2|_{s \leftrightarrow t}$$

in agreement with the IBP decomposition.

● Differential Equations

$$\begin{array}{c} \diagup \quad \diagdown \\ \square \\ \diagdown \quad \diagup \end{array} = K \int_{\mathcal{C}} \frac{u d^4 \mathbf{z}}{z_1 z_2 z_3 z_4}, \quad K = (-st(s+t))^{2-\frac{d}{2}}$$

$$\partial_s \begin{array}{c} \diagup \quad \diagdown \\ \square \\ \diagdown \quad \diagup \end{array} = a_1 \begin{array}{c} \diagup \quad \diagdown \\ \square \\ \diagdown \quad \diagup \end{array} + a_2 \begin{array}{c} \diagup \quad \diagdown \\ \bigcirc \\ \diagdown \quad \diagup \end{array} + a_3 \begin{array}{c} \diagup \quad \diagdown \\ \bigcirc \\ \diagdown \quad \diagup \end{array}$$

● $\text{Cut}_{\{1,3\}}$:

$$\partial_s \begin{array}{c} \diagup \quad \diagdown \\ \square \\ \diagdown \quad \diagup \end{array} = K \int_{\mathcal{C}} u_{1,3} \varphi_{1,3}, \quad \varphi_{1,3} = \hat{\varphi}_{1,3} dz_4 \wedge dz_2 \quad \hat{\varphi}_{1,3} = \frac{f}{z_2 z_4} \text{ and } f = \frac{1}{Ku} \frac{\partial(Ku)}{\partial s}$$

$$= a_1 \begin{array}{c} \diagup \quad \diagdown \\ \square \\ \diagdown \quad \diagup \end{array} + a_2 \begin{array}{c} \diagup \quad \diagdown \\ \bigcirc \\ \diagdown \quad \diagup \end{array}$$

$$a_1 = \sum_{j=1}^2 \langle \varphi_{1,3} | e_j^{(42)} \rangle (\mathbf{C}_{(42)}^{-1})_{j1} = \frac{(d-6)t - 2s}{2s(s+t)},$$

$$a_2 = \sum_{j=1}^2 \langle \varphi_{1,3} | e_j^{(42)} \rangle (\mathbf{C}_{(42)}^{-1})_{j2} = \frac{2(d-3)}{s^2(s+t)}.$$

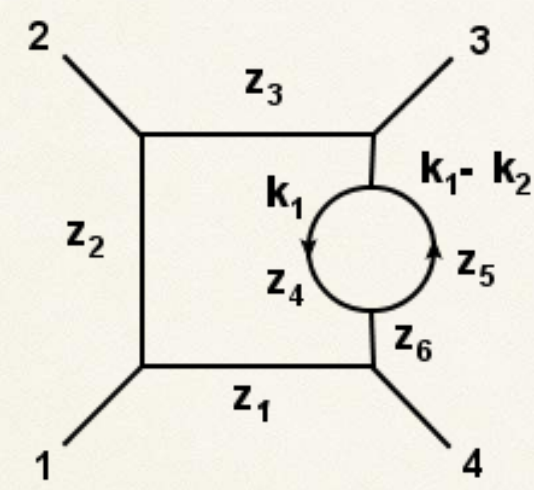
● $\text{Cut}_{\{2,4\}}$:

$$\partial_s \begin{array}{c} \diagup \quad \diagdown \\ \square \\ \diagdown \quad \diagup \end{array} = K \int_{\mathcal{C}} u_{2,4} \varphi_{2,4}, \quad \varphi_{2,4} = \hat{\varphi}_{2,4} dz_3 \wedge dz_1 \quad \hat{\varphi}_{2,4} = \frac{f}{z_1 z_3}.$$

$$= a_1 \begin{array}{c} \diagup \quad \diagdown \\ \square \\ \diagdown \quad \diagup \end{array} + a_3 \begin{array}{c} \diagup \quad \diagdown \\ \bigcirc \\ \diagdown \quad \diagup \end{array},$$

$$a_3 = \sum_{j=1}^2 \langle \varphi_{2,4} | e_j^{(31)} \rangle (\mathbf{C}_{(31)}^{-1})_{j2} = -\frac{2(d-3)}{st(s+t)}.$$

Box w/ self-energy



$$u(\mathbf{z}) = \mathcal{B}_1^{\frac{2-d}{2}} \mathcal{B}_2^{\frac{d-3}{2}} \mathcal{B}_3^{\frac{d-5}{2}} ,$$

$$\mathcal{B}_1 = z_6 , \quad \mathcal{B}_2 = 2(z_5+z_6)z_4 - z_4^2 - (z_5-z_6)^2 ,$$

$$\mathcal{B}_3 = t^2 z_1^2 + s^2 z_2^2 - 2t z_1 ((2s+t)z_3 + s(t-z_2-z_6)) - 2s z_2 (st - t z_3 + (s+2t)z_6) + (t z_3 + s(z_6-t))^2$$

● **Integral Decomposition**

$$\text{Box with self-energy} = c_1 \text{Box} + c_2 \text{Triangle} + c_3 \text{Bubble}$$

Example.

$$\text{Diagram} = \int_{\mathcal{C}} \frac{u \, d^6 \mathbf{z}}{z_1 z_2^2 z_3 z_4 z_5 z_6^2}$$

• **Cut**_{1,3,4,5} :

$$\text{Diagram} = \int_{\mathcal{C}} u_{1,3,4,5} \, \varphi_{1,3,4,5}, \quad \varphi_{1,3,4,5} = \hat{\varphi}_{1,3,4,5} \, dz_2 \wedge dz_6 \quad \hat{\varphi}_{1,3,4,5} = \frac{\hat{\omega}_2}{z_2 z_6^2}$$

$$u_{1,3,4,5} = z_2^{\rho_2} u(0, z_2, 0, 0, 0, z_6)$$

$$\nu_{(62)} = 2 \quad \hat{e}_1^{(62)} = \hat{h}_1^{(62)} = \frac{1}{z_2}, \quad \hat{e}_2^{(62)} = \hat{h}_2^{(62)} = 1,$$

$$\nu_{(2)} = 2 \quad \hat{e}_1^{(2)} = \hat{h}_1^{(2)} = \frac{1}{z_2}, \quad \hat{e}_2^{(2)} = \hat{h}_2^{(2)} = 1.$$

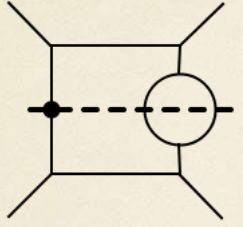
$$\text{Diagram} = c_1 \text{Diagram}_1 + c_2 \text{Diagram}_2,$$

$$c_1 = \sum_{j=1}^2 \langle \varphi_{1,3,4,5} | h_j^{(62)} \rangle (\mathbf{C}_{(62)}^{-1})_{j1} = \frac{-3(3d-16)(3d-14)(2s+t)}{2(d-6)st^3},$$

$$c_2 = \sum_{j=1}^2 \langle \varphi_{1,3,4,5} | h_j^{(62)} \rangle (\mathbf{C}_{(62)}^{-1})_{j2} = \frac{-3(3d-16)(3d-14)(3d-10)(2ds-10s-t)}{4(d-6)(d-5)(d-4)s^2t^3}.$$

in agreement with the IBP decomposition.

• **Cut**_{2,4,5} :



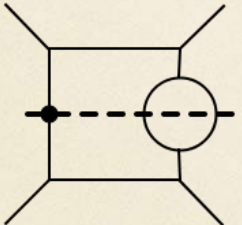
$$= \int_{\mathcal{C}} u_{2,4,5} \varphi_{2,4,5} , \quad \varphi_{2,4,5} = \hat{\varphi}_{2,4,5} dz_1 \wedge dz_3 \wedge dz_6 , \quad \hat{\varphi}_{2,4,5} = \frac{\hat{\omega}_2}{z_1 z_3 z_6^2} .$$

$$u_{2,4,5} = z_1^{\rho_1} z_3^{\rho_3} u(z_1, 0, z_3, 0, 0, z_6)$$

$$\nu_{(631)} = 2 \quad \hat{e}_1^{(631)} = \hat{h}_1^{(631)} = \frac{1}{z_1 z_3}, \quad \hat{e}_2^{(631)} = \hat{h}_2^{(631)} = 1 ,$$

$$\nu_{(31)} = 2 \quad \hat{e}_1^{(31)} = \hat{h}_1^{(31)} = z_1, \quad \hat{e}_2^{(31)} = \hat{h}_2^{(31)} = 1 ,$$

$$\nu_{(1)} = 2. \quad \hat{e}_1^{(1)} = \hat{h}_1^{(1)} = z_1, \quad \hat{e}_2^{(1)} = \hat{h}_2^{(1)} = 1 .$$



$$= c_1 \text{ (same diagram) } + c_3 \text{ (circle with a vertical line through its center and a dashed line on the left side) } ,$$

c_1 is the same as found in $\text{Cut}_{1,3,4,5}$

$$c_3 = \sum_{j=1}^2 \langle \varphi_{2,4,5} | h_j^{(631)} \rangle (\mathbf{C}_{(631)}^{-1})_{j2} = \frac{3(3d-16)(3d-14)(3d-10)(3d-8)}{2(d-6)^2(d-4)st^4} .$$

in agreement with the IBP decomposition.

Quadratic Relations

Riemann Twisted Periods Relations (RTPR)

- Completeness for forms

$$\sum_{i,j=1}^{\nu} |e_j\rangle (\mathbf{C}^{-1})_{ji} \langle e_i| = \mathbb{I}_c$$

$$\mathbf{C}_{ij} \equiv \langle e_i | e_j \rangle$$

- Completeness for contours

$$\sum_{i,j=1}^{\nu} [\mathcal{C}_j] (\mathbf{H}^{-1})_{ji} [\mathcal{C}_i| = \mathbb{I}_h$$

$$\mathbf{H}_{ij} \equiv [\mathcal{C}_i | \mathcal{C}_j]$$

- Riemann Twisted Period Relations

Cho, Matsumoto (1995)

$$\langle \varphi_L | \varphi_R \rangle = \sum_{i,j} \langle \varphi_L | \mathcal{C}_{R,j}] [\mathcal{C}_{L,j} | \mathcal{C}_{R,i}]^{-1} [\mathcal{C}_{L,i} | \varphi_R \rangle$$

$$[\mathcal{C}_L | \mathcal{C}_R] = \sum_{i,j} [\mathcal{C}_L | \varphi_{R,j} \rangle \langle \varphi_{L,j} | \varphi_{R,i} \rangle^{-1} \langle \varphi_L | \mathcal{C}_R]$$

Elliot's Identity from Intersections Matsumoto & P.M.

The complete elliptic integrals $\mathcal{K}$ and $\mathcal{E}$ of the first and second kind

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - r^2 \sin^2 \phi}} \qquad \mathcal{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 \phi} \, d\phi$$

● Legendre Identity

$$\mathcal{E}\mathcal{K}' + \mathcal{E}'\mathcal{K} - \mathcal{K}\mathcal{K}' = \frac{\pi}{2} \qquad \mathcal{K}'(r) = \mathcal{K}(r') \text{ and } \mathcal{E}'(r) = \mathcal{E}(r') \\ r^2 + r'^2 = 1$$

Elliot's Identity from Intersections Matsumoto & P.M.

The complete elliptic integrals $\mathcal{K}$ and $\mathcal{E}$ of the first and second kind

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - r^2 \sin^2 \phi}} \quad \mathcal{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 \phi} d\phi$$

● Legendre Identity

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$$r^2 + r'^2 = 1$$

● Elliot's Identity and Hypergeometric Functions

Balasubramanian, Naik, Ponnusamy, Vuorinen (2001)

$$\begin{aligned} & F\left(\frac{1}{2} + \lambda, -\frac{1}{2} - \nu, 1 + \lambda + \mu; r\right) F\left(\frac{1}{2} - \lambda, \frac{1}{2} + \nu, 1 + \mu + \nu; 1 - r\right) \\ & + F\left(\frac{1}{2} + \lambda, \frac{1}{2} - \nu, 1 + \lambda + \mu; r\right) F\left(-\frac{1}{2} - \lambda, \frac{1}{2} + \nu, 1 + \mu + \nu; 1 - r\right) \\ & - F\left(\frac{1}{2} + \lambda, \frac{1}{2} - \nu, 1 + \lambda + \mu; r\right) F\left(\frac{1}{2} - \lambda, \frac{1}{2} + \nu, 1 + \mu + \nu; 1 - r\right) \\ & = \frac{\Gamma(1 + \lambda + \mu)\Gamma(1 + \mu + \nu)}{\Gamma(\lambda + \mu + \nu + \frac{3}{2})\Gamma(\mu + \frac{1}{2})}. \end{aligned}$$

the choice $\lambda = \mu = \nu = 0$ gives the Legendre relation.

Elliot’s Identity from Intersections Matsumoto & P.M.

● *Hypothesis: too close to RTPR to be accidental*

● *Proof*

$$u(t) = t^{1/2+\lambda}(1-t)^{-1/2+\mu}(1-rt)^{1/2+\nu},$$

$$\varphi_1 = \frac{dt}{t}, \quad \varphi_2 = \frac{dt}{t(1-rt)} = \left(\frac{1}{t} - \frac{1}{t-1/r}\right)dt,$$

$$\psi_1 = \frac{dt}{1-t} = \frac{-dt}{t-1}, \quad \psi_2 = \frac{dt}{t(1-t)} = \left(\frac{1}{t} - \frac{1}{t-1}\right)dt.$$

$$\gamma = (0,1)\otimes u(t) \text{ and } \delta = (-\infty,0)\otimes 1/u(t)$$

● *Riemann Twisted Period Relation*

$${}^t\Pi_\omega\, {}^tH_c^{-1}\Pi_{-\omega} = H_h.$$

$$\left(\int_0^1 u(t)\varphi_1, \int_0^1 u(t)\varphi_2\right) {}^tH_c^{-1}\begin{pmatrix} \int_{-\infty}^0 \frac{1}{u(t)}\psi_1 \\ \int_{-\infty}^0 \frac{1}{u(t)}\psi_2 \end{pmatrix} = \frac{-1}{e^{2\pi\sqrt{-1}\lambda}+1}.$$

$$\left(F(\frac{1}{2}+\lambda,-\frac{1}{2}-\nu,1+\lambda+\mu;r),F(\frac{1}{2}+\lambda,\frac{1}{2}-\nu,1+\lambda+\mu;r)\right)\cdot\begin{pmatrix}1&0\\-1&1\end{pmatrix}\cdot\left(\begin{matrix}F(\frac{1}{2}-\lambda,\frac{1}{2}+\nu,1+\mu+\nu;1-r)\\F(-\frac{1}{2}-\lambda,\frac{1}{2}+\nu,1+\mu+\nu;1-r)\end{matrix}\right)$$

$$=\frac{\Gamma(\lambda+\mu+1)\Gamma(\mu+\nu+1)}{\Gamma(\lambda+\mu+\nu+\frac{3}{2})\Gamma(\mu+\frac{1}{2})}$$

Elliot’s Identity from Intersections Matsumoto & P.M.

● *Hypothesis: too close to RTPR to be accidental*

● *Proof*

$$u(t) = t^{1/2+\lambda}(1-t)^{-1/2+\mu}(1-rt)^{1/2+\nu},$$

$$\begin{aligned}\varphi_1 &= \frac{dt}{t}, & \varphi_2 &= \frac{dt}{t(1-rt)} = \left(\frac{1}{t} - \frac{1}{t-1/r}\right)dt, \\ \psi_1 &= \frac{dt}{1-t} = \frac{-dt}{t-1}, & \psi_2 &= \frac{dt}{t(1-t)} = \left(\frac{1}{t} - \frac{1}{t-1}\right)dt.\end{aligned}$$

$$\gamma = (0,1)\otimes u(t) \text{ and } \delta = (-\infty,0)\otimes 1/u(t)$$

● *Riemann Twisted Period Relation*

$${}^t\Pi_\omega\, {}^tH_c^{-1}\Pi_{-\omega} = H_h.$$

$$\left(\int_0^1 u(t)\varphi_1,\int_0^1 u(t)\varphi_2\right){}^tH_c^{-1}\begin{pmatrix}\int_{-\infty}^0\frac{1}{u(t)}\psi_1\\ \int_{-\infty}^0\frac{1}{u(t)}\psi_2\end{pmatrix}=\frac{-1}{e^{2\pi\sqrt{-1}\lambda}+1}.$$

$$\left(F(\frac{1}{2}+\lambda,-\frac{1}{2}-\nu,1+\lambda+\mu;r),F(\frac{1}{2}+\lambda,\frac{1}{2}-\nu,1+\lambda+\mu;r)\right)\cdot\begin{pmatrix}1&0\\-1&1\end{pmatrix}\cdot\begin{pmatrix}F(\frac{1}{2}-\lambda,\frac{1}{2}+\nu,1+\mu+\nu;1-r)\\ F(-\frac{1}{2}-\lambda,\frac{1}{2}+\nu,1+\mu+\nu;1-r)\end{pmatrix}$$

$$=\frac{\Gamma(\lambda+\mu+1)\Gamma(\mu+\nu+1)}{\Gamma(\lambda+\mu+\nu+\frac{3}{2})\Gamma(\mu+\frac{1}{2})}$$

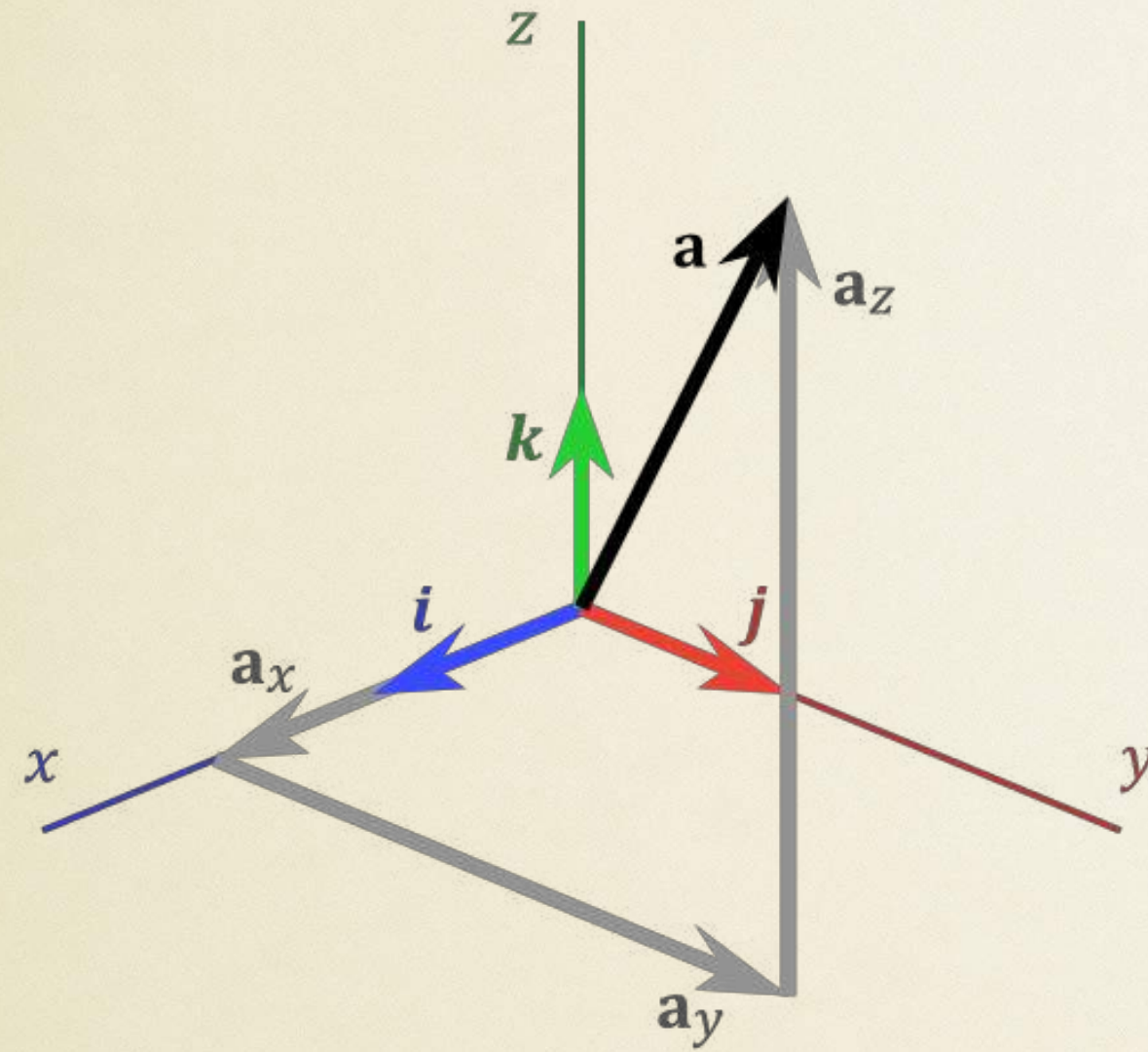
● Paradigmatic case for studying **quadratic relations for Feynman Integrals**

Broadhurst, Roberts (2018)
Lee, Pomeranski (2019)

To Conclude:

Amplitudes Decomposition:

the algebraic way



$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}$$

📌 **Basis:** $\{\mathbf{i} \ \mathbf{j} \ \mathbf{k}\}$

📌 **Scalar product/Projection:**
to extract the components

$$a_x = \mathbf{a} \cdot \mathbf{i} \quad a_y = \mathbf{a} \cdot \mathbf{j} \quad a_z = \mathbf{a} \cdot \mathbf{k}$$

Summary

- **Novel Mathematical ideas for Quantum Field Theory**

- 📌 De Rham (co)Homology and Intersection Theory
- 📌 Stokes' and Cauchy's Residue theorem at work
- 📌 Rich theory :: Differential and Algebraic Geometry, Topology, Number Theory

- **Novel Property Discovered**

- 📌 The **algebra** of Feynman Integrals (and not only) is controlled by **Intersection Numbers**
- 📌 Intersection Numbers ~ Scalar Product/Projection between Feynman Integrals
- 📌 Exploiting the geometric properties of the integrands, dictated by graph polynomials

- **Novel Formulae for Multivariate Intersection number**

- 📌 Useful in Physics and Math
- 📌 Owing to Equivalence Classes, Amplitudes are better behaving (in UV and IR) w.r.t. what appears from their integrands: this seems the proper framework to study hidden symmetries.

- **"The number of master integrals" from being the *question* to being the *answer***

- 📌 Q: What is the dimension of the co-homology group for Feynman Integrals
- 📌 A: the number of Master Integrals

Definition. Physics is a part of mathematics devoted to the calculation of integrals of the form $\int g(x)e^{f(x)}dx$. Different branches of physics are distinguished by the range of the variable x and by the names used for $f(x)$, $g(x)$ and for the integral. For example, in classical statistical physics x runs over a symplectic manifold, $f(x)$ is called the Hamiltonian function and the integral has the meaning of a partition function or of a correlation function. In a d -dimensional quantum field theory x runs over the space of functions on a d -dimensional manifold (the space of fields) and $f(x)$ is interpreted as an action functional.

Of course this is a joke, physics is not a part of mathematics. However, it is true that the main mathematical problem of physics is the calculation of integrals of the form

$$I(g) = \int g(x)e^{-f(x)}dx$$

[...] If f can be represented as $f_0 + \lambda V$ where f_0 is a negative quadratic form, then the integral $\int g(x)e^{f(x)}dx$ can be calculated in the framework of perturbation theory with respect to the formal parameter λ . We will fix f and consider the integral as a functional $I(g)$ taking values in $\mathbb{R}[[\lambda]]$. It is easy to derive from the relation

$$\int \partial_a(h(x)e^{f(x)})dx = 0$$

that the functional $I(g)$ vanishes in the case when g has the form

$$g = \partial_a h + (\partial_a f)h.$$

The unreasonable effectiveness of mathematics

E. Wigner

Wigner was referring to the mysterious phenomenon in which areas of pure mathematics, originally constructed without regard to application, are suddenly discovered to be exactly what is required to describe the structure of the physical world.

M. Berry

Extra

Univariate Intersection Number

Matsumoto (1996)

Mizera (2017)

1. Regularized Forms

Logarithmic twisted cocycles φ_L can have simple poles only at z_i 's. To construct φ_L^c with compact support, we must find a cocycle in the sme cohomology class, which vanishes in a small tubular neighborhood around each z_i .

Let's divide the space $X = \mathbb{CP}^1 \setminus \cup_{i=1}^k \{z = z_i\}$, into regions:

where V_i and U_i are discs centered in z_i with small radii $0 < \epsilon_V < \epsilon_U$. For convenience, let us define the annulus $D_i = U_i \setminus V_i$.

We introduce the regulating function

$$h_i = h_i(z, \bar{z}) \equiv \begin{cases} 0, & \text{on } U_i \\ 0 < h_i < 1, & \text{on } D_i = U_i \setminus V_i \\ 1, & \text{on } V_i \end{cases} \quad (1.1)$$

and define

$$\varphi_L^c \equiv \varphi_L - \sum_{z_i \in \mathcal{P}_\omega} \nabla_\omega(h_i \psi_i) \quad (1.2)$$

For notation ease, we omit the sum over the poles of ω , and restore it at the end. Observe that,

$$\nabla_\omega(h_i \psi_i) = (d + \omega)(h_i \psi_i) = \psi_i(dh_i) + h_i(d\psi_i) + h_i \omega \psi_i = \psi_i(dh_i) + h_i \nabla_\omega \psi_i \quad (1.3)$$

Therefore,

$$\varphi_L^c \equiv \varphi_L - (\psi_i(dh_i) + h_i \nabla_\omega \psi_i) \quad (1.4)$$

Iff

$$\nabla_\omega \psi_i = \varphi_L, \quad \text{for } z \rightarrow z_i, \quad \text{namely on } U_i \setminus \{z_i\} \quad (1.5)$$

then

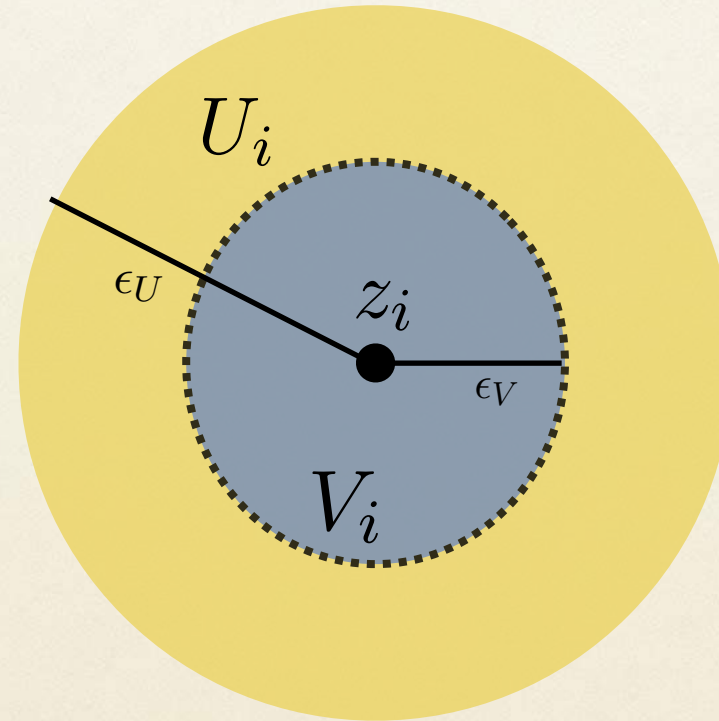
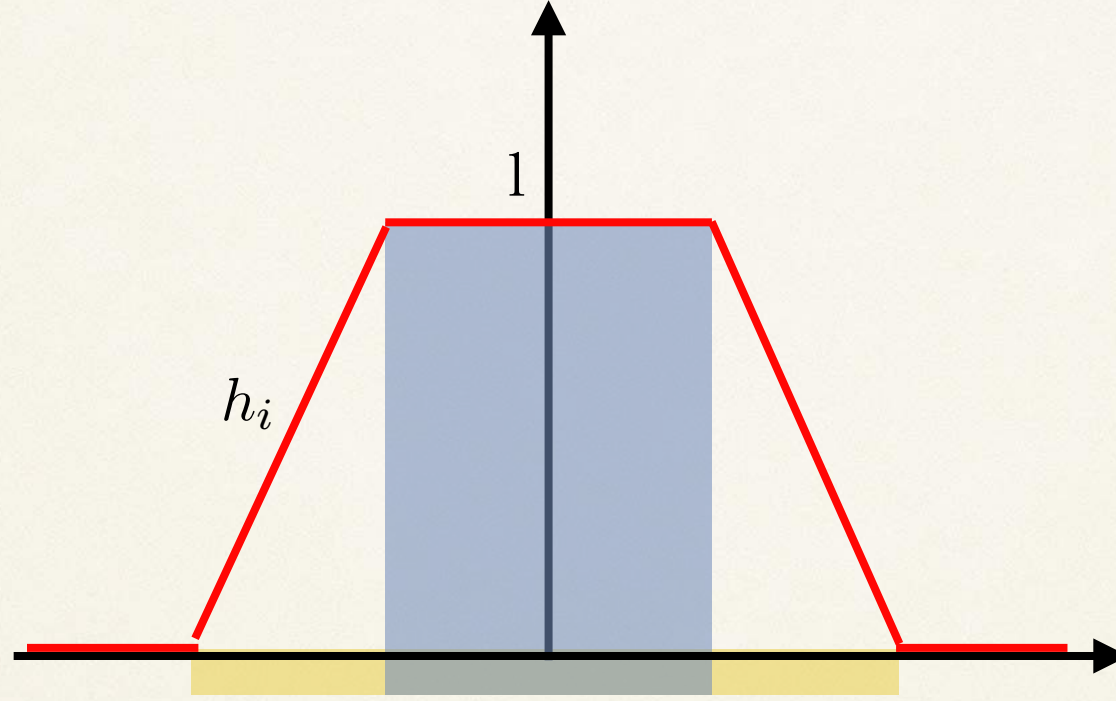
$$\varphi_L^c \equiv \begin{cases} 0, & \text{on } V_i \\ \varphi_L - (\psi_i(dh_i) + h_i \varphi_L), & \text{on } D_i = U_i \setminus V_i \\ \varphi_L, & \text{on } X \setminus U_i \end{cases} \quad (1.6)$$

hence φ_L^c has *compact support*, because $\varphi_L^c = 0$ on $\cup_{i=1}^k V_i$.

Let us consider the following two identities:

1. Since $\varphi_L^c = \varphi_L$, on $X \setminus U_i$,

$$\int_{X \setminus U_i} \varphi_L^c \wedge \varphi_R = 0 \quad (1.7)$$



2. Useful identity

$$\begin{aligned} d(h_i \psi_i \varphi_R) &= d(h_i \psi_i) \wedge \varphi_R + h_i \underbrace{\psi_i d\varphi_R}_{=0} = \psi_i dh_i \wedge \varphi_R + h_i \underbrace{d\psi_i \wedge \varphi_R}_{\varphi_L - \psi_i \omega}_{=0} \\ &= \psi_i dh_i \wedge \varphi_R \end{aligned} \quad (1.8)$$

We are now in the position for defining the *intersection number* as the pairing

$$\begin{aligned} \langle \varphi_L | \varphi_R \rangle &= \int_X \varphi_L^c \wedge \varphi_R \\ &= \underbrace{\int_{V_i} \varphi_L^c \wedge \varphi_R}_{=0} + \int_{X \setminus V_i} \varphi_L^c \wedge \varphi_R \\ &= (\text{by adding and subtracting } D_i) \\ &= \int_{D_i} \varphi_L^c \wedge \varphi_R + \underbrace{\int_{X \setminus U_i} \varphi_L^c \wedge \varphi_R}_{=0} \\ &= \int_{D_i} (\varphi_L - \psi_i(dh_i) - h_i \varphi_L) \wedge \varphi_R \\ &= - \int_{D_i} \psi_i(dh_i) \wedge \varphi_R \\ &= - \int_{D_i} d(h_i \psi_i \varphi_R) = - \int_{\partial D_i} h_i \psi_i \varphi_R \\ &= - \int_{\partial U_i} \underbrace{h_i}_{=0} \psi_i \varphi_R + \int_{\partial V_i} \underbrace{h_i}_{=1} \psi_i \varphi_R \\ &= \int_{\partial V_i} \psi_i \varphi_R = \oint \psi_i \varphi_R = (2\pi i) \text{Res}_{z=z_i} \{ \psi_i \varphi_R \}. \end{aligned} \quad (1.9)$$

Let us recap our result, which, after reinserting the sum over poles, reads as:

$$\langle \varphi_L | \varphi_R \rangle = (2\pi i) \sum_i \text{Res}_{z=z_i} \{ \psi_i \varphi_R \}, \quad (1.10)$$

where ψ_i is the solution of the differential equation $\nabla_\omega \psi = \varphi_L$ around the pole $z = z_i$.

1.1 Alternative definition

$$\epsilon_V = \epsilon_U \equiv \epsilon$$

Mizera (2019)

$$h_i = h_i(z, \bar{z}) \equiv \theta(|z - z_i|^2 - \epsilon^2)$$

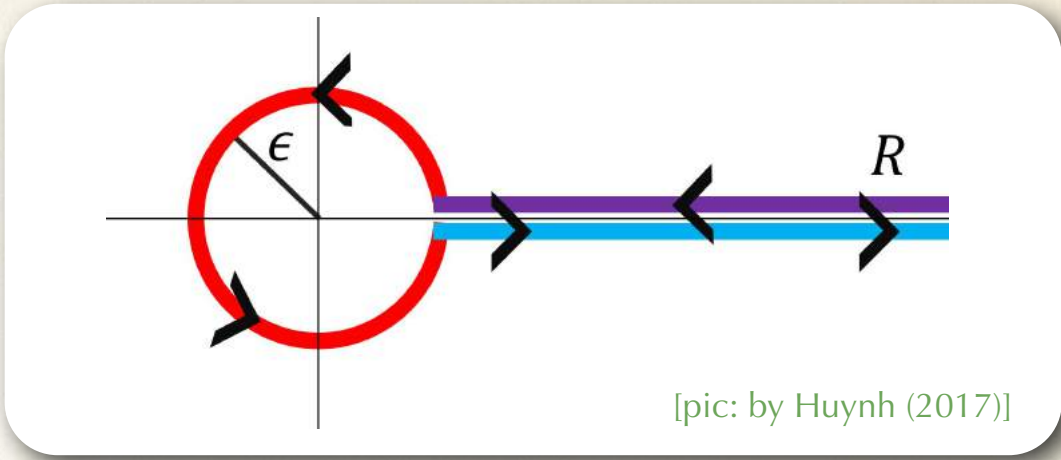
Regularised Twisted Cycles

● **Analytic Continuation of Gamma Function** [Hankel Contour]

$$\Gamma(s) = \frac{1}{e^{2\pi is} - 1} \oint_C t^{s-1} e^{-t} dt.$$

Due to the branch cut and the denominator of $e^{2\pi is} - 1$, this representation is defined for complex numbers s where $s \notin \mathbb{R}_{\geq 0} \cup \mathbb{Z}_{<0}$.

$$\oint_C t^{s-1} e^{-t} dt = \lim_{R \rightarrow \infty, \epsilon \rightarrow 0} \left(\underbrace{\int_R^\epsilon x^{s-1} e^{-x} dx}_{\text{[purple line]}} + \underbrace{\int_0^{2\pi} (\epsilon e^{i\theta})^{s-1} e^{-\epsilon e^{i\theta}} i \epsilon e^{i\theta} d\theta}_{\text{[red circle]}} + \underbrace{\int_\epsilon^R (x e^{2\pi i})^{s-1} e^{-x e^{2\pi i}} dx}_{\text{[blu line]}} \right) = \lim_{R \rightarrow \infty} \left(- \int_0^R x^{s-1} e^{-x} dx + 0 + e^{2\pi is} e^{-2\pi i} \int_0^R x^{s-1} e^{-x} dx \right) = (e^{2\pi is} - 1) \Gamma(s).$$



[pic: by Huynh (2017)]

● **Analytic Continuation of Beta Function** Yoshida; Matsumoto; Mizera [1706.08527]

$$\oint_\gamma z^s (1-z)^t \varphi(z) = \left(1 - e^{2\pi it} + e^{2\pi i(s+t)} - e^{2\pi is} \right) \int_0^1 z^s (1-z)^t \varphi(z), \quad \text{reg } \overrightarrow{(0,1)} := \frac{S(\varepsilon, 0)}{e^{2\pi is} - 1} + \overrightarrow{(\varepsilon, 1 - \varepsilon)} - \frac{S(1 - \varepsilon, 1)}{e^{2\pi it} - 1} = \bigcirc_{z=0} \longrightarrow \bigcirc_{z=1}.$$

● **Intersection Numbers for Twisted cycles:** graphical rules :: **signs, circles** and **monodromy factors** ::

$$\begin{array}{c} \gamma^\vee \\ \swarrow \quad \nearrow \\ \gamma \end{array} = +1 \quad \text{or} \quad \begin{array}{c} \gamma \\ \swarrow \quad \nearrow \\ \gamma^\vee \end{array} = -1.$$

$$C(1234) = \{0 < z_2 < 1\} \otimes z_2^{s_{12}} (1 - z_2)^{s_{23}} = \overrightarrow{(0,1)} \otimes z^s (1 - z)^t,$$

$$\begin{aligned} \langle C(1234), C(1234) \rangle &= \begin{array}{c} \text{Diagram 1: Two circles with points } z=0 \text{ and } z=1. \text{ A horizontal line connects them with arrows pointing right. Above the line is } \overrightarrow{(0,1)}_{\text{sin}} \text{ and below is } \overrightarrow{(0,1)}_{\text{reg}}. \end{array} \\ &= -\frac{1}{e^{2\pi is} - 1} - 1 - \frac{1}{e^{2\pi it} - 1}. \\ &= \begin{array}{c} \text{Diagram 2: Similar to Diagram 1, but with arcs connecting the circles. Labels include } \overrightarrow{(0,1)}_{\text{sin}}, \overrightarrow{(0,1)}_{\text{reg}}, \text{ and } \overrightarrow{(0,1)}_{\text{arc}}. \end{array} \\ &= -\frac{e^{2\pi is}}{e^{2\pi is} - 1} + 1 - \frac{e^{2\pi it}}{e^{2\pi it} - 1}. \\ &= \begin{array}{c} \text{Diagram 3: Similar to Diagram 2, with different arc orientations. Labels include } \overrightarrow{(0,1)}_{\text{arc}}, \overrightarrow{(0,1)}_{\text{reg}}, \text{ and } \overrightarrow{(0,1)}_{\text{sin}}. \end{array} \\ &= -\frac{1}{e^{2\pi is} - 1} - \frac{e^{2\pi it}}{e^{2\pi it} - 1}, \\ &= \begin{array}{c} \text{Diagram 4: Similar to Diagram 3, with different arc orientations. Labels include } \overrightarrow{(0,1)}_{\text{reg}}, \overrightarrow{(0,1)}_{\text{sin}}, \text{ and } \overrightarrow{(0,1)}_{\text{arc}}. \end{array} \\ &= -\frac{e^{2\pi is}}{e^{2\pi is} - 1} - \frac{1}{e^{2\pi it} - 1}. \\ &= \frac{i}{2} \left(\frac{1}{\tan \pi s} + \frac{1}{\tan \pi t} \right) \end{aligned}$$