The High Temperature Limit of QFT

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This presentation is based on arXiv: 2005.03676 with Noam Chai, Soumyadeep Chaudhuri, Chang-Ha Choi , Eliezer Rabinovici, and Misha Smolkin.

And also ongoing work.

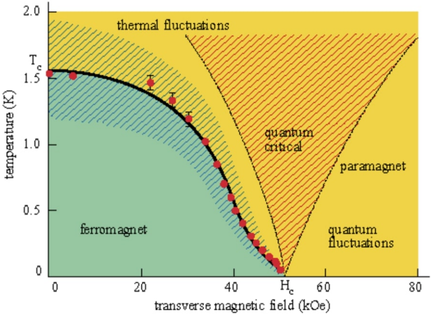
Many Hamiltonians can exhibit symmetry breaking at zero temperature. For instance, ferromagnets, massless QCD, the Neél phase etc. We usually think that if we heat these systems up, i.e. study instead of the vacuum the thermal state

$$e^{-\beta H}$$

then all the symmetries are restored for sufficiently small β. (I am talking about ordinary symmetries only.)

Indeed, most phase diagrams for quantum critical points look like this (phase diagram of $LiHoF_4$ as measured by Bitko and co-workers)

Figure 1: **Quantum criticality in a ferromagnet.**



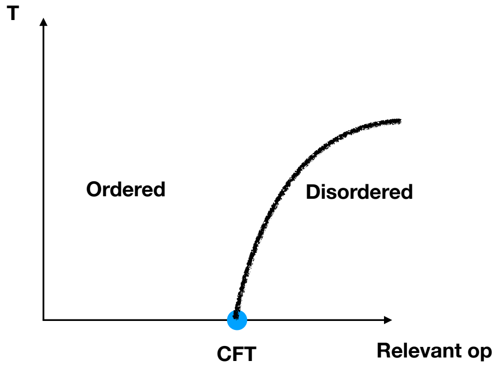
It is the ordered phase that is capped off not the disordered. Symmetry should be restored at high enough temperature.

- One reason is that at finite temperature we minimize

$$F = E - ST \ .$$

  At large T the dominant contribution is from high entropy states and those are disordered. Or so we are taught in school.
- A much more highbrow reason is that finite temperature CFT is sometimes dual to a black brane in AdS. For the latter, the AdS/CMT community proved essentially a no-go theorem – it has no hair and hence no symmetry breaking in the CFT.

The question is therefore clear: Consider a CFT in d+1 space-time dimensions and turn on some temperature T. The physics is independent of T as long as T is nonzero. Can symmetry breaking take place? If so the phase diagram would have to look like the following:



We can also start from a CFT with some chemical potential μ for our symmetry and temperature T. Then there is a nontrivial phase diagram as a function of T/μ. The typical situation is

$$T \ll \mu : \qquad \text{superfluid} + \text{fluid} - -\text{broken symmetry}$$

$$T \gg \mu : \qquad \text{fluid} - -\text{all symmetries are restored}$$

This kind of situation was studied extensively in the AdS/CMT literature. The low temperature phase is a hairy BH, the hair coming from symmetry breaking (bulk superconductivity) and the high temperature phase is a standard RN black hole.

In summary: experiments, the no-hair theorem, and thermodynamic arguments all suggest that the expectation values of order parameters must vanish at high temperature

$$\beta < \beta_c : \quad Tr(Oe^{-\beta H}) = 0 \ .$$

Is this really true?

Weinberg constructed in 74' a model with "intermediate symmetry breaking" – that is a situation where there is an RG flow and for some intermediate temperatures there is spontaneous symmetry breaking while at $T = 0$ there is none. It was not possible to analyze it at very high temperatures since it was not UV complete so the question we are after could not be posed.

There are also materials such as the sodium potassium tartrate ($KNaC_4H_4O_6 \cdot 4H_2O$) which has a higher crystal symmetry between $-18^o C$-$24^o C$ than at lower temperatures.

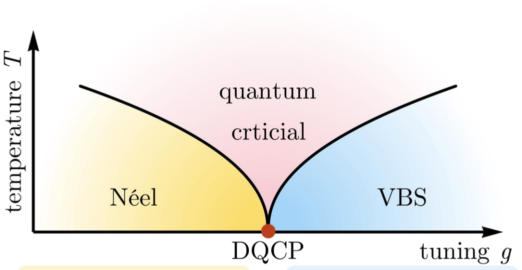
Here we want to ask about the ultimate high temperature limit, which translates to a well defined problem in the space of allowed CFTs.

The question is:

Are there unitary, local, nontrivial CFTs (with finitely many dofs) which break a global symmetry at finite temperature?

Here we construct an example in $4-\epsilon$ space-time dimensions that does so, for $0<\epsilon<\epsilon_c$. Since CFTs in fractional dimensions are not full fledged unitary theories, this is not yet a definitive solution of the problem. The theory we construct has several conceptually interesting properties and some of the results carry over to $\epsilon=1$.

- Free CFTs: trivial.
- Experimentally studied CFTs: Ising, O(2), some deconfined critical points, all display normal behavior, with a disordered phase above the CFT.
- Weakly coupled CFTs where we may hope to compute the answer.
- AdS constructions...
- Maybe general theorems?! (we will see some!)



Because for some purposes finite temperature is the same as the theory on a circle, one can draw some immediate conclusions:

- In 1+1 dimensions no symmetry breaking can occur at finite temperature. This follows also from modular invariance right away.
- In 2+1 dimensions no continuous symmetry breaking can occur at finite temperature (Coleman-Mermin-Wagner).

There are familiar subtleties with QFT on a circle. We review them through the ϕ^4 model in 3+1 dimensions.

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{4!}\lambda\phi^4 \ .$$

At zero temperature the model is free at long distances. Now take a circle of radius $\frac{\beta}{2\pi}$ and Fourier expand. The most important terms are

$$\frac{1}{2}(\partial\phi_0)^2 - \frac{1}{4!}\lambda\beta^{-1}\phi_0^4 + \frac{\lambda\beta^{-1}}{2}\phi_0^2\sum_{n\neq 0}|\phi_n|^2 \ .$$

The dynamics of ϕ_0 is now in three dimensions and the quartic interactions becomes strong and non-perturbative!

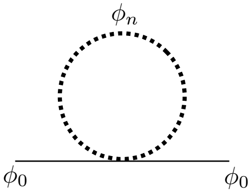
$$\frac{1}{2}(\partial\phi_0)^2 - \frac{1}{4!}\lambda\beta^{-1}\phi_0^4 + \frac{\lambda\beta^{-1}}{2}\phi_0^2\sum_{n\neq 0}|\phi_n|^2 \ .$$

The strong coupling scale of ϕ_0 is $\Lambda \sim \lambda\beta^{-1}$. This is the source of the famous infrared issues in thermal field theory – the zero mode dynamics may be strong even if the original model is tractable at zero temperature.

$$\frac{1}{2}(\partial\phi_0)^2 - \frac{1}{4!}\lambda\beta^{-1}\phi_0^4 + \frac{\lambda\beta^{-1}}{2}\phi_0^2\sum_{n\neq 0}|\phi_n|^2 \ .$$

Luckily in this model we are saved from strong coupling dynamics thanks to the loops of ϕ_n. These loops generate a mass for ϕ_0:

$$m_0^2 = \lambda\beta^{-1}\sum_{n\neq 0}\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2+\frac{4\pi^2n^2}{\beta^2}} = -\frac{\lambda}{2}\beta^{-2}\sum_{n\neq 0} n = \frac{\lambda}{24}\beta^{-2} \ .$$



The mass is at the scale $m_0 \sim \sqrt{\lambda}\beta^{-1}$ while the strong coupling scale is $\Lambda \sim \lambda\beta^{-1}$ so we are saved by the mere factor of $\sqrt{\lambda}$. The mode ϕ_0 is massive and it decouples before the interactions become strong. We can then safely conclude that since $m_0^2 > 0$ the thermal vacuum is at the origin and the $\mathbb{Z}_2$ symmetry is unbroken.

Our ultimate interest is in the thermal behavior of interacting CFTs so let us cover some of the possible constructions:

- Vector Models in the ϵ Expansion or large N limit: this is what we will study today.
- Weakly coupled conformal gauge theories of the Banks-Zaks type: Some comments at the end.
- 2+1 dimensional fixed points with lots of matter or large Chern-Simons coefficients. Some comments at the end.
- AdS constructions (very interesting recent work by Buchel).
- A thermal bootstrap approach – if time permits.

We consider models with N scalar fields ϕ_i, $i = 1, ..., N$ and potential

$$V = \frac{1}{4!}\tilde{\lambda}_{ijkl}\phi_i\phi_j\phi_k\phi_l$$

in $4 - \epsilon$ space-time dimensions. We will first take $\epsilon << 1$ the smallest parameter in the problem.

$$\epsilon\tilde{\lambda}_{ijkl} = \frac{1}{16\pi^2}\left(\tilde{\lambda}_{ijmn}\tilde{\lambda}_{mnkl} + 2\ \text{permutations}\right)\ .$$

It is convenient to rescale out the factors of ϵ and $\frac{1}{16\pi^2}$ by defining $\lambda = \frac{\tilde{\lambda}}{16\pi^2\epsilon}$ in terms of which the fixed point equations become

$$\lambda_{ijkl} = \lambda_{ijmn}\lambda_{mnkl} + 2\ \text{permutations}$$

These are rather complicated equations and the solutions are not classified. But there are lots of known classes of solutions. Some of the solutions correspond to fixed points which are theoretically and experimentally interesting.

Upon turning on temperature, one generates thermal masses given by

$$M_{ij}^2 = \frac{\beta^{-2}}{24}\tilde{\lambda}_{ijkk} = \frac{2}{3}\pi^2\epsilon\beta^{-2}\lambda_{ijkk}$$

From the fixed point equation

$$M_{ij}^2 \sim \lambda_{ijmn}\lambda_{mnkk} + 2\lambda_{ikmn}\lambda_{mnjk}$$

The last term is obviously positive definite. The first term is not necessarily positive definite.

$$\lambda_{ijkl} = \lambda_{ijmn}\lambda_{mnkl} + 2 \text{ permutations}$$

A useful way to attack these equations is by the symmetry group of the solution. The maximal possible symmetry group is $O(N)$ and this is preserved for

$$\lambda_{ijkl} = \alpha\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

with $\alpha = \frac{1}{N+8}$, which is the famous $O(N)$ invariant fixed point. The thermal mass is $\frac{N+2}{N+8}\beta^{-2}\sum_i \phi_i\phi_i$. Clearly the vacuum is at the origin and we have a standard thermal gap ("Debye screening").

$$\lambda_{ijkl} = \lambda_{ijmn}\lambda_{mnkl} + 2 \text{ permutations}$$

We can discuss solutions which preserve some subgroup $G < O(N)$. Suppose that the fundamental representation of $O(N)$ is irreducible as a representation of G. This is the same as assuming that the only quadratic invariant of G is $\sum_i \phi_i\phi_i$. In this case we can prove a no-go theorem: no symmetry breaking occurs at finite temperature!

This no-go theorem covers a large class of examples, e.g. the $O(N)$ models, the cubic, tetrahedral, bi-fundamental, MN, tetragonal, and the Michel fixed points. See [Rychkov-Stergiou] for more information about these various classes.

A class of models that has two quadratic Casimirs are the bi-conical models with symmetry group $O(m) \times O(N-m)$:

$$V = 2\pi^2\epsilon \left[\alpha(\vec{\phi}_1^2)^2 + \beta(\vec{\phi}_2^2)^2 + 2\gamma\vec{\phi}_1^2\vec{\phi}_2^2\right]$$

where $\vec{\phi}_1$ is a vector of length m and $\vec{\phi}_2$ is of length $N-m$. The fixed point equations are (for nonzero γ):

$$\alpha = (m+8)\alpha^2 + (N-m)\gamma^2 \ ,$$

$$\beta = (N-m+8)\beta^2 + m\gamma^2 \ ,$$

$$1 = \alpha(m+2) + \beta(N-m+2) + 4\gamma \ .$$

The easiest case is the equal rank case $2m = N$. The equations are explicitly solvable and one finds the $O(N)$ point $(\alpha = \beta = \gamma = \frac{1}{N+8})$ as well as a new one!

$$\alpha = \beta = \frac{m}{2m^2 + 16}$$

$$\gamma = \frac{4 - m}{2m^2 + 16}$$

The thermal mass eigenvalues are both $\frac{6m}{2m^2+16} > 0$ and hence the symmetry is unbroken again.

The non-equal rank case is not as explicitly solvable. But there is a very nice way to understand the physics of it through the large rank limit. Rescale the couplings so that there is a convenient large rank limit:

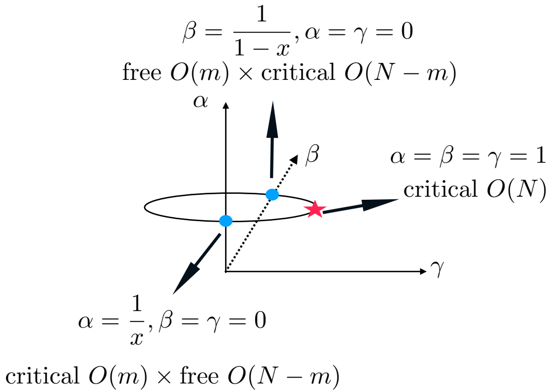
$$\tilde{\alpha} = N\alpha \ , \quad \tilde{\beta} = N\beta \ , \quad \tilde{\gamma} = N\gamma$$

We will also denote

$$x = m/N \ .$$

Some curious facts hold true in the leading order of the large rank limit:

Fact 1: For any x, there is a "circle" of fixed points!



Fact 2: At any point on the conformal manifold with $\gamma < 0$, there is a moduli space of vacua at zero temperature. This is because $\alpha\beta = \gamma^2$ is always satisfied. Hence the zero temperature potential is

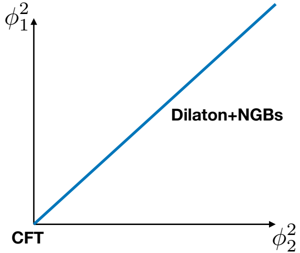
$$V \sim \left(\sqrt{\alpha}\phi_1^2 - \sqrt{\beta}\phi_2^2\right)^2 \ .$$

We therefore have a moduli space

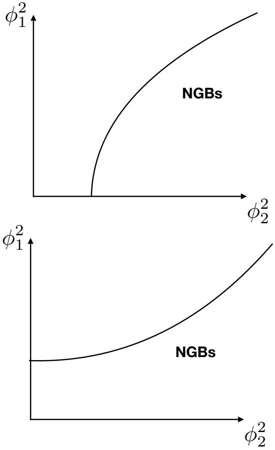
$$\left\{\phi_1^2 = \frac{\sqrt{\beta}}{\sqrt{\alpha}}\phi_2^2\right\}$$

This moduli space is connected to the origin.

As usual at the origin of the moduli space we have a (large rank) conformal theory while away from the origin we have some NGBs and a dilaton.



What happens to this moduli space of vacua as we turn on temperature β^{-1}? One should certainly expects that it would disappear but instead it is deformed!



$$\phi_1^2 - \phi_2^2 = CN\beta^{-2}$$

C is an $O(1)$ function on the half circle $\gamma < 0$. It may vanish at some isolated points. Whether $C < 0$ or $C > 0$ or $C = 0$ is very important – it tells us which symmetry breaking patterns are allowed.

These facts about the large rank theories are correct at any finite ϵ.

When we take finite rank corrections into account only ONE fixed point survives with $\gamma < 0$. And out of the thermal moduli space of vacua, only ONE vacuum survives.

More elaborate calculations are required to understand the phase diagram completely for all finite ϵ and large finite N. See the recent work by [Chai, Rabinovici, Sinha, Smolkin].

$x = \frac{1}{2}$ is an "atypical" example of this. It turns out that only the fixed point $\alpha = \beta = 1$, $\gamma = -1$ survives with $\gamma < 0$. But that one happens to have $C = 0$ so the moduli space of vacua $\phi_1^2 = \phi_2^2$ is not deformed in the large rank limit. To understand whether symmetry breaking takes place or not we need to go beyond the large rank limit and the answer is $\phi_1^2 = \phi_2^2 = 0$.

That $C = 0$ is the case for equal rank can be seen also from our explicit solution for the thermal masses:

$$M^2 \sim \frac{6N}{N^2 + 32} \sim O(1/N) \ .$$

By contrast, barring cancelations, the thermal masses should be $O(1)$ in the large rank limit.

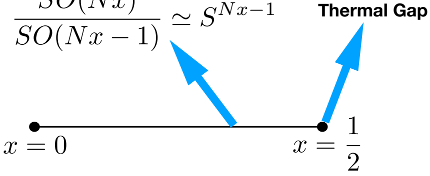
For $x \neq \frac{1}{2}$ the fixed point that survives the finite rank corrections has $C \neq 0$. Therefore symmetry breaking at finite temperature can be established from the leading large N computation. If $x > 1/2$ then $C < 0$ and if $x < 1/2$ then $C > 0$. This also naturally explains why $C = 0$ for $x = 1/2$.

Now we know which hyperbola we get for $x \in (0,1)$ we need to decide where the vacuum is on the hyperbola. This requires another computation beyond the large rank limit. But the answer is very simple! It is the vertex of the hyperbola: For $x > 1/2$ it is of the form $(\phi_1^2, \phi_2^2) = (0, -C\beta^{-2})$ and for $x < 1/2$ it is of the form $(\phi_1^2, \phi_2^2) = (C\beta^{-2}, 0)$.

So, for $O(m) \times O(N-m)$ symmetry, the smaller group of the two is broken and if they are equal then none is broken.

Suppose we approach $x = 1/2$ from below. $\sim N$ NGBs disappear. The radius of the coset shrinks as we get close to $x = 1/2$. But at $x = 1/2$ the theory is gapped at finite temperature. This would make no sense as a phase diagram for continuous x. In the planar theory, where x is continuous, $x = 1/2$ has a gapless critical point.

$$\frac{SO(Nx)}{SO(Nx-1)} \simeq S^{Nx-1}$$



We have given a construction of a CFT which break a global symmetry at any finite β. We have approached the problem by studying the vector model $O(m) \times O(N-m)$ at large N with fixed m/N and showed that the moduli space of vacua is thermally deformed to a hyperbola.

These conclusions hold for small finite ϵ. It would be nice to know if this holds all the way to $\epsilon < 1$. This requires a certain resummation, but it is entirely doable.

Since we are talking about continuous symmetry breaking

$$O(m) \times O(N-m) \longrightarrow O(m-1) \times O(N-m) ,$$

$(m < N/2)$ this cannot occur at finite temperature in 2+1 dimensions. It can only be true for $\epsilon < 1$. At $\epsilon = 1$ the thermal Goldstone bosons living on S^{m-1} are lifted by non-perturbative effects. Their mass is tiny $\sim e^{-N}$. So this is an interesting violation of the idea that the gap at sufficiently high temperatures should be of the order of the temperature!

What we have achieved here is a construction of conformal vector models in the ϵ expansion that behave counter-intuitively at finite temperature. But we have not constructed an example in integer dimensions.

Note the special case of $m = 1$ – this looks promising even in 2+1 dimensions. We are currently studying it. Quite remarkably the answer depends on the $\langle \phi^2 \phi^2 \phi^2 \rangle$ OPE coefficient in the usual critical $O(N)$ model. But for some unknown reason this vanishes all the way up to order $1/N^{5/2}$ (the order $1/N^{3/2}$ was recently heroically computed by Goykhman-Smolkin). Recall that a similar OPE coefficient vanishes in 2d due to KW duality.

The $m = 1$ case therefore strangely remains unclear in 3d. In fact at present it is unclear whether an example exists for more general Chern-Simons matter theories for the same reason.

Choi, Rabinovici, and Sumyadeep have tried to construct an explicit 4d Banks-Zaks like fixed point that breaks a symmetry at finite T. They have many a priori promising candidates but as far as I know each one of them strangely fails to produce an example in strictly 4d. Their upcoming paper constructs examples at strictly $N = \infty$ in $d = 4$ but no examples are known for finite N.

The problem of symmetry restoration at high temperature is closely related to the problem of deconfinement at high temperature. Indeed in 3d, if we gauge an ordinary $\mathbb{Z}_2$ symmetry we get a dual one-form $\mathbb{Z}_2$ symmetry and if the former is broken the latter would be unrbroken and vice versa.

So in 2+1d the "no-hair theorem" is really equivalent to high temperature deconfinement.

We learned that it is not impossible that a critical point would be in a broken phase upon heating it up. At least not as far as models in $4-\epsilon$ dimensions are concerned. Are there such gauge theories in 3+1 dimensions? In 2+1 dimensions? Is there a proof that this is impossible? How come these models in fractional dimensions exist? Why do they violate our intuition from thermodynamics?

Thank You!

Consider a conformal theory in 2+1 dimensions. We put it in a box with sides L_x, L_y, L_z. For $L_z \ll L_x, L_y$ we can think about it as the high temperature limit and hence from thermodynamics:

$$\log Z = e^{fL_xL_y/L_z^2} \ .$$

with some $f > 0$.

It is useful to interpret this in the L_x direction (L_y is similar). In this case the ground state energy is L_y/L_z^2. It arises from integrating out the KK modes on the z circle. So in the L_x direction this is a Casimir energy effect and the energy density in the y direction is L_z^{-2}.

Furthermore, if there is a gap of the theory on the cylinder then the effective action after reducing on the cylinder ought to be

$$fL_z^{-2}\int dxdy$$

and no further power corrections are possible due to locality. This means that $\log Z = e^{fL_xL_y/L_z^2}$ is exact up to exponentially smaller terms.

Next we assume that there is a $\mathbb{Z}_2$ symmetry. Furthermore we assume it is broken at finite temperature in infinite space. In other words we take $L_x, L_y = \infty$ and the claim is that $Tr(Oe^{-L_z H}) \neq 0$. It is best to think about it first as a statement about the theory quantized in the L_x direction.

This means that the theory in the directions $x - y$ that we obtain after reducing on the direction z has two vacua. In each the energy density is of order fL_z^{-2} and f is the same in the two vacua. In finite volume in the $x - y$ plane there are now two approximate ground states each with a gap that scales like L_z^{-1} and their energy difference is tiny:

$$\Delta E \sim L_z^{-1} e^{-cL_y/L_z}$$

with some positive c. The tension of the domain wall is simply c/L_z.

We can therefore write the partition function from this quantization in the direction L_x. Neglecting exponentially small corrections the two vacua have identical energies and we find

$$Z = 2e^{fL_xL_y/L_z^2}$$

the factor of 2 in front is due to the two-fold degeneracy. This should be contrasted with the previous expression, which did not have this log2 correction in the free energy (and in fact could not have had any corrections besides the exponentially small ones).

It is useful to include the exponentially small correction arising due to the domain wall. Remember that the splitting of the two states is symmetric:

$$Z = e^{fL_xL_y/L_z^2 - L_x\Delta E/2} + e^{fL_xL_y/L_z^2 + L_x\Delta E/2} = 2e^{fL_xL_y/L_z^2}\cosh(L_x\Delta E/2)\ .$$

This is approximated by

$$Z = 2e^{fL_xL_y/L_z^2}(1 + L_x^2\Delta E^2/8)\ .$$

This represents exponentially small corrections of order e^{-2cL_y/L_z}. The exponentially small corrections from ordinary particles in each of these vacua are of order e^{-L_x/L_z}. These two types of corrections should together combine to a Euclidean invariant partition function where L_y and L_x are interchangeable.

We therefore see that $-2log(\Delta E)/L_y$ must actually coincide with the gap around each vacuum! In estimating ΔE from two separated domain walls we are alluding to an instanton gas approximation which may not hold true in general. But the relation between ΔE and the lightest particle around each vacuum is more general.

For instance in the massive Schwinger model at $\theta = \pi$ the domain wall is an electron and the simplest excitation about the vacuum is an electron-positron bound state. So we see that the relationship works at weak coupling.

We are now ready to interpret all of this in the L_z quantization. Up to exponentially small corrections we have $Z = 2e^{fL_xL_y/L_z^2}$. The $\log 2$ correction is not consistent with the usual high temperature effective theory, indicating that there are two vacua at high temperature. We can use this result also on $S^2 \times S^1$ where this $\log 2$ entails a certain doubling of the operators at high dimension. This means that the finite temperature theory is an ordinary fluid appended by some $\mathbb{Z}_2$ hair.

It is now useful to consider g insertions. We can insert the charge operator in essentially two distinct ways. One is such that it wraps L_z and the other that it does not.

It is easier to begin with the case where it wraps $z - y$. We can study this in terms of the L_x quantization. After the dimensional reduction this looks like a $\mathbb{Z}_2$ charge for the theory living on $x - y$. Therefore we get

$$Z_{g_{zy}} = -e^{fL_xL_y/L_z^2 - L_x\Delta E/2} + e^{fL_xL_y/L_z^2 + L_x\Delta E/2} = 2e^{fL_xL_y/L_z^2}\sinh(L_x\Delta E/2)\ .$$

which approximately is given by

$$Z_{g_{zy}} = 2e^{fL_xL_y/L_z^2}L_x\Delta E/2 = e^{fL_xL_y/L_z^2 - cL_y/L_z}\ .$$

Recall that we also have contributions from massive particles in each vacuum. These lead to exponentially smaller terms.

It is instructive to interpret $e^{fL_xL_y/L_z^2-cL_y/L_z}$ also in the L_z and L_y quantizations. In L_y quantization this looks like a defect Hilbert space and we are at very low temperatures. Hence the contribution is from the ground state only. We see that the ground state energy has a negative Casimir energy density f/L_z^2 and there is a positive correction c/L_z to the energy which can be viewed as the ground state energy difference due to the twist in this channel, i.e. c/L_z is the domain wall tension, which is positive.

It is also interesting to interpret this in L_z quantization. We are now studying a CFT in finite large volume and high temperature. But we are in a defect Hilbert space. Here the puzzling thing is that the contribution to the free energy (and entropy) from the defect is negative: $-cL_y/L_z$. The defect creates a straight domain wall in the two space dimensions extended along y. It is therefore reasonable that the entropy be extensive along y. This contribution is very much analogous to the g coefficient in 1+1 dimensions which is also obtained from a high temperature limit.

A defect with a negative contribution to the entropy / heat capacity / free energy is not in principle disallowed – for instance, $\log g < 0$ for the Dirichlet boundary conditions in the 1+1 dimensional Ising model. Here the situation is a little more extreme since the defect also has negative heat capacity and not just negative entropy (this is if one can sensibly separate the heat capacity of the defect from the bulk which is not obvious).

The contribution $-cL_y/L_z$ come from the term $-c/L_z \int dy$ in the effective action in the presence of a defect.